Iterative methods for image formation in MRI

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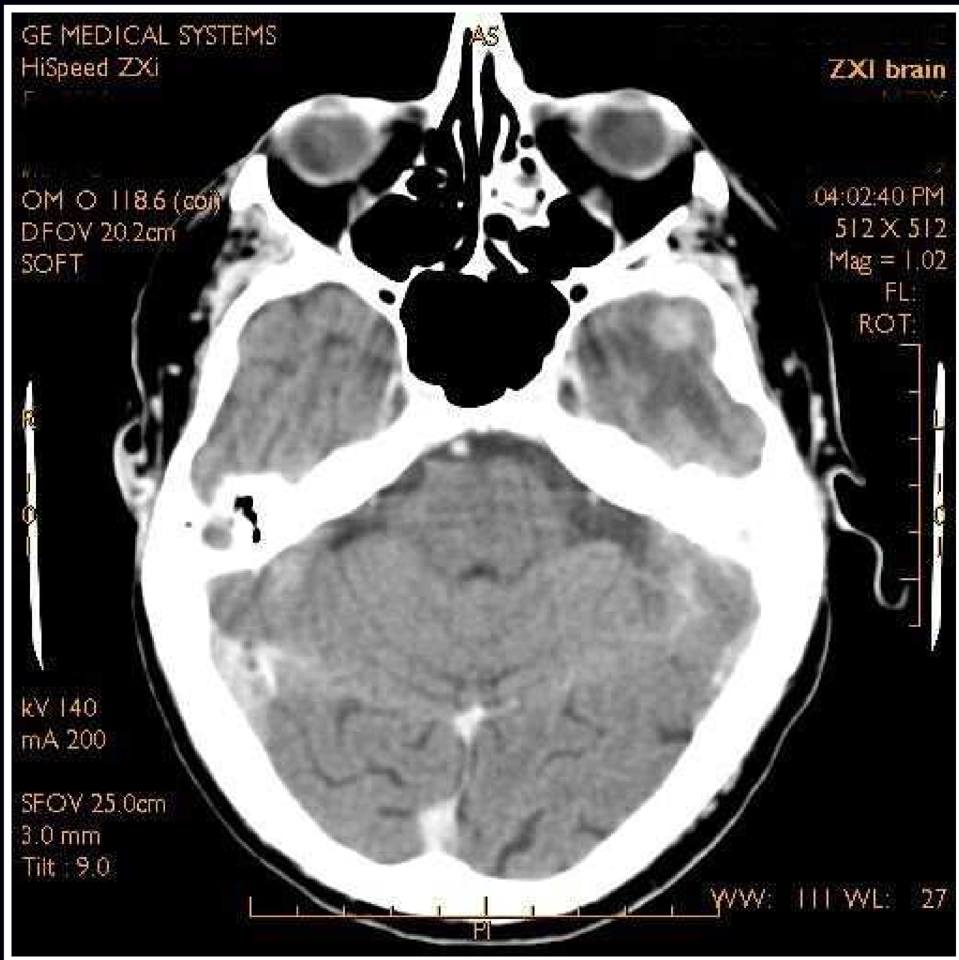
The Ends



X-ray CT

www.gehealthcare.com

MRI

www.cis.rit.edu

MRI: excellent soft tissue contrast, and no ionizing radiation.
(But, expensive, slow, big, small bone signal...)

Outline

- MR imaging physics
- MR image reconstruction problem description
- Overview of image reconstruction methods
- MR image reconstruction introduction
- Conventional reconstruction
- Model-based image reconstruction
- Iterations and computation (NUFFT etc.)
- Regularization
- Field inhomogeneity correction
- Parallel (sensitivity encoded) imaging
- Iterative methods for RF pulse design

Image reconstruction toolbox:
`http://www.eecs.umich.edu/~fessler`

Physics

Bloch Equation - Overview

PHYSICAL REVIEW VOLUME 70, NUMBERS 7 AND 8 OCTOBER 1 AND 15, 1946

Nuclear Induction

F. BLOCH
Stanford University, California

Time evolution (phenomenological) of local magnetization $\boldsymbol{M}(\boldsymbol{r},t)$:

$$\frac{d\boldsymbol{M}}{dt} = \boldsymbol{M} \times \gamma \boldsymbol{B} - \frac{M_x \boldsymbol{i} + M_y \boldsymbol{j}}{T_2} - \frac{(M_z - M_0)\boldsymbol{k}}{T_1}$$

Precession ↑

Relaxation ↑ ↑

Equilibrium ↑

Bloch Equation and Imaging

$$\frac{d\boldsymbol{M}(\boldsymbol{r},t)}{dt} = \boldsymbol{M}(\boldsymbol{r},t) \times \gamma \boldsymbol{B}(\boldsymbol{r},t) - \frac{M_x \boldsymbol{i} + M_y \boldsymbol{j}}{T_2(\boldsymbol{r})} - \frac{(M_z - M_0(\boldsymbol{r}))\boldsymbol{k}}{T_1(\boldsymbol{r})}$$

Image properties depend on:

- Spin (Hydrogen) density $M_0(\boldsymbol{r})$
- Longitudinal (spin-lattice) relaxation $T_1(\boldsymbol{r})$
- Transverse (spin-spin) relaxation $T_2(\boldsymbol{r})$
- Chemical shift
  (resonant frequency of H is ≈ 3.5 ppm lower in fat than in water)

Applied field $\boldsymbol{B}(\boldsymbol{r},t)$ includes three components we can control:

- Main field B_0
- RF field $\boldsymbol{B}_1(t)$
- Field gradients $\boldsymbol{r} \cdot \boldsymbol{G}(t) = xG_x(t) + yG_y(t) + zG_z(t)$

$$\boldsymbol{B}(\boldsymbol{r},t) = B_0 + \boldsymbol{B}_1(t) + \boldsymbol{r} \cdot \boldsymbol{G}(t)\boldsymbol{k}$$

Systems view of MRI

Applied field $\boldsymbol{B}(\boldsymbol{r},t)$ $\longrightarrow$ Patient $\longrightarrow$ magnetization pattern $\boldsymbol{M}(\boldsymbol{r},t)$ $\longrightarrow$ RF coil(s) (Faraday induction) $\longrightarrow$ received signal $s_r(t)$

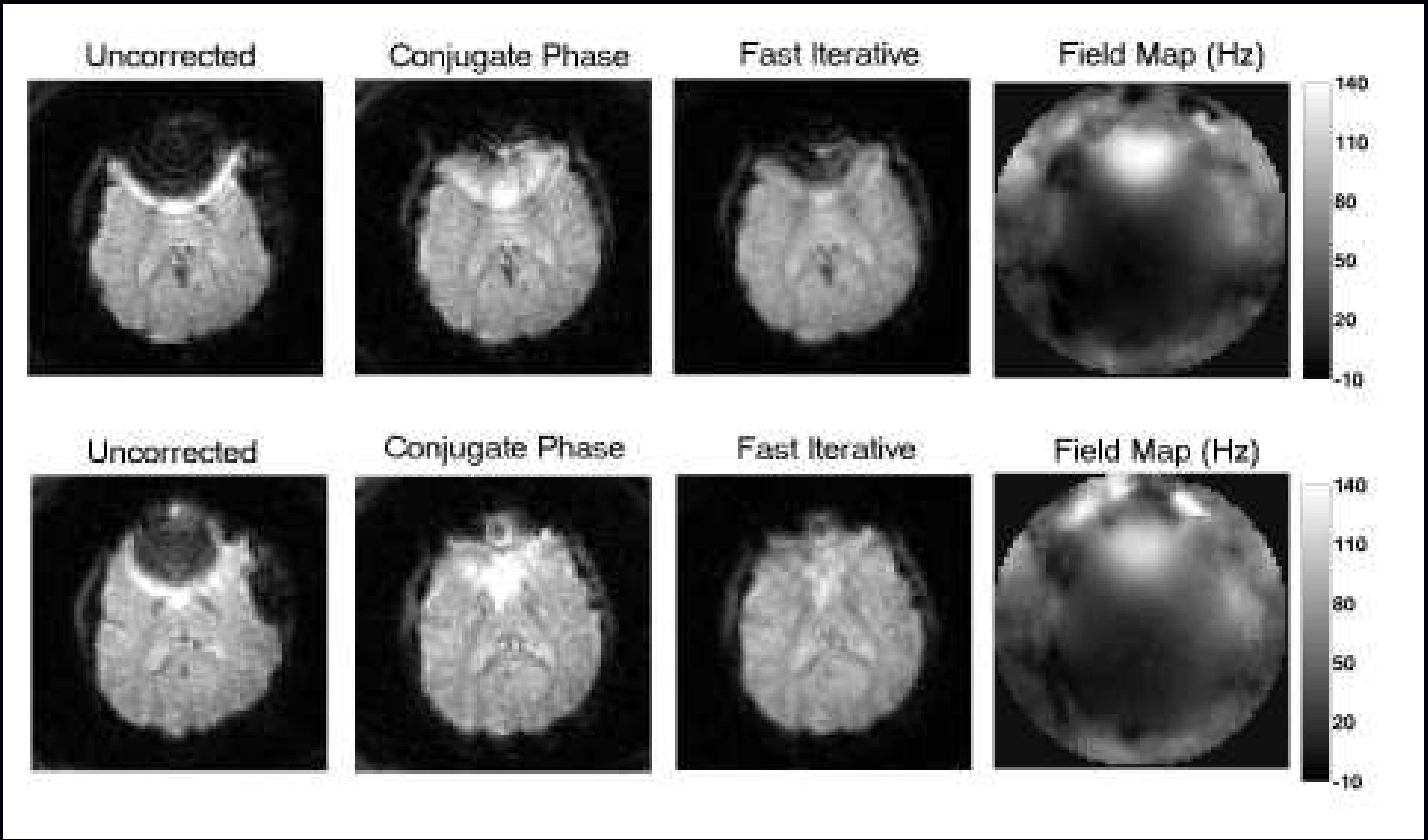
$\longrightarrow$ demodulate (Larmor frequency) $\longrightarrow$ baseband signal $s(t)$ $\longrightarrow$ sample (A/D) $\longrightarrow$ recorded data $y_i,\ i = 1,\ldots,M$

