Fast Variance Prediction for Iteratively Reconstructed CT with Arbitrary Geometries

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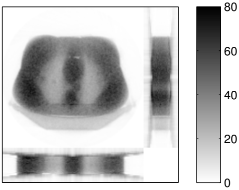
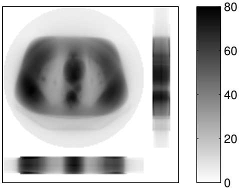


Empirical σ



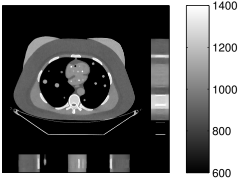
Predicted σ

Motivation

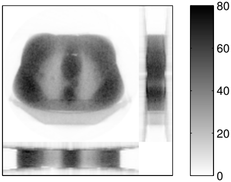
- Statistical image reconstruction (SIR) methods for CT are nonlinear, complicating analysis of noise (co)variance
- Predicted (co)variance can inform:
  - Regularization design
  - Reconstruction analysis
  - Tube current modulation
- Prior methods exist for fast variance prediction for 2DCT and for some limited 3DCT geometries (Zhang-O'Connor 2007)

Contribution

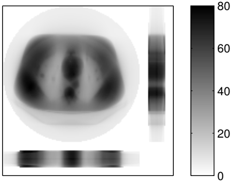
- Fast variance prediction for SIR with arbitrary CT geometries
  - Frequency response approximation for $\mathbf{A}'\mathbf{W}\mathbf{A}$ operator (projection, statistical weighting, and backprojection)
  - Extends previous 2D fan-beam methods to any CT geometry
  - Further simplification to 3DCT with small cone angles
  - Evaluation of methods with simulated and real sinogram data
- Example with real data



SIR for 3D CT



Empirical σ
(100 hours)



Predicted σ
(11 min)

Prior work on variance prediction for SIR

* J. A. Fessler, "Mean and variance of implicitly defined biased estimators (such as penalized maximum likelihood): Applications to tomography," *IEEE Trans. Im. Proc.*, vol. 5, no. 3, 493–506, Mar. 1996.
* J. Qi and R. M. Leahy, "A theoretical study of the contrast recovery and variance of MAP reconstructions from PET data," *IEEE Trans. Med. Imag.*, vol. 18, no. 4, 293–305, Apr. 1999.
* Y. Zhang-O'Connor and J. A. Fessler, "Fast predictions of variance images for fan-beam transmission tomography with quadratic regularization," *IEEE Trans. Med. Imag.*, vol. 26, no. 3, 335–46, Mar. 2007.
* ——,"Fast variance predictions for 3D cone-beam CT with quadratic regularization," in *Proc. SPIE 6510 Medical Imaging 2007: Phys. Med. Im.*, 2007, 65105W:1–10.
* S. M. Schmitt and J. A. Fessler, "Fast variance computation for iterative reconstruction of 3D helical CT images," in *Proc. Intl. Mtg. on Fully 3D Image Recon. in Rad. and Nuc. Med*, 2013, 162–5.

- Too slow or too geometry specific

Outline

SIR formulation

- Reconstruct image $\hat{\mathbf{x}}$ via a (strictly convex) minimization:

$$\hat{\mathbf{x}} = \operatorname{argmin}_{\mathbf{x}} L(\mathbf{Y}; \mathbf{x}) + \alpha R(\mathbf{x})$$

- Data-fit term assumes independent observations $\{Y_i\}$:

$$L(\mathbf{Y}; \mathbf{x}) = \sum_{i=1}^{N_{\mathrm{Y}}} L_i(Y_i; [\mathbf{A}\mathbf{x}]_i), \qquad \mathbf{A} \text{ is system matrix}$$

For PWLS: $L_i(Y_i; y) = \frac{1}{2} w_i \left(\log(I_{0,i}/Y_i) - y\right)^2$

- Regularizer has general form:

$$R(\mathbf{x}) = \sum_d r_d \sum_k \psi([\mathbf{C}_d \mathbf{x}]_k)$$

  - r_d : direction-dependent regularization parameters;
  - ψ : edge-preserving potential function (smooth), $\ddot{\psi}(0) = 1$
  - $\mathbf{C}_d$: finite-differencing matrix in dth direction

