Comparing Estimator Covariances at Matched Spatial Resolutions for Imaging System Design

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General Problem

Determine "best" imaging system parameters

- aperture geometry (parallel / fan / cone)
- aperture openings (resolution vs sensitivity)
- dwell times
- ...

Image formation model:

$$\boxed{\text{Object}} \xrightarrow{\text{Energy}} \boxed{\text{Sensors}} \xrightarrow{\text{Measurements } \underline{Y}} \boxed{\text{Estimator}} \longrightarrow \text{Image } \underline{\hat{X}}$$

Goal (aka holy grail):

Specify sensor properties/parameters to

"maximize the information in the image about the object."

Possible Approaches

- Cramer-Rao Bound
  - Fisher information depends on system properties
  - Estimator independent
  - Unnatural for ill-conditioned problems
- Uniform Cramer-Rao Bound
  - Estimator independent
  - Allows for regularization-induced bias
  - Selection of bias-gradient norm and interpretation nontrivial
- Mutual Information
  - Global measure. Indirect relation to reconstruction errors
- Detection Task Performance
  - Task dependent
- Estimator performance analysis
  - Conclusions are estimator-dependent

Implicitly Defined Estimators

$$\underline{\hat{X}} \stackrel{\triangle}{=} \arg\max_{\underline{x}} \Phi(\underline{x}, \underline{Y})$$

Covariance (IEEE T-IP 5(3) 1996)

$$\text{Cov}\left\{\underline{\hat{X}}\right\} \approx [-\nabla^{20}\Phi]^{-1}[\nabla^{11}\Phi]\text{Cov}\left\{\underline{Y}\right\}\left[\nabla^{11}\Phi\right]'[-\nabla^{20}\Phi]^{-1}$$

Local impulse response (IEEE T-IP 5(9) 1996)

$$\lim_{\delta\to 0}\frac{E[\underline{\hat{X}}(\underline{Y})|\underline{x}+\delta\underline{e}_j] - E[\underline{\hat{X}}(\underline{Y})|\underline{x}]}{\delta} \approx [-\nabla^{20}\Phi]^{-1}[\nabla^{11}\Phi]\frac{\partial E[\underline{Y}|\underline{x}]}{\partial x_j}$$

- Both $\text{Cov}\left\{\underline{Y}\right\}$ and Φ depend on the imaging system
- Φ depends on the estimator
- Shape and width of local impulse response changes as system parameters vary!

Regularized Least Squares

Gaussian measurement model:

$$\underline{Y} \sim \mathcal{N}\big(\boldsymbol{A}_\alpha \underline{x}^{\text{true}}, \boldsymbol{K}_\alpha\big)$$

System matrix $\boldsymbol{A}_\alpha$ and covariance $\boldsymbol{K}_\alpha$ depend on system parameters α.

Regularized least-squares estimator:

$$\begin{aligned} \hat{\underline{X}} &= \arg\min_{\underline{x}} (\underline{Y} - \boldsymbol{A}_\alpha \underline{x})' \boldsymbol{K}_\alpha^{-1} (\underline{Y} - \boldsymbol{A}_\alpha \underline{x}) + \underline{x}' \boldsymbol{R} \underline{x} \\ &= [\boldsymbol{F}_\alpha + \boldsymbol{R}^{\text{sym}}]^{-1} \boldsymbol{A}_\alpha' \boldsymbol{K}_\alpha^{-1} \underline{Y} \end{aligned}$$

where

- Fisher information matrix: $\boldsymbol{F}_\alpha \triangleq \boldsymbol{A}_\alpha' \boldsymbol{K}_\alpha^{-1} \boldsymbol{A}_\alpha$
- Symmetric component of $\boldsymbol{R}$: $\boldsymbol{R}^{\text{sym}} \triangleq \frac{1}{2}(\boldsymbol{R} + \boldsymbol{R}')$

Resolution and Noise

For the regularized LS estimator:

$$E[\underline{\hat{X}}] = [\boldsymbol{F}_\alpha + \boldsymbol{R}^{\text{sym}}]^{-1} \boldsymbol{A}'_\alpha \boldsymbol{K}_\alpha^{-1} \boldsymbol{A}_\alpha \underline{x}^{\text{true}} = \boldsymbol{P} \underline{x}^{\text{true}}$$

where the *PSF matrix* is $\boxed{\boldsymbol{P} \triangleq [\boldsymbol{F}_\alpha + \boldsymbol{R}^{\text{sym}}]^{-1} \boldsymbol{F}_\alpha.}$
(The jth column of $\boldsymbol{P}$ is the local impulse response.)