$\longrightarrow$ reconstruction algorithm $\longrightarrow$ displayed image $f(\vec{r})$

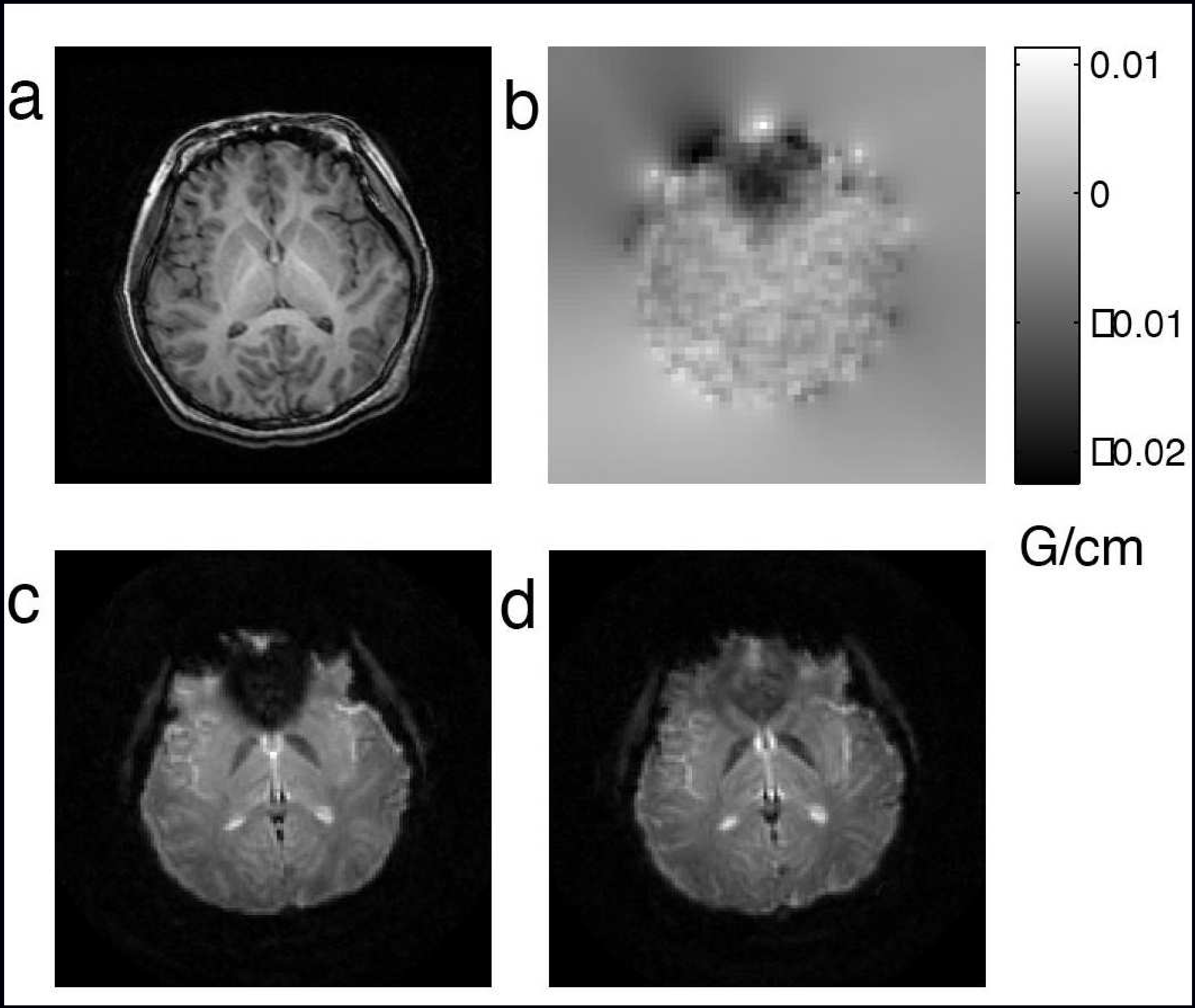
Research areas:

- design of RF pulses / gradient waveforms (*many* possibilities!)
- coil design
- reconstruction algorithm development