Covariance approximation - Matrix form

- Covariance matrix of reconstruction $\hat{\mathbf{x}}$ is approximately (JF '96):

$$\text{cov}(\hat{\mathbf{x}}) \approx \left[\bar{\mathbf{F}} + \alpha \nabla^2 R(\check{\mathbf{x}})\right]^{-1} \hat{\mathbf{F}} \left[\bar{\mathbf{F}} + \alpha \nabla^2 R(\check{\mathbf{x}})\right]^{-1}$$

$$\bar{\mathbf{F}} \triangleq \mathbf{A}'\bar{\mathbf{W}}\mathbf{A} \text{ (Fisher information)}$$

$$\hat{\mathbf{F}} \triangleq \mathbf{A}'\hat{\mathbf{W}}\mathbf{A}$$

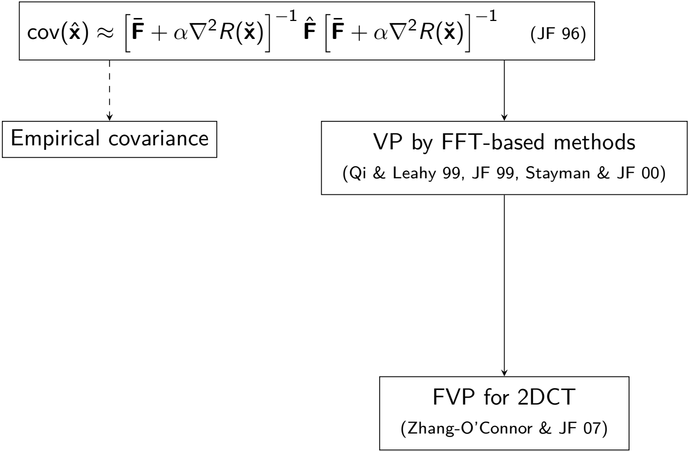
- $\nabla^2 R$: Hessian of regularizer
- $\check{\mathbf{x}}$: reconstruction from noiseless (!) data
- Diagonal weighting matrices $\bar{\mathbf{W}}$ and $\hat{\mathbf{W}}$:

$$[\bar{\mathbf{W}}]_{ii} \triangleq \left. \frac{\partial^2}{\partial y^2} L_i(Y_i; y) \right|_{y=[\mathbf{A}\check{\mathbf{x}}]_i}$$

$$[\hat{\mathbf{W}}]_{ii} \triangleq \text{var}(Y_i) \cdot \left. \frac{\partial^2}{\partial y \, \partial Y_i} L_i(Y_i; y) \right|_{y=[\mathbf{A}\check{\mathbf{x}}]_i}$$

($\hat{\mathbf{W}} = \bar{\mathbf{W}}$ for PWLS with appropriate statistical weighting.)

Towards Tractable (Co)Variance Prediction



Local Frequency Response

- $\bar{\mathbf{F}}$ and $\mathbf{R} \triangleq \sum_d r_d \mathbf{C}_d' \mathbf{C}_d$ are approximately locally shift invariant near the jth voxel,
- so the DFT $\mathbf{Q}$ approximately locally diagonalizes them:

$$\bar{\mathbf{F}} \approx \mathbf{Q}' \operatorname{diag}\left\{\bar{F}_j(\vec{\nu}_k)\right\} \mathbf{Q} \qquad \text{(local frequency response)}$$

$$\mathbf{R} \approx \mathbf{Q}' \operatorname{diag}\left\{R(\vec{\nu}_k)\right\} \mathbf{Q} \qquad \vec{\nu}_k : \text{ spatial frequencies}$$

- Diagonalization (somewhat) simplifies variance prediction:

$$\operatorname{var}(\hat{x}_j) = \mathbf{e}_j' \operatorname{cov}(\hat{\mathbf{x}}) \mathbf{e}_j \approx \mathbf{e}_j' \left[\bar{\mathbf{F}} + \alpha \mathbf{R}\right]^{-1} \hat{\mathbf{F}} \left[\bar{\mathbf{F}} + \alpha \mathbf{R}\right]^{-1} \mathbf{e}_j$$

$$\approx \mathbf{e}_j' \mathbf{Q}' \operatorname{diag}\left\{S_j(\vec{\nu}_k)\right\} \mathbf{Q} \mathbf{e}_j = \frac{1}{N_{\mathrm{x}}} \sum_k S_j(\vec{\nu}_k)$$

$$\approx \int_{[-\frac{1}{2},\frac{1}{2}]^d} S_j(\vec{\nu}) \, \mathrm{d}\vec{\nu}, \quad \text{local NPS: } S_j(\vec{\nu}) \triangleq \frac{\hat{F}_j(\vec{\nu})}{(\bar{F}_j(\vec{\nu}) + \alpha R(\vec{\nu}))^2}$$