$$\boxed{\text{Cov}\left\{\underline{\hat{X}}\right\} = [\boldsymbol{F}_\alpha + \boldsymbol{R}^{\text{sym}}]^{-1} \boldsymbol{F}_\alpha [\boldsymbol{F}_\alpha + \boldsymbol{R}^{\text{sym}}]^{-1}}$$

We would like to choose the system parameters α so as to minimize the "noise," subject to some constraint on spatial resolution.

As we vary α, both the covariance *and* the resolution properties change.

Pre-specified PSF Matrix

(for *exactly* matched spatial resolution)

Suppose we insist that the PSF matrix be a pre-determined matrix $\boldsymbol{P}_0$.

- Can we find a regularizer $\boldsymbol{R}$ that achieves that specification $\forall \alpha$?
- How do we minimize $\mathrm{Cov}\left\{\underline{\hat{X}}\right\}$ over α subject to that constraint?
- How does the minimum covariance vary as a function of $\boldsymbol{P}_0$?

Achievability of PSF Specification

Recall $\boldsymbol{P} \triangleq [\boldsymbol{F}_\alpha + \boldsymbol{R}^{\text{sym}}]^{-1} \boldsymbol{F}_\alpha$.
Rearranging and solving yields the regularization matrix:

$$\boldsymbol{R}_\alpha^{\text{sym}} \triangleq \boldsymbol{F}_\alpha [\boldsymbol{P}_0^{-1} - \boldsymbol{I}].$$

Requirements:

- $\boldsymbol{P}_0$ must be invertible
- $\boldsymbol{F}_\alpha + \boldsymbol{R}_\alpha^{\text{sym}} = \boldsymbol{F}_\alpha \boldsymbol{P}_0^{-1}$ must be invertible, $\therefore$ $\boldsymbol{F}_\alpha^{-1}$ must exist
- $\boldsymbol{R}_\alpha^{\text{sym}}$ must be symmetric, $\therefore$ $\boldsymbol{F}_\alpha \boldsymbol{P}_0^{-1} = \boldsymbol{P}_0^{-T} \boldsymbol{F}_\alpha$,
  Sufficient condition: $\boldsymbol{P}_0$ symmetric and $\boldsymbol{F}_\alpha$ and $\boldsymbol{P}_0$ commute

Otherwise *no* regularization matrix $\boldsymbol{R}$ provides the desired PSF matrix.

Counter-intuitive?

Covariance with Constrained PSF

If above conditions hold, and $\boldsymbol{R}_\alpha = \boldsymbol{F}_\alpha[\boldsymbol{P}_0^{-1} - \boldsymbol{I}]$, then

$$
\begin{aligned}
\operatorname{Cov}\left\{\underline{\hat{X}}\right\} &= [\boldsymbol{F}_\alpha + \boldsymbol{R}^{\mathrm{sym}}]^{-1}\boldsymbol{F}_\alpha[\boldsymbol{F}_\alpha + \boldsymbol{R}^{\mathrm{sym}}]^{-1} \\
&= [\boldsymbol{F}_\alpha + \boldsymbol{R}^{\mathrm{sym}}]^{-1}\boldsymbol{F}_\alpha\boldsymbol{F}_\alpha^{-1}\boldsymbol{F}_\alpha[\boldsymbol{F}_\alpha + \boldsymbol{R}^{\mathrm{sym}}]^{-1} \\
&= \boxed{\boldsymbol{P}_0\boldsymbol{F}_\alpha^{-1}\boldsymbol{P}_0',}
\end{aligned}
$$

a simple function of Fisher information $\boldsymbol{F}_\alpha$ and PSF matrix $\boldsymbol{P}_0$.