Example: Iterative Reconstruction under ΔB_0

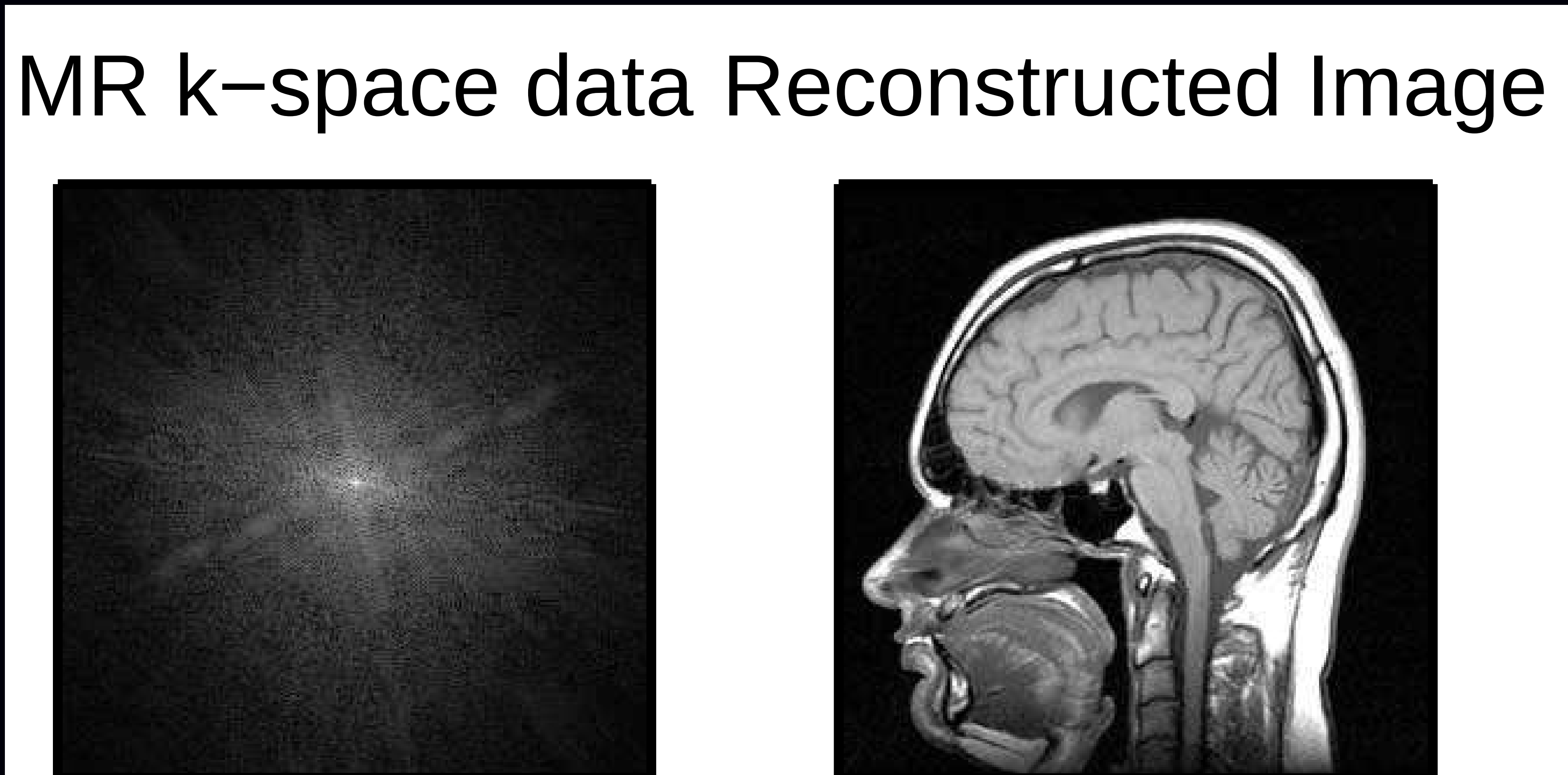


Example: Iterative RF Pulse Design



Introduction to Reconstruction

Standard MR Image Reconstruction



Cartesian sampling in k-space. An inverse FFT. End of story.

Commercial MR system quotes 400 FFTs (256^2) per second.

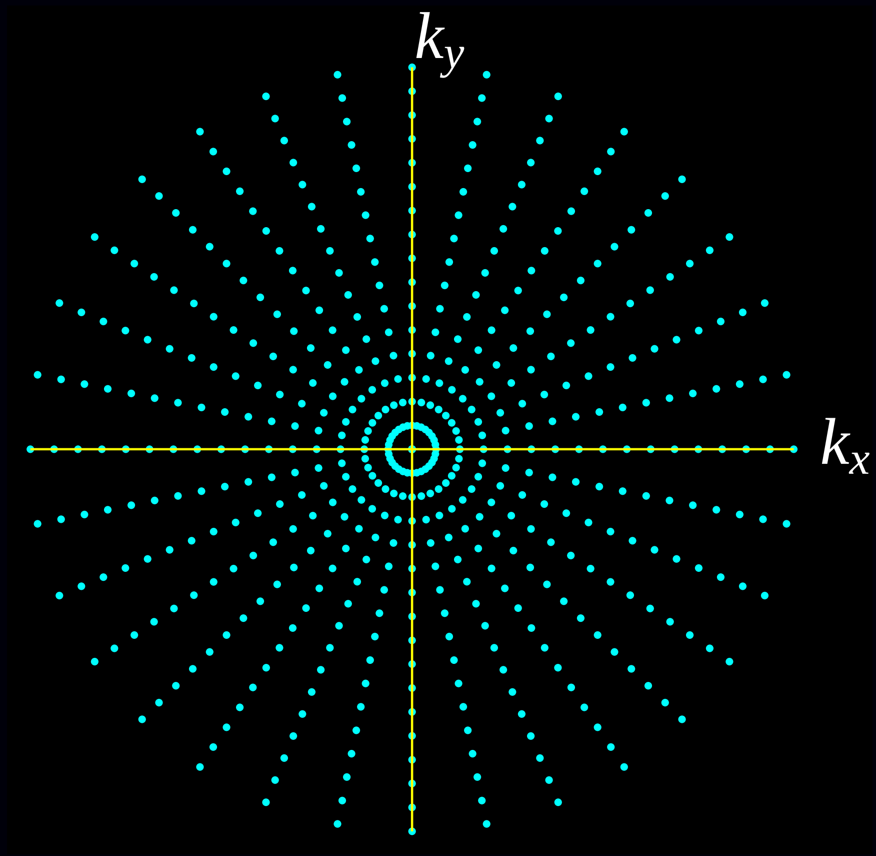
Non-Cartesian MR Image Reconstruction

"k-space" data

$\boldsymbol{y} = (y_1, \ldots, y_M)$

image

$f(\vec{r})$



$\Longrightarrow$

k-space trajectory:

$\vec{\kappa}(t) = (k_x(t), k_y(t))$

spatial coordinates:

$\vec{r} \in \mathbb{R}^{\bar{d}}$

Textbook MRI Measurement Model

Ignoring *lots* of things, the standard measurement model is:

$$y_i = s(t_i) + \text{noise}_i, \qquad i = 1, \ldots, M$$

$$s(t) = \int f(\vec{r})\, e^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}}\, d\vec{r} = F(\vec{\kappa}(t)).$$

$\vec{r}$: spatial coordinates
$\vec{\kappa}(t)$: k-space trajectory of the MR pulse sequence
$f(\vec{r})$: object's unknown transverse magnetization
$F(\vec{\kappa})$: Fourier transform of $f(\vec{r})$. We get noisy samples of this!
$e^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}}$ provides spatial information $\Longrightarrow$ Nobel Prize

Goal of image reconstruction: find $f(\vec{r})$ from measurements $\{y_i\}_{i=1}^{M}$.

The unknown object $f(\vec{r})$ is a continuous-space function,
but the recorded measurements $\boldsymbol{y} = (y_1, \ldots, y_M)$ are finite.

Under-determined (ill posed) problem $\Longrightarrow$ no canonical solution.

All MR scans provide only "partial" k-space data.

Image Reconstruction Strategies

- Continuous-continuous formulation

  Pretend that a continuum of measurements are available:

$$F(\vec{\kappa}) = \int f(\vec{r})\, e^{-\imath 2\pi \vec{\kappa}\cdot\vec{r}}\, d\vec{r}.$$

  The "solution" is an inverse Fourier transform:

$$f(\vec{r}) = \int F(\vec{\kappa})\, e^{\imath 2\pi \vec{\kappa}\cdot\vec{r}}\, d\vec{\kappa}.$$

  Now discretize the integral solution:

$$\hat{f}(\vec{r}) = \sum_{i=1}^{M} F(\vec{\kappa}_i)\, e^{\imath 2\pi \vec{\kappa}_i\cdot\vec{r}}\, w_i \approx \sum_{i=1}^{M} y_i w_i\, e^{\imath 2\pi \vec{\kappa}_i\cdot\vec{r}},$$

  where w_i values are "sampling density compensation factors."
  Numerous methods for choosing w_i values in the literature.
  For Cartesian sampling, using $w_i = 1/N$ suffices,
  and the summation is an inverse FFT.
  For non-Cartesian sampling, replace summation with gridding.

- Continuous-discrete formulation

  Use many-to-one linear model:

$$\boldsymbol{y} = \mathcal{A}\, f + \boldsymbol{\varepsilon}, \text{ where } \mathcal{A} : \mathcal{L}_2(\mathbb{R}^{\bar{d}}) \to \mathbb{C}^M.$$

  Minimum norm solution (*cf.* “natural pixels”):

$$\min_{\hat{f}} \left\| \hat{f} \right\| \text{ subject to } \boldsymbol{y} = \mathcal{A}\, \hat{f}$$

$$\hat{f} = \mathcal{A}^*(\mathcal{A}\mathcal{A}^*)^{-1}\boldsymbol{y} = \textstyle\sum_{i=1}^M c_i \, \mathrm{e}^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}}, \text{ where } \mathcal{A}\mathcal{A}^*\boldsymbol{c} = \boldsymbol{y}.$$