- Tolerable computation for one voxel; impractical for many/all

Regularizer Hessian Approximation

- (Co)variance prediction needs regularizer Hessian:

$$\nabla^2 R(\mathbf{x}) = \sum_d r_d \mathbf{C}_d' \ddot{\mathbf{\Psi}}_d(\mathbf{x}) \mathbf{C}_d, \qquad \left[\ddot{\mathbf{\Psi}}_d(\mathbf{x})\right]_{kk} = \ddot{\psi}([\mathbf{C}_d \mathbf{x}]_k)$$

- Near edges, predicted variance sensitive to (unknown) $\check{\mathbf{x}}$
- Even when $\check{\mathbf{x}}$ is known (simulations), variance predictions near edges are inaccurate due to local shift variance
- To remove $\check{\mathbf{x}}$ dependence and simplify, we approximate: $\ddot{\psi}([\mathbf{C}_d \check{\mathbf{x}}]_k) \approx \ddot{\psi}(0) = 1$. In matrix form: $\ddot{\mathbf{\Psi}}_d(\check{\mathbf{x}}) \approx \mathbf{I}$.
- Shift-invariant regularizer Hessian approximation:

$$\nabla^2 R(\check{\mathbf{x}}) \approx \sum_d r_d \mathbf{C}_d' \mathbf{C}_d \triangleq \mathbf{R}, \qquad R(\vec{\nu}) \approx \|\vec{\nu}\|_2^2$$

Approximation is accurate (only) away from edges.

Outline

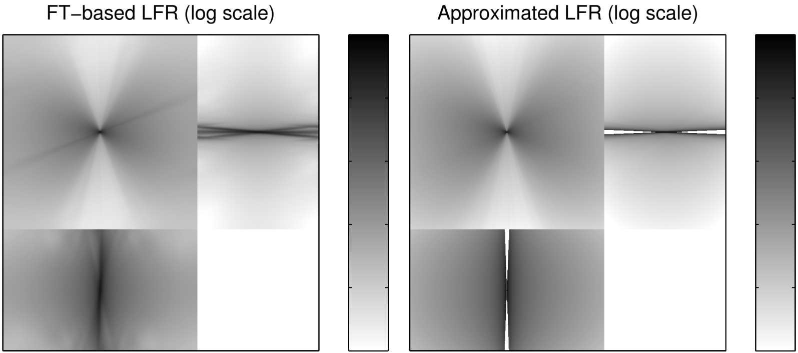
Local Frequency Response Factorization

- Need local frequency response $\bar{F}_j(\vec{\nu})$ of $\mathbf{A}'\bar{\mathbf{W}}\mathbf{A}$ near jth voxel
- Skipping long derivation...
- We derive a factorization of the local frequency response:

$$\begin{aligned} \bar{F}_j(\vec{\nu}) &= \sum_k [\bar{\mathbf{F}}]_{kj} \exp\left(-\imath 2\pi \vec{\nu} \cdot (\vec{n}_k - \vec{n}_j)\right) && \text{(DFT-based)} \\ &\approx J(\vec{\nu})\, \bar{E}_j(\vec{\Theta}) && \text{(Approximation)} \end{aligned}$$

- J is independent of system geometry, voxel location, weighting
- $\bar{E}_j$ depends on angle $\vec{\Theta} = \vec{\nu}/||\vec{\nu}||$, not on $\varrho = ||\vec{\nu}||$
- Applicable to arbitrary CT geometries (generalizes F3D 2013)
- Also useful for regularization design (Cho & JF, F3D, 2013)

Local Frequency Response



2D slices through 3D LFR $\bar{F}_j(\vec{\nu})$

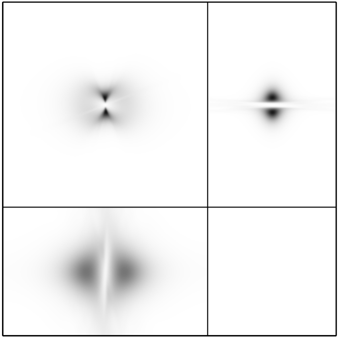
- For parallel-beam CT with $\mathbf{W} = \mathbf{I}$ these would look like $1/\rho$.
- Infamous "missing cone" is evident