Annoyingly simple, in fact, since it is just the covariance of post-smoothed *unregularized* weighted least squares:

$$\underline{\hat{X}} = \boldsymbol{P}_0\underline{\hat{X}}_{\mathrm{WLS}} = \boldsymbol{P}_0[\boldsymbol{A}_\alpha'\boldsymbol{K}_\alpha^{-1}\boldsymbol{A}_\alpha]^{-1}\boldsymbol{A}_\alpha'\boldsymbol{K}_\alpha^{-1}\boldsymbol{Y}.$$

So Why Regularize?

- Faster converging algorithms (better condition number)
- Constraints (*e.g.* nonnegativity) could change conclusions
- Nonquadratic regularization not equivalent to post-smoothing.

But, it is still disappointing that (under the above assumptions)

regularized least squares / penalized-likelihood / MAP reconstruction

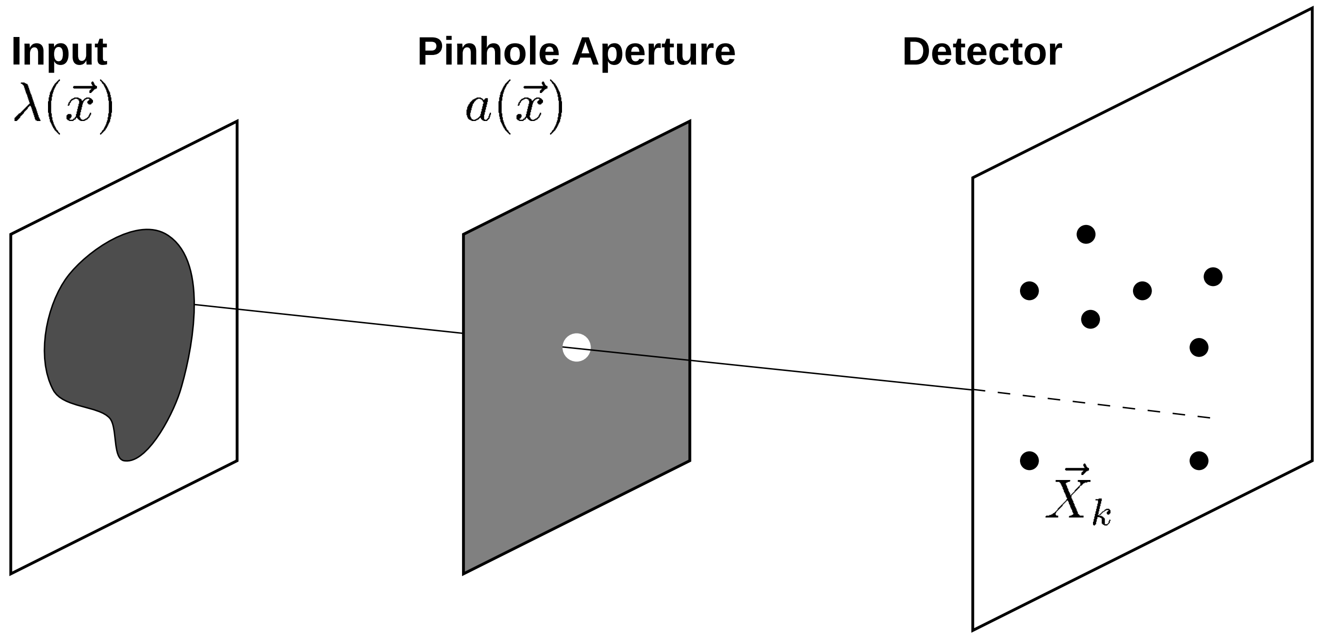
$$\equiv$$

post-smoothed least squares

when spatial resolution is exactly matched.

Pinhole Imaging Problem

Pinhole imaging system with position-sensitive detector.



Goal: find aperture function $a(\vec{x})$ that minimizes variance of reconstructed object estimate, subject to a spatial resolution constraint.