- Discrete-discrete formulation

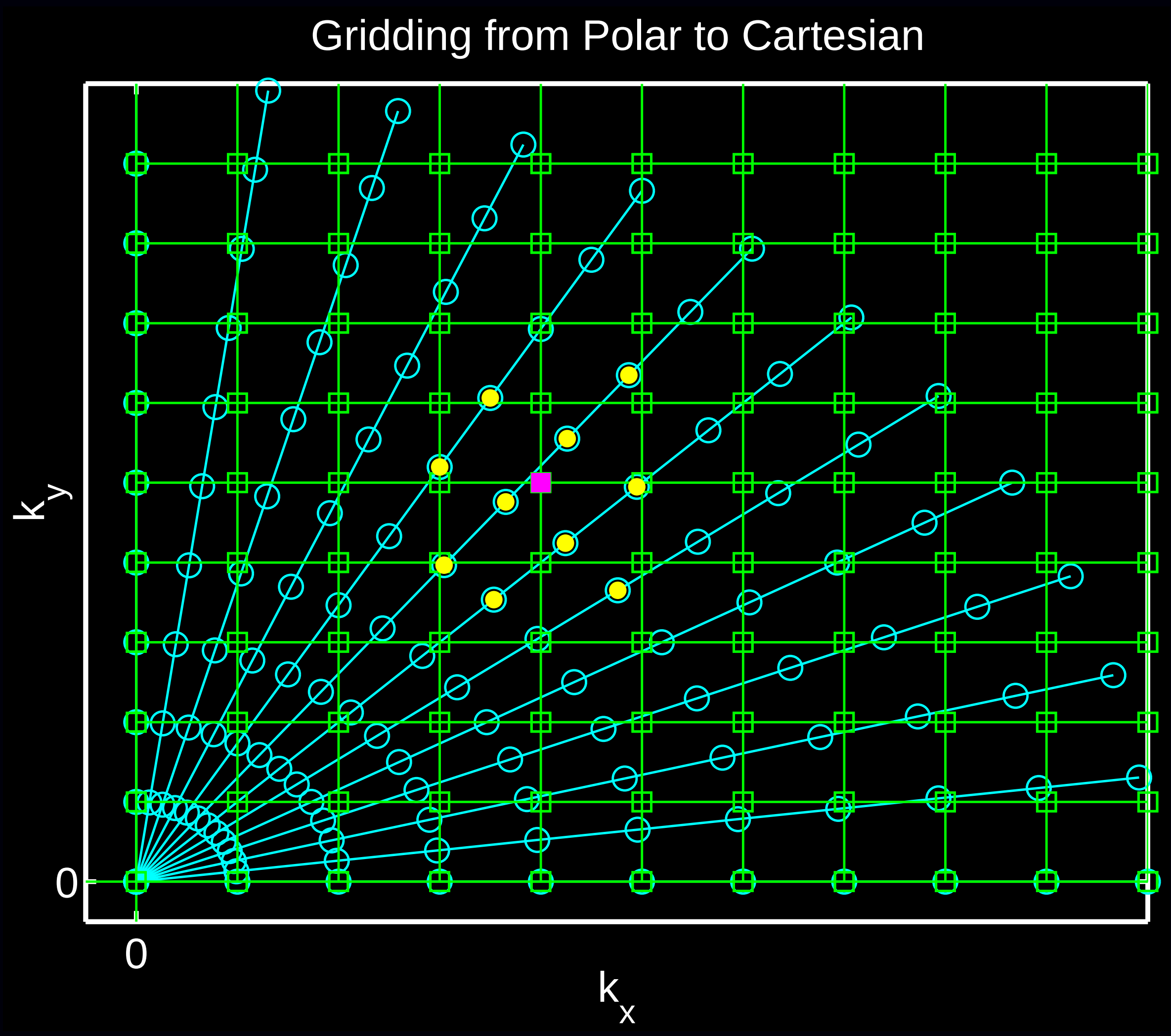
  Assume parametric model for object:

$$f(\vec{r}) = \sum_{j=1}^{N} f_j \, p_j(\vec{r}) \,.$$

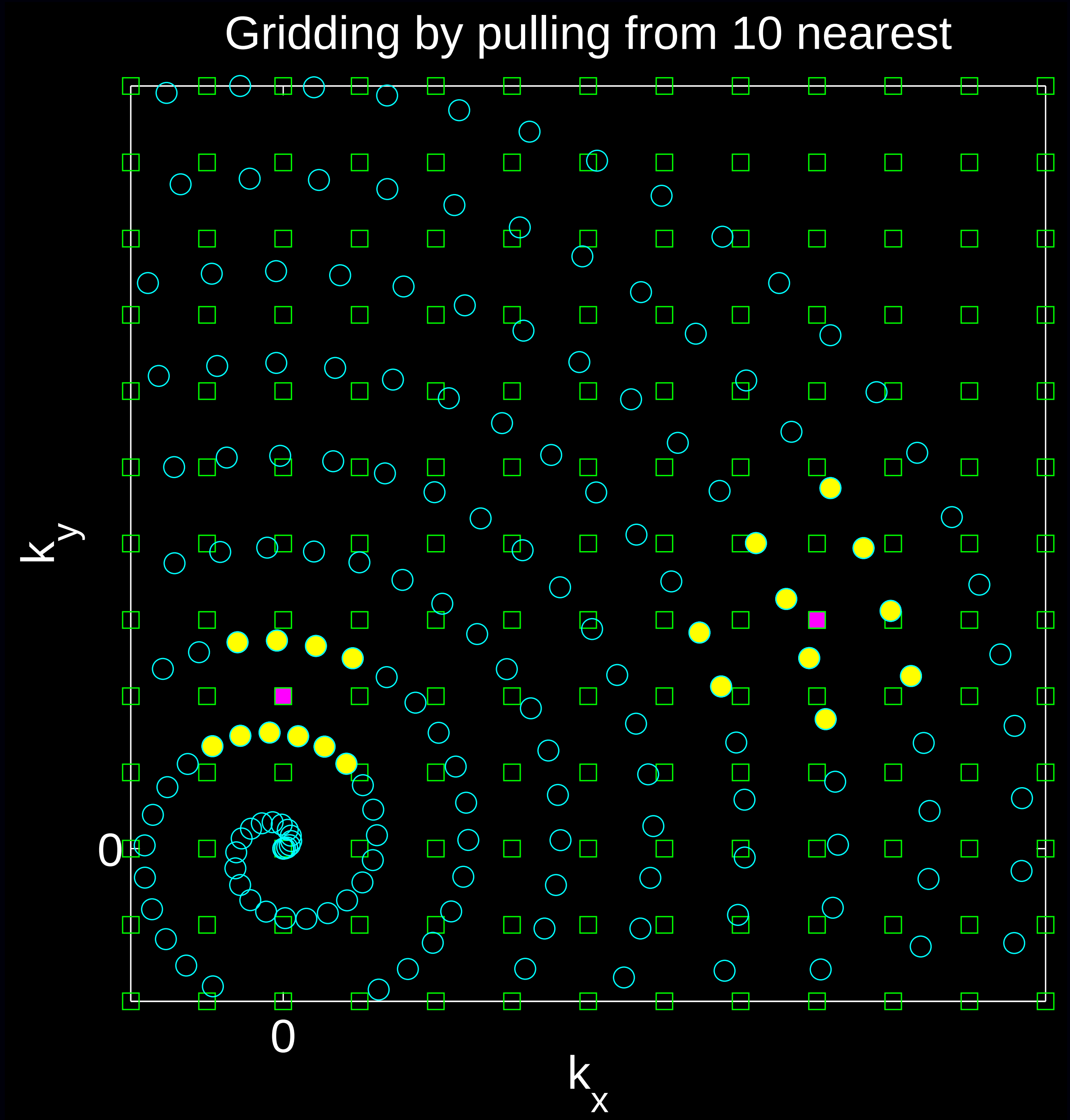
  Estimate parameter vector $\boldsymbol{f} = (f_1, \ldots, f_N)$ from data vector $\boldsymbol{y}$.