Outline

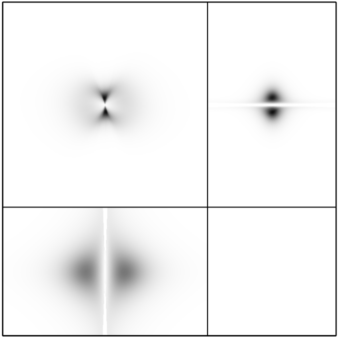
Local Noise Power Spectrum (NPS)

$$\text{Local NPS: } S_j(\vec{\nu}) = \frac{\hat{F}_j(\vec{\nu})}{(\bar{F}_j(\vec{\nu}) + \alpha R(\vec{\nu}))^2}$$

FT–based NPS (linear scale)



Approximated NPS



(reverse color scale: NPS is small at DC and for large frequencies)

Variance Prediction — Arbitrary Geometries

- Recall that noise variance is integral of local NPS:

$$\text{var}(\hat{x}_j) \approx \int_{[-\frac{1}{2},\frac{1}{2}]^d} S_j(\vec{\nu})\,\mathrm{d}\vec{\nu} = \int_{[-\frac{1}{2},\frac{1}{2}]^d} \frac{\hat{F}_j(\vec{\nu})}{(\bar{F}_j(\vec{\nu}) + \alpha R(\vec{\nu}))^2}\,\mathrm{d}\vec{\nu}$$

- Using LFR factorization and rewriting in spherical coordinates:

$$\text{var}(\hat{x}_j) \approx \frac{1}{\alpha} \int_{\mathbb{S}^d} \frac{\hat{E}_j(\vec{\Theta})}{\bar{E}_j(\vec{\Theta})} G(\alpha^{-1}\bar{E}_j(\vec{\Theta}), \vec{\Theta})\,\mathrm{d}\vec{\Theta}$$

- Tabulate based on voxel basis and regularizer:

$$G(\gamma, \vec{\Theta}) \triangleq \int_0^{\varrho_{\max}(\vec{\Theta})} \frac{\gamma J(\varrho, \vec{\Theta})}{(\gamma J(\varrho, \vec{\Theta}) + R(\varrho, \vec{\Theta}))^2} \varrho^{d-1}\,\mathrm{d}\varrho$$

Variance Prediction — 3DCT

- For 3DCT geometries with small cone angles, factor LFR using cylindrical spatial frequency coordinates:

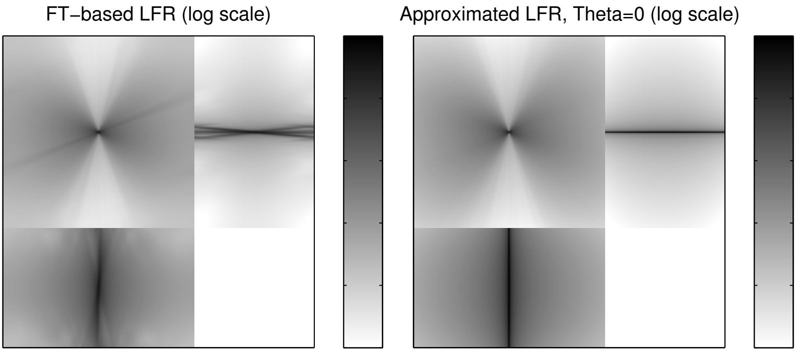
$$\bar{F}_j(\vec{\nu}) \approx J_{\mathsf{cyl}}(\vec{\nu}) \bar{E}_j^{\mathrm{cyl}}(\Phi)$$

- In cylindrical coordinates, the NPS integral becomes 1D:

$$\mathsf{var}(\hat{x}_j) \approx \frac{1}{\alpha} \int_0^{2\pi} \frac{\hat{\bar{E}}_j^{\mathrm{cyl}}(\Phi)}{\bar{E}_j^{\mathrm{cyl}}(\Phi)} G_{\mathsf{cyl}}(\Phi, \alpha^{-1} \bar{E}_j^{\mathrm{cyl}}(\Phi)) \, \mathrm{d}\Phi$$