Small pinhole $\Rightarrow$ better resolution, but fewer photons (and vice versa)

Assumptions

(Big leap to continuous problem)

- Emission process is a Poisson point process with rate $\lambda(\vec{x})$
- Number N of detected photons is Poisson
- Shift invariant system response $a(\vec{x})$ (*e.g.* scanned pinhole)
- Perfect position-sensitive detector
- System sensitivity $\propto \left[\int |a(\vec{x})|^2 \, d\vec{x}\right]^{-1} = \left[\int |A(u)|^2 \, du\right]^{-1}$
- Kernel-based density estimator:

$$\hat{\lambda}(\vec{x}) = \frac{1}{N} \sum_{k=1}^{N} g(\vec{x} - \vec{X}_k)$$

... (Optics Express, 1998)

- Resolution:

$$E[\hat{\lambda}(\vec{x})] = \lambda(\vec{x}) * a(\vec{x}) * g(\vec{x})$$

so the PSF is $p(\vec{x}) = a(\vec{x}) * g(\vec{x})$. In frequency domain:

$$P(u) = A(u)G(u), \ \therefore \ G(u) = P(u)/A(u).$$

- Variance:

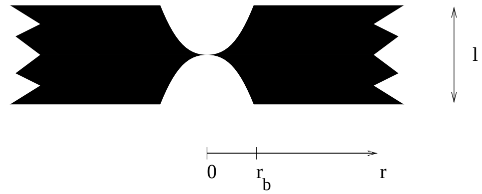
$$\mathrm{Var}\left\{\hat{\lambda}(\vec{x})\right\} \propto \int \left|\frac{P(u)}{A(u)}\right|^2 \, du \cdot \int |A(u)|^2 \, du.$$

Minimizing variance with respect to $A(u)$ by variational calculus yields:

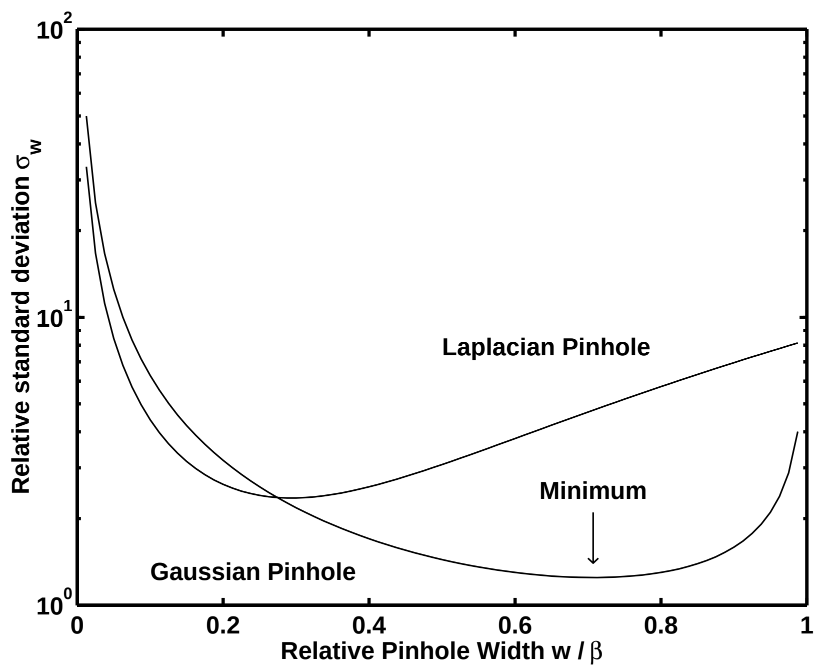
$$\boxed{|A(u)| = \sqrt{|P_0(u)|}.}$$

If $p_0(\vec{x})$ is Gaussian, then $a(\vec{x})$ is also Gaussian, with FHWM/$\sqrt{2}$.