Conventional MR Image Reconstruction

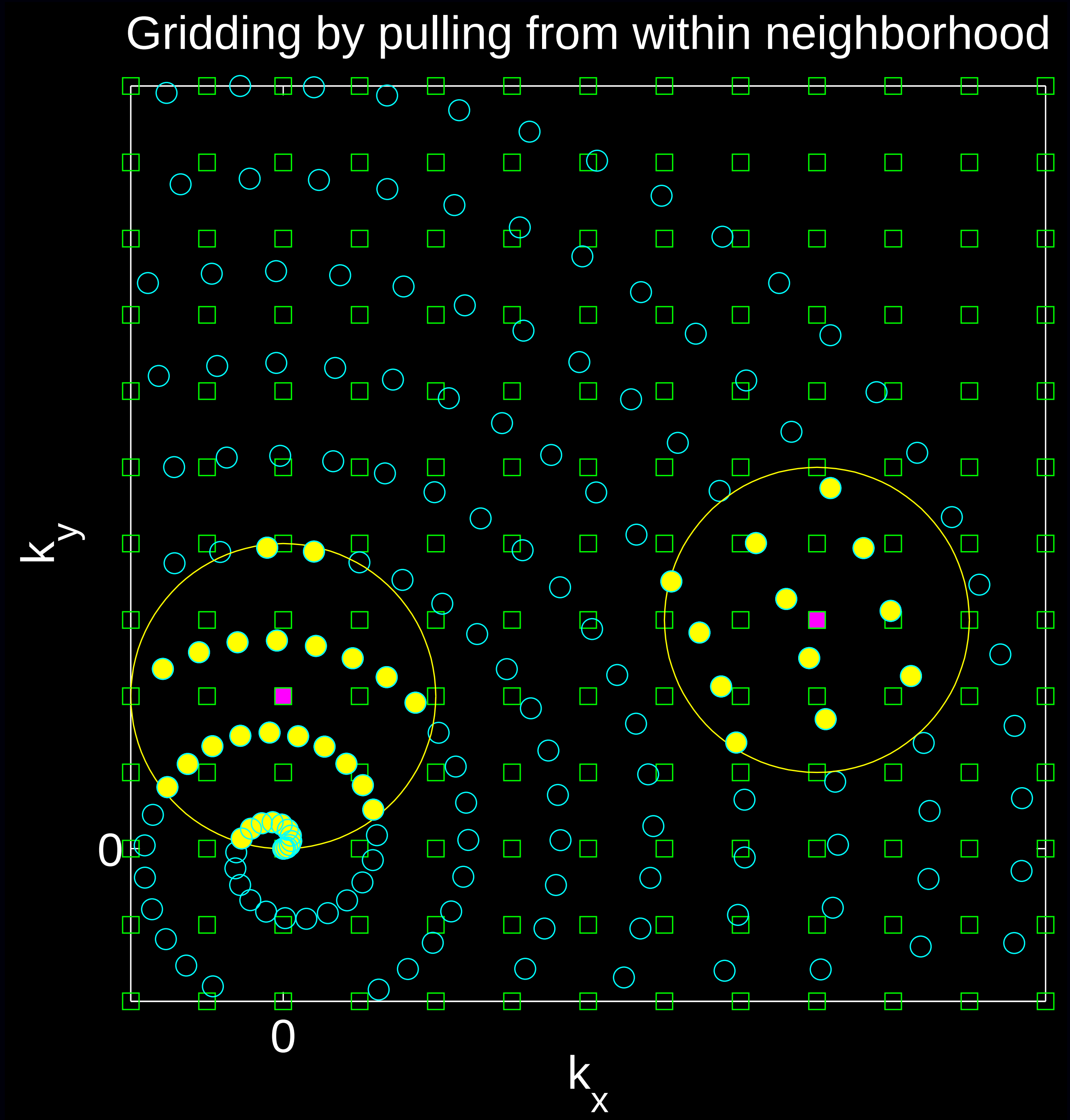
1. Interpolate measurements onto rectilinear grid ("gridding")
2. Apply inverse FFT to estimate samples of $f(\vec{r})$



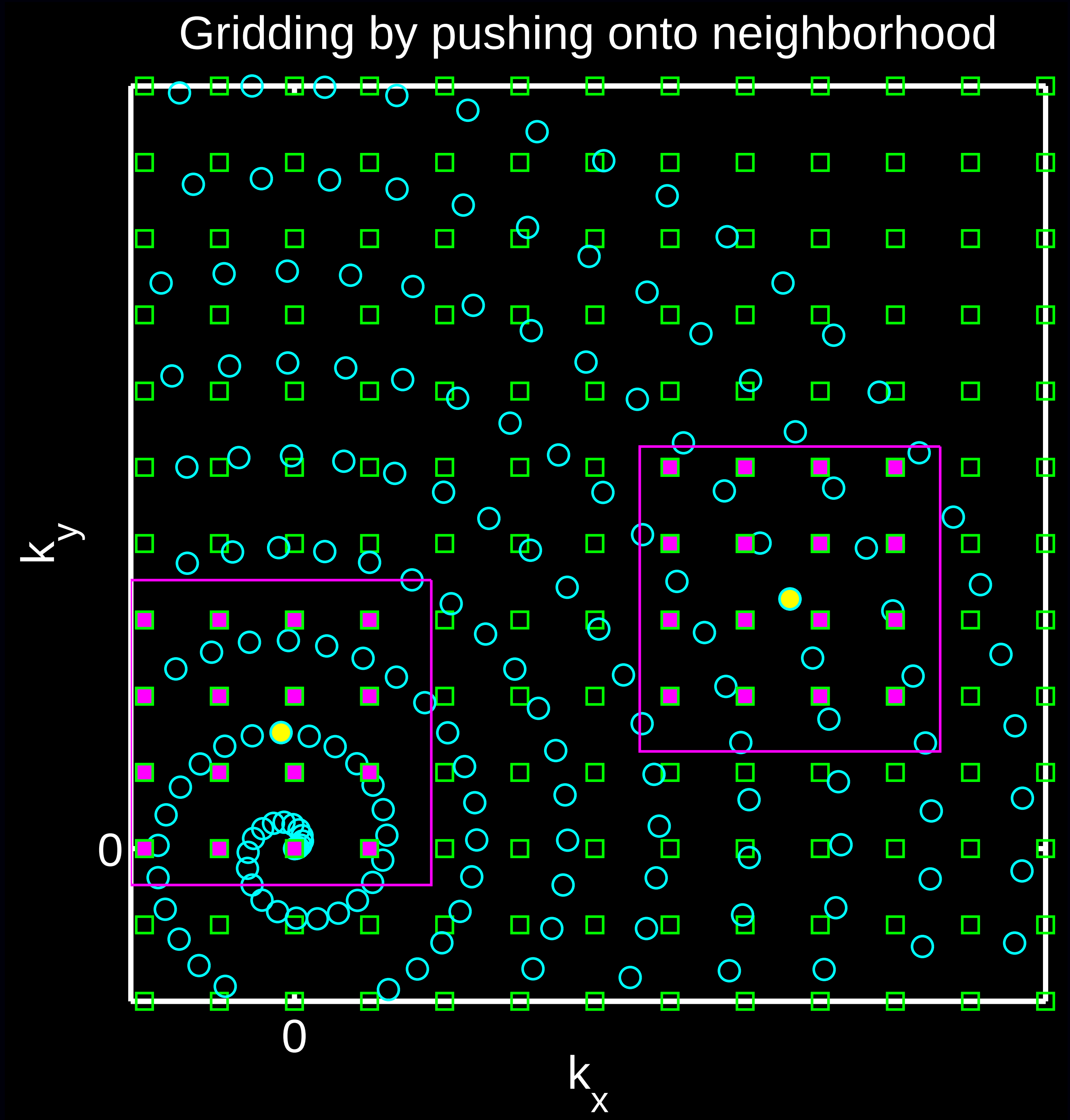
Gridding Approach 1: Pull from K nearest



Gridding Approach 2: Pull from neighborhood



Gridding Approach 3: Push to neighborhood



Gridding Approaches

Ignore noise: $y_i = F(\vec{\kappa}_i)$

Pull:
for each Cartesian grid point, use weighted average
of nonuniform k-space samples within some neighborhood

- Does not require density compensation
- Requires cumbersome search/indexing to find neighbors

Push:
each nonuniform k-space sample onto a Cartesian neighborhood

$$\hat{F}(\vec{\kappa}) = \sum_{i=1}^{M} y_i w_i C(\vec{\kappa} - \vec{\kappa}_i)$$

- $C(\vec{\kappa})$ denotes the *gridding kernel*, typically separable Kaiser-Bessel
  Jackson *et al.*, IEEE T-MI, 1991
- "$*$" denotes convolution.
- $\delta(\cdot)$ denotes the Dirac impulse
- density compensation factors w_i essential

Post-iFFT Gridding Correction

Gridding as convolution in k-space:

$$\hat{F}(\vec{\kappa}) = \sum_{i=1}^{M} y_i w_i C(\vec{\kappa} - \vec{\kappa}_i) = C(\vec{\kappa}) * \sum_{i=1}^{M} y_i w_i \delta(\vec{\kappa} - \vec{\kappa}_i).$$