- Tabulate G_{cyl} using voxel basis and regularizer.
- Variance approximation requires just 1D integral (per voxel) akin to back-projection

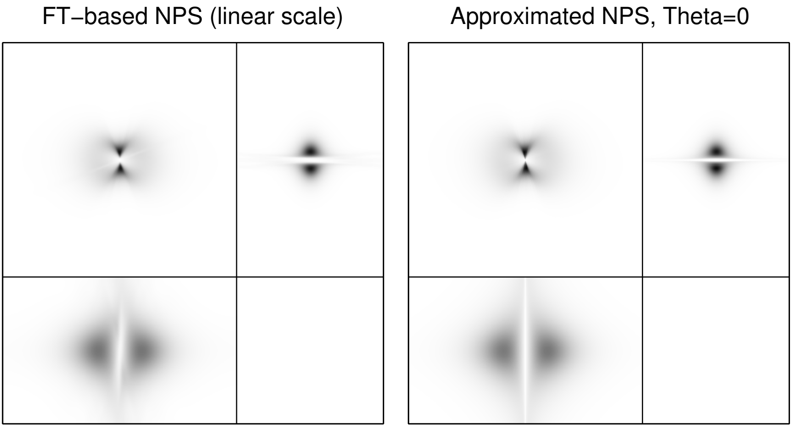
Local Frequency Response — Cylindrical Factorization



$$\bar{F}_j(\vec{\nu})$$

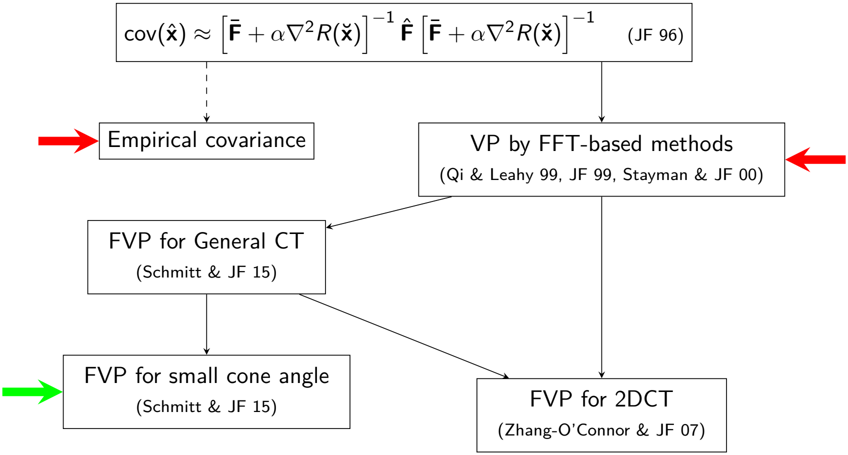
- Still reasonable agreement, albeit somewhat less so
- Infamous "missing cone" is absent

Local NPS — Cylindrical Factorization



NPS $S_j(\vec{\nu})$

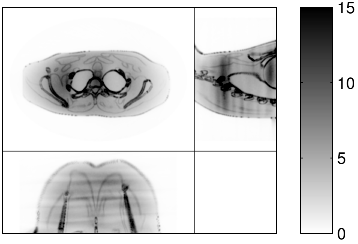
Towards Tractable (Co)Variance Prediction



Results — Simulation

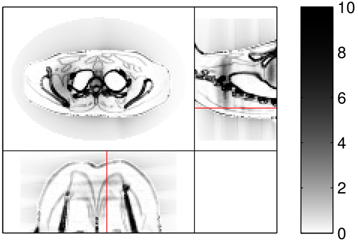
- $512 \times 512 \times 320$ voxel section of an XCAT phantom (Segars 08)
- Voxel size $0.976 \times 0.976 \times 0.625$ mm
- $888 \times 64 \times 2952$ sinogram (3 turn helix, pitch = 1)
- Detector element size 1.024×1.096 mm
- Huber potential, $\delta = 10$HU: $\psi(x) = \begin{cases} x^2/2, & |x| \leq \delta \\ \delta|x| - \delta^2/2, & |x| > \delta, \end{cases}$
- Empirical SD from many realizations,
  smoothed with a 3-voxel FWHM kernel
- Two regularization penalties:
  - Space-varying regularization for uniform resolution (Fessler 96), 111 realizations
  - Uniform (conventional) regularization 61 realizations
- Variance prediction computed once per $4 \times 4 \times 4$ block

Results: Simulation — Space-Varying Regularization

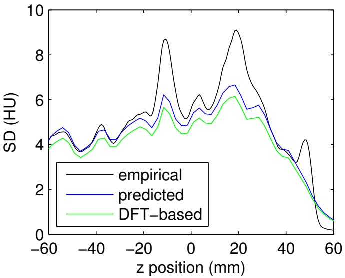


(a) Empirical (190 days)

(b) Predicted (1207 sec.)