Example: Gaussian Pinhole



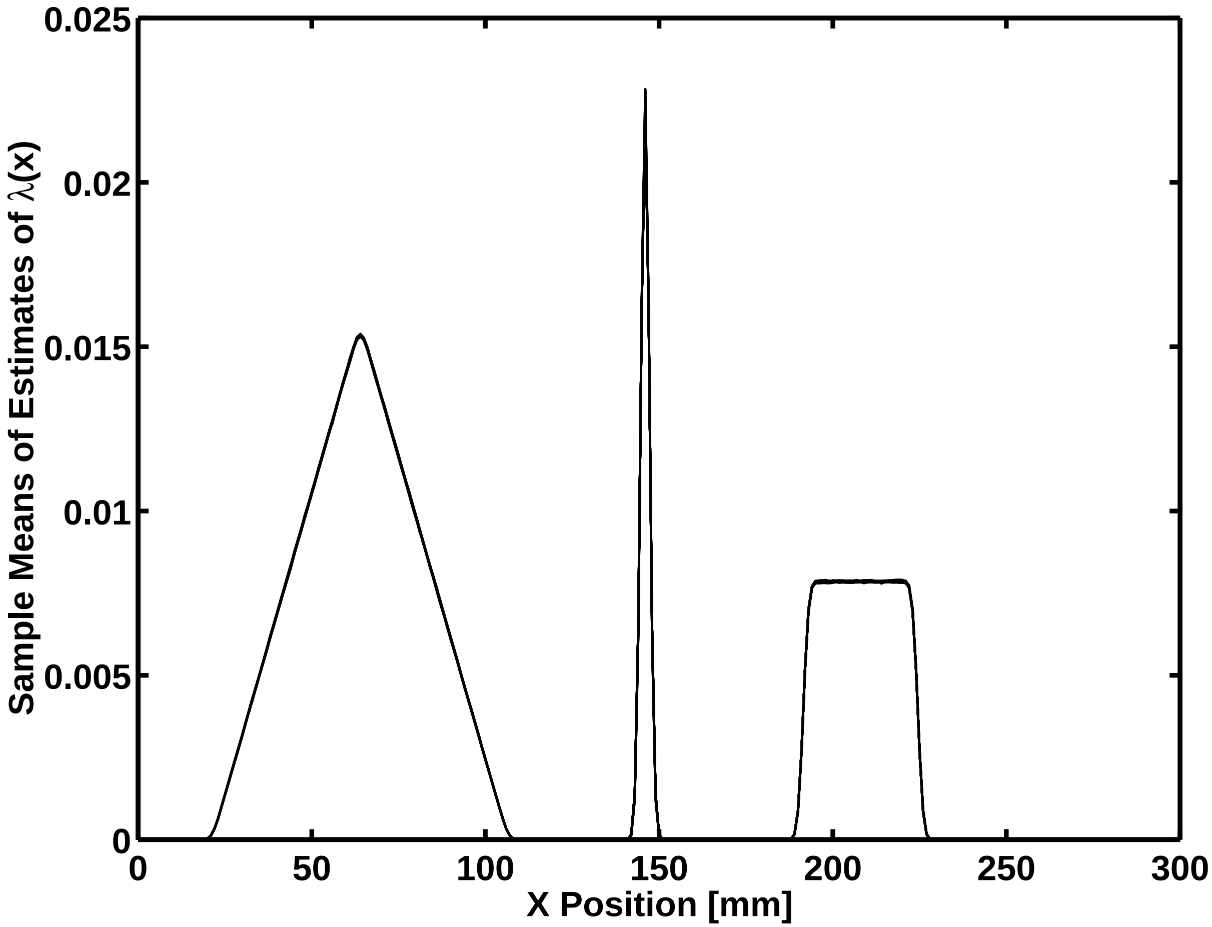
For a Gaussian-apodized inverse filter: $w_{\min} = \beta/\sqrt{2}.$