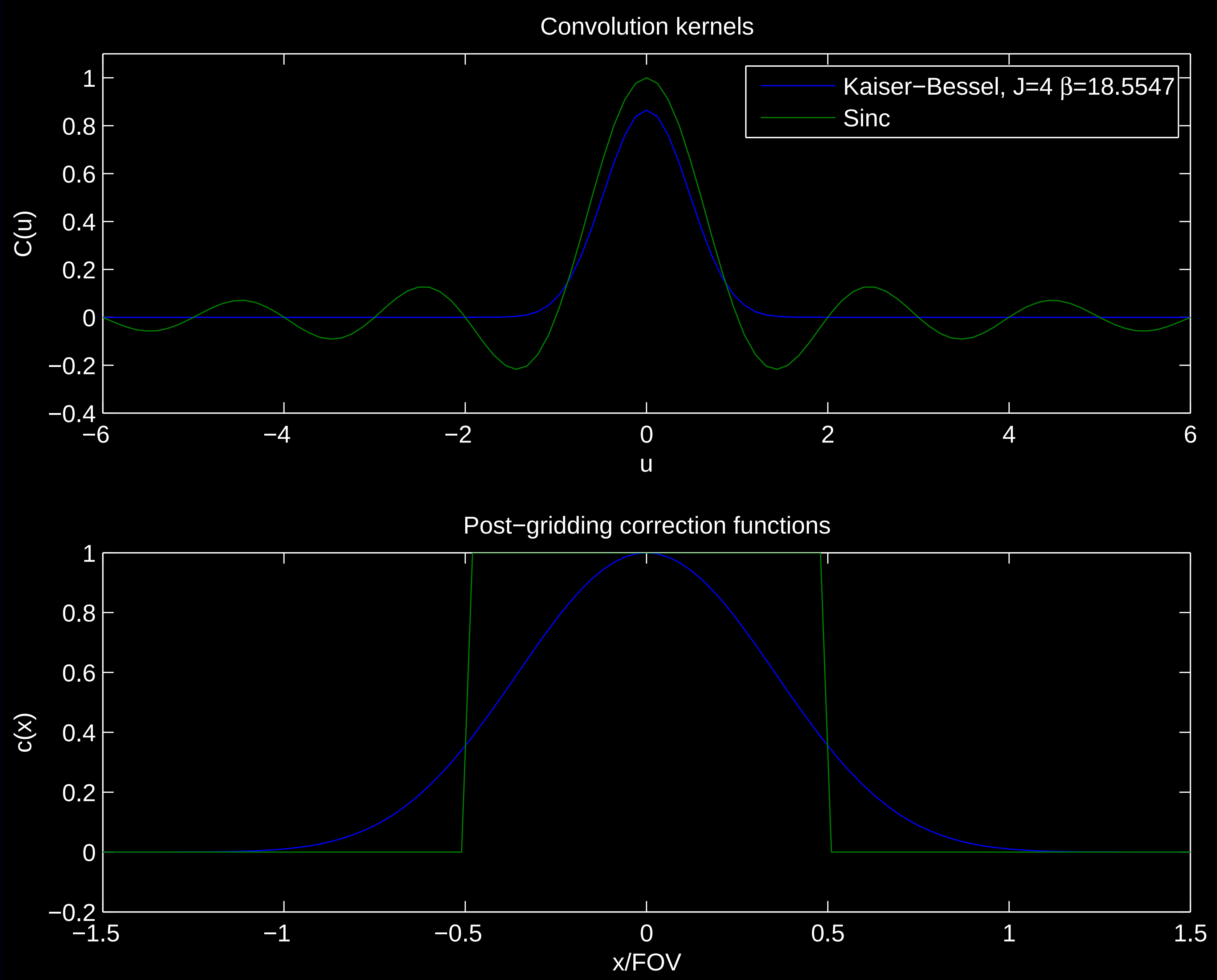
Inverse FT reconstruction:

$$\hat{f}_{\text{initial}}(\vec{r}) = \mathcal{F}^{-1}\{\hat{F}(\vec{\kappa})\} = c(\vec{r}) \sum_{i=1}^{M} y_i w_i \, e^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}}.$$

Post-correction:

$$\hat{f}_{\text{final}}(\vec{r}) = \frac{\hat{f}_{\text{initial}}(\vec{r})}{c(\vec{r})}.$$

Gridding Kernels and Post-corrections



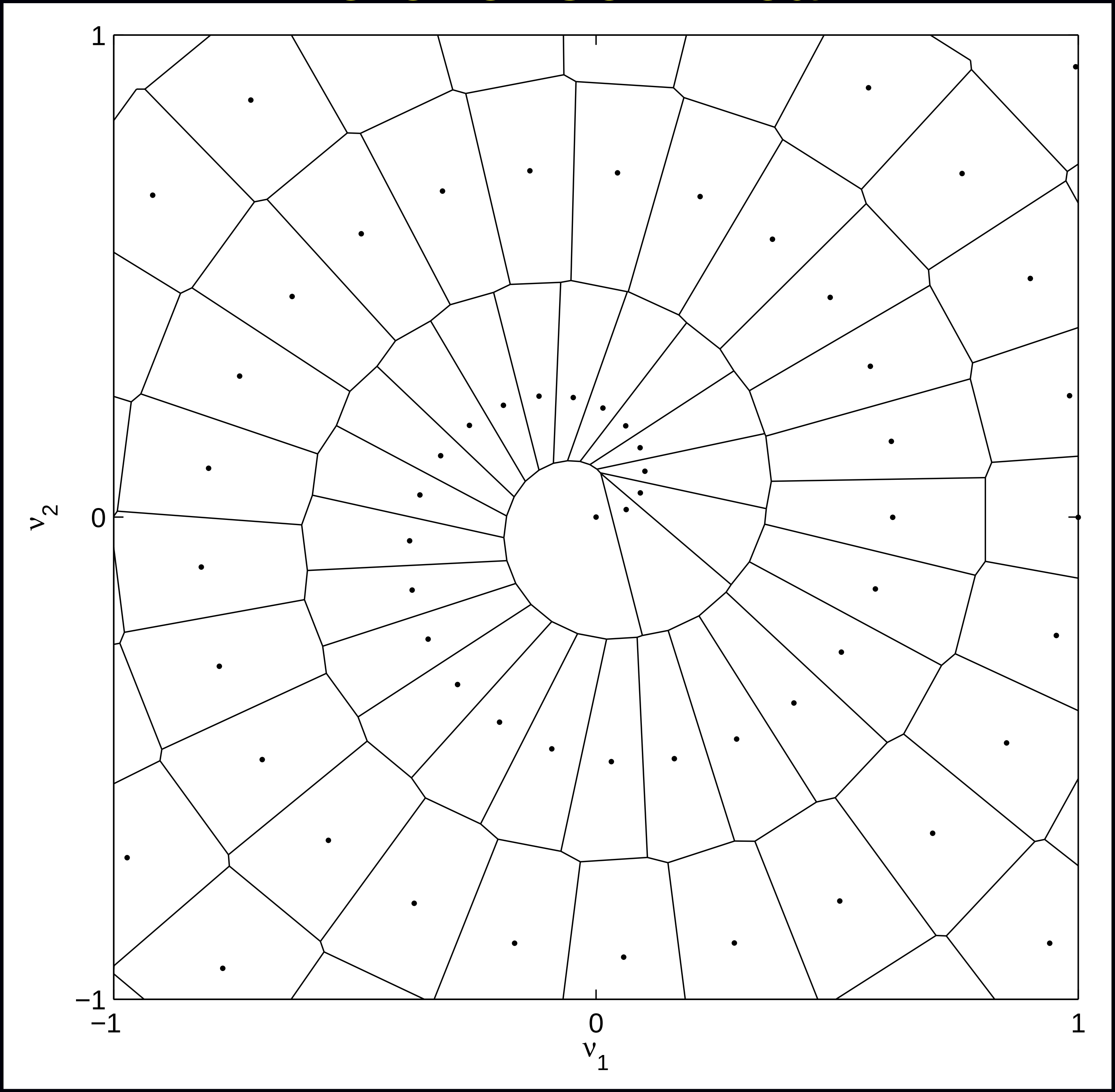
Density Compensation

$$f(\vec{r}) = \int F(\vec{\kappa})\, e^{\imath 2\pi \vec{\kappa}\cdot\vec{r}}\, d\vec{\kappa} \approx \sum_{i=1}^{M} y_i\, e^{\imath 2\pi \vec{\kappa}_i\cdot\vec{r}}\, w_i.$$

- Voronoi cell area
  Bracewell, 1973, Astrophysical Journal; Rasche *et al.*, IEEE T-MI, 1999
- Jacobians — Norton, IEEE T-MI, 1987
- Jackson's area density — Jackson, IEEE T-MI, 1991
- Iterative methods — Pipe and Menon, MRM, 2000
- ...

Tradeoffs between simplicity and accuracy.

Voronoi Cell Area



Limitations of Gridding Reconstruction

1. Artifacts/inaccuracies due to interpolation
2. Contention about sample density “weighting”
3. Oversimplifications of Fourier transform signal model:
   - Magnetic field inhomogeneity
   - Magnetization decay (T_2)
   - Eddy currents
   - ...
4. Sensitivity encoding ?
5. ...

(But it is faster than iterative methods...)