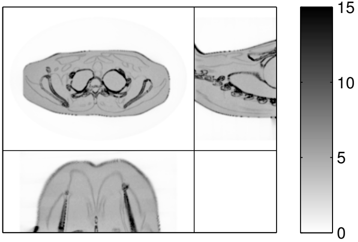


(c) Absolute Error

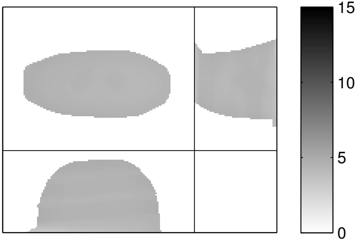


(d) Axial profile

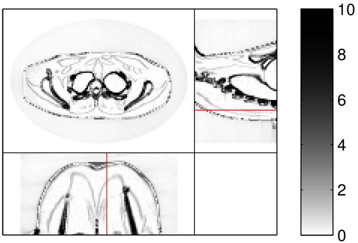
Results: Simulation — Uniform Regularization



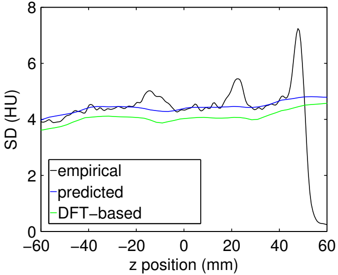
(a) Empirical



(b) Predicted



(c) Absolute Error

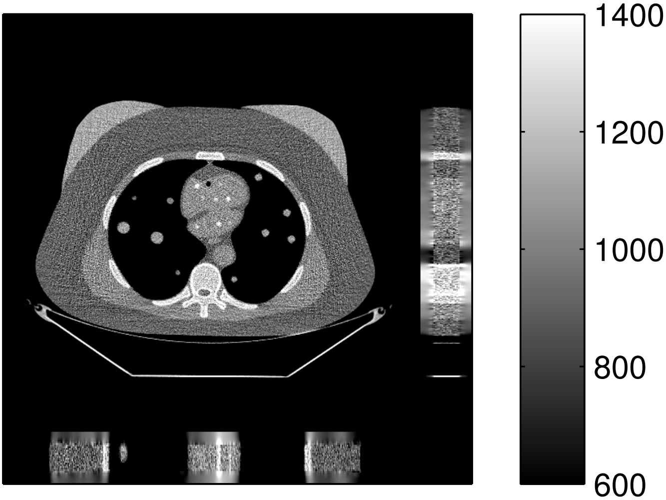


(d) Axial profile

Results: Real CT scans — Setup

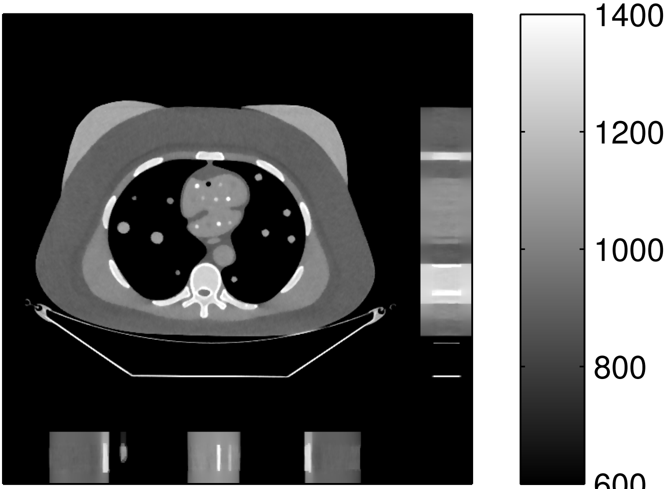
- GE Discovery CT750 HD scanner;
  $888 \times 16 \times 984$ sinogram with detector element size 1.024×1.096 mm; 40mA tube current
- $512 \times 512 \times 32$ voxel reconstruction of a chest phantom
- Voxel size $0.976 \times 0.976 \times 0.625$ mm
- Empirical SD from 10 reconstructions, each slice smoothed with a 3-voxel FWHM kernel
- Two regularization penalties:
  - Space-varying regularization, quadratic penalty
  - Uniform regularization, Huber penalty, $\delta = 10$HU
- Variance prediction computed once per $4 \times 4 \times 1$ block