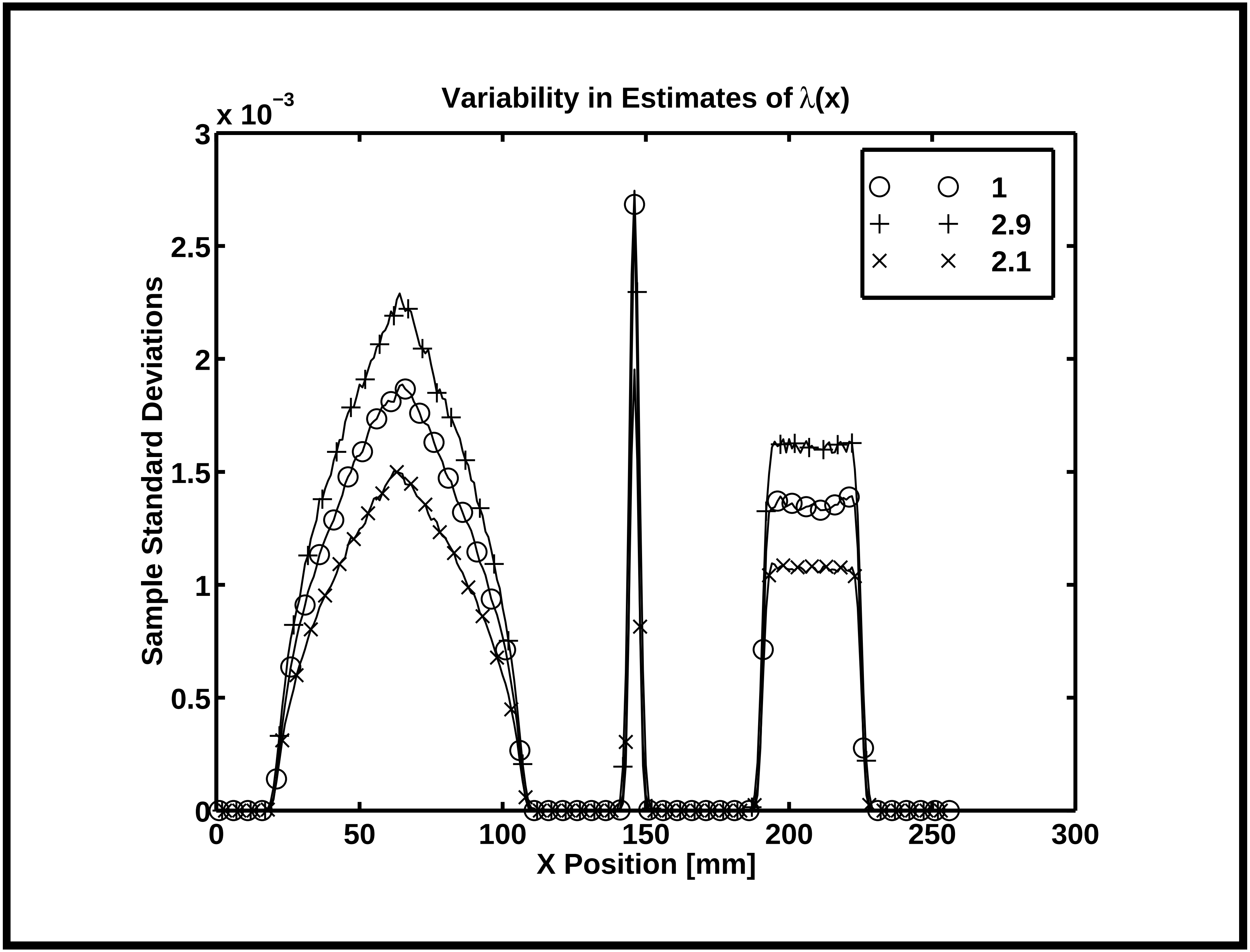


1D Simulation

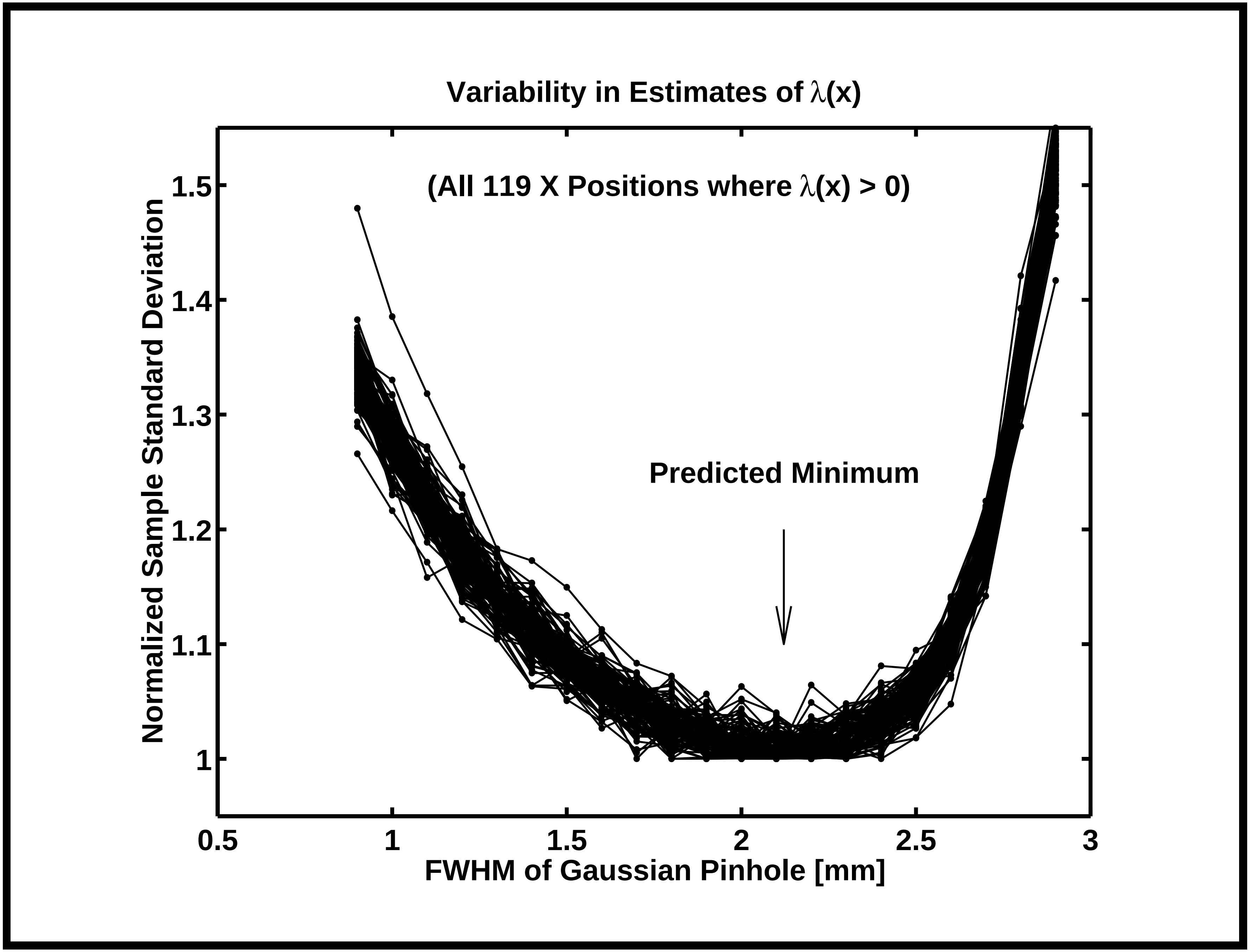
- $\lambda(\vec{x}) \propto 9\delta(x-146) + \mathrm{rect}((x-208)/64) + 2\Lambda((x-64)/44)$
- Target resolution: $\beta = 3$ mm.
- $a(\vec{x})$: 1D Gaussian pinhole, FWHM $w \in [0.9, 2.9]$ mm.
- 4000 realizations per pinhole size
- Mean number of photons per realization: $100w$ (*i.e.* the sensitivity increased linearly with pinhole size)
- Gaussian-apodized inverse filter, with FWHM $\sqrt{\beta^2 - w^2}$.
- Theoretically predicted variance-minimizing pinhole size is $w = \beta/\sqrt{2} \approx 2.1$ mm

Spatial Resolution Check (w=0.9:0.1:2.9)





Variability in Estimates of λ(x)
x 10^−3
3
2.5
2
1.5
1
0.5
0
Sample Standard Deviations
0
50
100
150
200
250
300
X Position [mm]
1
2.9
2.1


Variability in Estimates of λ(x)
(All 119 X Positions where λ(x) > 0)
Predicted Minimum
Normalized Sample Standard Deviation
FWHM of Gaussian Pinhole [mm]
1.5
1.4
1.3
1.2
1.1
1
0.5
1
1.5
2
2.5
3

Conclusions (?)

- Regularized WLS $\equiv$ post-filtered unregularized WLS when spatial resolution is exactly matched, with covariance $\boldsymbol{P}_0 \boldsymbol{F}_\alpha^{-1} \boldsymbol{P}_0'$.

Open questions

- How to minimize $\boldsymbol{P}_0 \boldsymbol{F}_\alpha^{-1} \boldsymbol{P}_0$ over α for 2d/3d problems?
- How does $\min_\alpha \boldsymbol{P}_0 \boldsymbol{F}_\alpha^{-1} \boldsymbol{P}_0$ vary with α?
- What happens when $\boldsymbol{P}_0$ and $\boldsymbol{F}_\alpha$ do not commute?
- What if we relax the PSF matrix specification to be more like filter design specs?
- Other estimators for pinhole problem?
- Extension of uniform CR bound to nonparametric estimation problems