Model-Based Image Reconstruction: Overview

Model-Based Image Reconstruction

MR signal equation with more complete physics:

$$s(t) = \int f(\vec{r})\, s^{\text{coil}}(\vec{r})\, \mathrm{e}^{-\imath\,\omega(\vec{r})\,t}\, \mathrm{e}^{-R_2^*(\vec{r})\,t}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t)\cdot\vec{r}}\, \mathrm{d}\vec{r}$$

$$y_i = s(t_i) + \text{noise}_i, \qquad i = 1, \ldots, M$$

- $s^{\text{coil}}(\vec{r})$ Receive-coil sensitivity pattern(s) (for SENSE)
- $\omega(\vec{r})$ Off-resonance frequency map (due to field inhomogeneity / magnetic susceptibility)
- $R_2^*(\vec{r})$ Relaxation map

Other physical factors (?)

- Eddy current effects; in $\vec{\kappa}(t)$
- Concomitant gradient terms
- Chemical shift
- Motion

Goal?

(it depends)

Field Inhomogeneity-Corrected Reconstruction

$$s(t) = \int f(\vec{r})\, s^{\text{coil}}(\vec{r})\, \mathrm{e}^{-\imath \omega(\vec{r}) t}\, \mathrm{e}^{-R_2^*(\vec{r})\, t}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}}\, \mathrm{d}\vec{r}$$

Goal: reconstruct $f(\vec{r})$ given field map $\omega(\vec{r})$.
(Assume all other terms are known or unimportant.)

Motivation

Essential for functional MRI of brain regions near sinus cavities!

(Sutton *et al.*, ISMRM 2001; T-MI 2003)

Sensitivity-Encoded (SENSE) Reconstruction

$$s(t) = \int f(\vec{r}) s^{\text{coil}}(\vec{r}) \, \mathrm{e}^{-\imath \omega(\vec{r}) t} \, \mathrm{e}^{-R_2^*(\vec{r}) t} \, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}} \, d\vec{r}$$

Goal: reconstruct $f(\vec{r})$ given sensitivity maps $s^{\text{coil}}(\vec{r})$.
(Assume all other terms are known or unimportant.)

Can combine SENSE with field inhomogeneity correction "easily."

(Sutton *et al.*, ISMRM 2001, Olafsson *et al.*, ISBI 2006)

Joint Estimation of Image and Field-Map

$$s(t) = \int f(\vec{r})\, s^{\text{coil}}(\vec{r})\, \mathrm{e}^{-\imath\omega(\vec{r})t}\, \mathrm{e}^{-R_2^*(\vec{r})\,t}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t)\cdot\vec{r}}\, \mathrm{d}\vec{r}$$

Goal: estimate *both* the image $f(\vec{r})$ and the field map $\omega(\vec{r})$
(Assume all other terms are known or unimportant.)

Analogy:
joint estimation of emission image and attenuation map in PET.

(Sutton *et al.*, ISMRM Workshop, 2001; ISBI 2002; ISMRM 2002; ISMRM 2003; MRM 2004)

The Kitchen Sink

$$s(t) = \int f(\vec{r})\, s^{\text{coil}}(\vec{r})\, \mathrm{e}^{-\imath\omega(\vec{r})t}\, \mathrm{e}^{-R_2^*(\vec{r})\,t}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t)\cdot\vec{r}}\, \mathrm{d}\vec{r}$$

Goal: estimate image $f(\vec{r})$, field map $\omega(\vec{r})$, *and* relaxation map $R_2^*(\vec{r})$

Requires “suitable” k-space trajectory.

(Sutton *et al.*, ISMRM 2002; Twieg, MRM, 2003)

Estimation of Dynamic Maps

$$s(t) = \int f(\vec{r})\, s^{\text{coil}}(\vec{r})\, e^{-\imath \omega(\vec{r}) t}\, e^{-R_2^*(\vec{r}) t}\, e^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}}\, d\vec{r}$$

Goal: estimate dynamic field map $\omega(\vec{r})$ and “BOLD effect” $R_2^*(\vec{r})$ given baseline image $f(\vec{r})$ in fMRI.

Motion...

Model-Based Image Reconstruction: Details

Basic Signal Model

$$y_i = s(t_i) + \varepsilon_i, \qquad \mathsf{E}[y_i] = s(t_i) = \int f(\vec{r})\, e^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}}\, d\vec{r}$$

Goal: reconstruct $f(\vec{r})$ from $\boldsymbol{y} = (y_1, \ldots, y_M)$.

Series expansion of unknown object:

$$f(\vec{r}) \approx \sum_{j=1}^{N} f_j\, p(\vec{r} - \vec{r}_j) \longleftarrow \text{usually 2D rect functions.}$$

Substituting into signal model yields

$$\mathsf{E}[y_i] = \int \left[\sum_{j=1}^{N} f_j\, p(\vec{r} - \vec{r}_j) \right] e^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}}\, d\vec{r} = \sum_{j=1}^{N} \left[\int p(\vec{r} - \vec{r}_j)\, e^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}}\, d\vec{r} \right] f_j$$

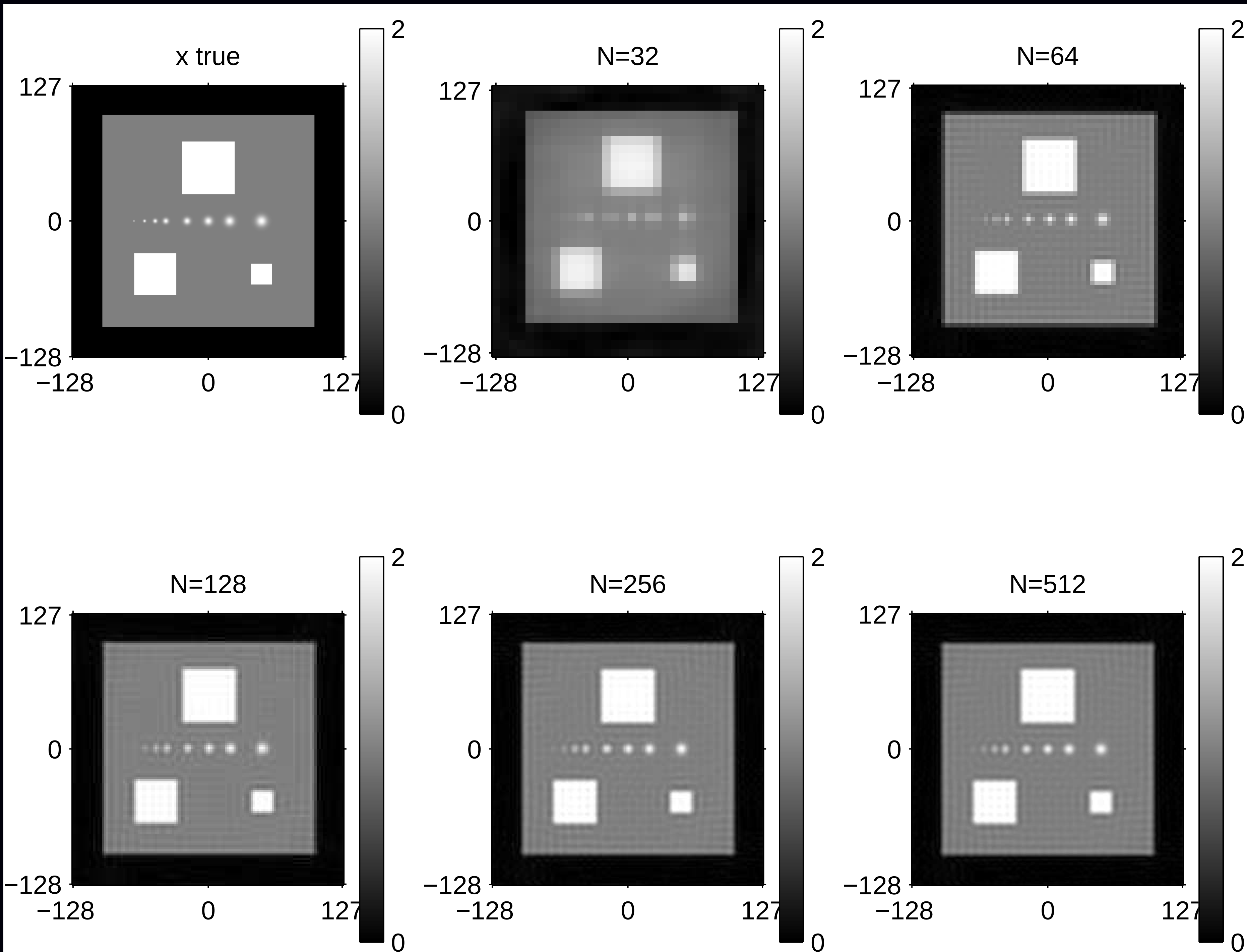
$$= \sum_{j=1}^{N} a_{ij} f_j, \qquad a_{ij} = P(\vec{\kappa}_i)\, e^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}_j}, \qquad p(\vec{r}) \overset{\text{FT}}{\Longleftrightarrow} P(\vec{\kappa}).$$