Results: Real CT Scans — Example Reconstruction



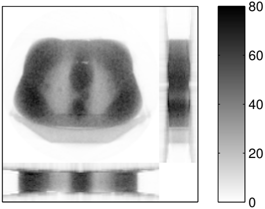
Space-Varying α, Quadratic potential

Results: Real CT Scans — Example Reconstruction

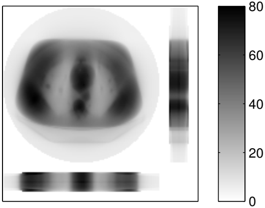


Conventional α, Huber edge-preserving potential

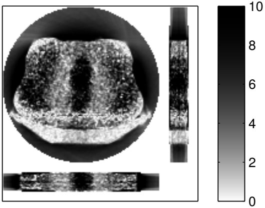
Results: Real CT — Space-Varying Regularization



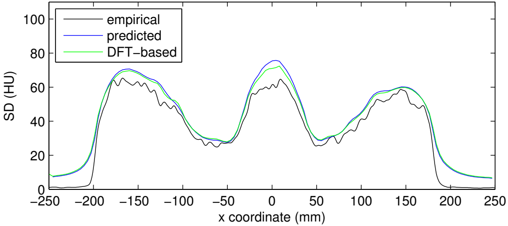
(a) Empirical (4.2 days)



(b) Predicted (673 sec.)

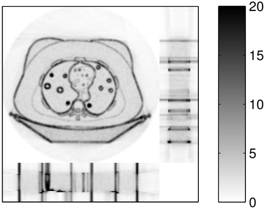


(c) Absolute Error

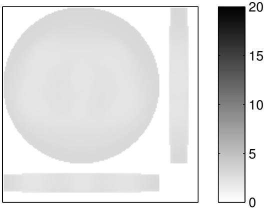


(d) Profile

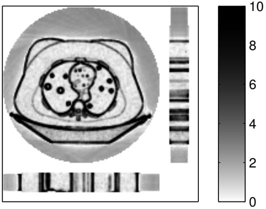
Results: Real CT — Conventional Regularization



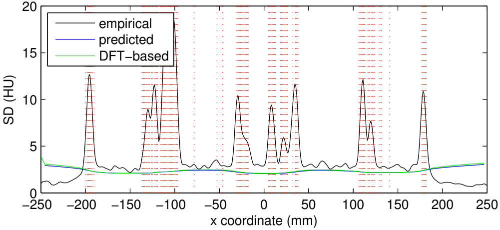
(a) Empirical (5.0 days)



(b) Predicted (661 sec.)



(c) Absolute Error



(d) Profile

Summary

- Analytical LFR and NPS expressions for SIR in general CT
- Fast variance prediction – akin to a back-projection
- Accurate for various regularizers except near edges
- Prediction near edges remains open problem (Ahn and Leahy, 08)
- Most error is due to local shift invariance approximation
- Proposed method is faster by orders of magnitude than both DFT-based methods and empirical methods
- Local frequency response (LFR) approximation useful in other statistical applications, such as regularization design and prediction of observer performance (Schmitt 15 thesis)
- Application to tube current modulation is work-in-progress

Results: Real CT Scans — Compute Time

Total computation time (CPU-seconds)

	Empirical	DFT-based	Proposed
Simulation	$1.64 \cdot 10^7$	$7.23 \cdot 10^8$	$1.21 \cdot 10^3$
111 realizations, $512 \times 512 \times 320$ image, $888 \times 64 \times 2952$ detector			
Real	$3.63 \cdot 10^5$	$1.07 \cdot 10^8$	$6.73 \cdot 10^2$
10 realizations, $512 \times 512 \times 32$ image, $888 \times 16 \times 984$ detector			

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