Discrete-discrete measurement model with system matrix $\boldsymbol{A} = \{a_{ij}\}$:

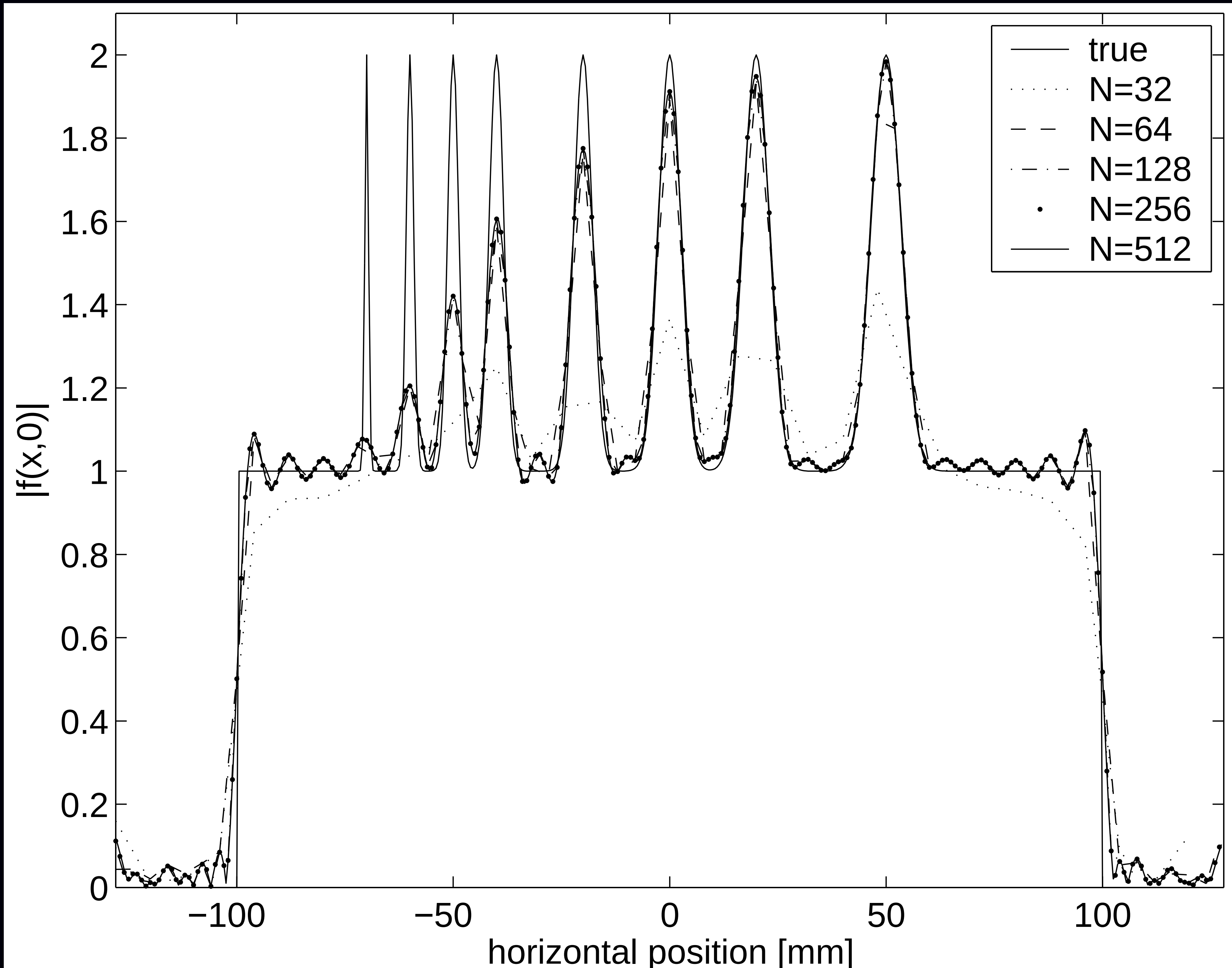
$$\boxed{\boldsymbol{y} = \boldsymbol{A}\boldsymbol{f} + \boldsymbol{\varepsilon}.}$$

Goal: estimate coefficients (pixel values) $\boldsymbol{f} = (f_1, \ldots, f_N)$ from $\boldsymbol{y}$.

Small Pixel Size Need Not Matter



Profiles



true
N=32
N=64
N=128
N=256
N=512
|f(x,0)|
horizontal position [mm]
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−100
−50
0
50
100

Regularized Least-Squares Estimation

Estimate object by minimizing a cost function:

$$\hat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f} \in \mathbb{C}^N} \Psi(\boldsymbol{f}), \qquad \Psi(\boldsymbol{f}) = \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{f}\|^2 + \alpha \mathsf{R}(\boldsymbol{f})$$

- data fit term $\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{f}\|^2$
  corresponds to negative log-likelihood of Gaussian distribution
- regularizing term $\mathsf{R}(\boldsymbol{f})$ controls noise by penalizing roughness,

$$\textit{e.g.}: \quad \mathsf{R}(\boldsymbol{f}) \approx \int \|\nabla f\|^2 \, \mathrm{d}\vec{r}$$

- regularization parameter $\alpha > 0$
  controls tradeoff between spatial resolution and noise
- Equivalent to Bayesian MAP estimation with prior $\propto \mathrm{e}^{-\alpha \mathsf{R}(\boldsymbol{f})}$

Issues:

- choosing $\mathsf{R}(f)$
- choosing α
- computing minimizer rapidly.

Quadratic regularization

1D example: squared differences between neighboring pixel values:

$$\mathsf{R}(f) = \sum_{j=2}^{N} \frac{1}{2} |f_j - f_{j-1}|^2.$$

In matrix-vector notation, $\mathsf{R}(\boldsymbol{f}) = \frac{1}{2} \|\boldsymbol{C}\boldsymbol{f}\|^2$ where

$$\boldsymbol{C} = \begin{bmatrix} -1 & 1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 \\ & & & \ddots & \ddots & \\ 0 & \dots & 0 & 0 & -1 & 1 \end{bmatrix}, \text{ so } \boldsymbol{C}\boldsymbol{f} = \begin{bmatrix} f_2 - f_1 \\ \vdots \\ f_N - f_{N-1} \end{bmatrix}.$$

For 2D and higher-order differences, modify differencing matrix $\boldsymbol{C}$.

Leads to closed-form solution:

$$\begin{aligned} \hat{\boldsymbol{f}} &= \arg\min_{\boldsymbol{f}} \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{f}\|^2 + \alpha \|\boldsymbol{C}\boldsymbol{f}\|^2 \\ &= \left[\boldsymbol{A}'\boldsymbol{A} + \alpha \boldsymbol{C}'\boldsymbol{C}\right]^{-1} \boldsymbol{A}'\boldsymbol{y}. \end{aligned}$$

(a formula of limited practical use for computing $\hat{\boldsymbol{f}}$)

Choosing the Regularization Parameter

Spatial resolution analysis (Fessler & Rogers, IEEE T-IP, 1996):

$$\hat{\boldsymbol{f}} = \left[\boldsymbol{A}'\boldsymbol{A} + \alpha \boldsymbol{C}'\boldsymbol{C}\right]^{-1} \boldsymbol{A}'\boldsymbol{y}$$

$$\mathsf{E}\left[\hat{\boldsymbol{f}}\right] = \left[\boldsymbol{A}'\boldsymbol{A} + \alpha \boldsymbol{C}'\boldsymbol{C}\right]^{-1} \boldsymbol{A}'\,\mathsf{E}[\boldsymbol{y}]$$

$$\mathsf{E}\left[\hat{\boldsymbol{f}}\right] = \underbrace{\left[\boldsymbol{A}'\boldsymbol{A} + \alpha \boldsymbol{C}'\boldsymbol{C}\right]^{-1} \boldsymbol{A}'\boldsymbol{A}}_{\text{blur}}\, \boldsymbol{f}$$

$\boldsymbol{A}'\boldsymbol{A}$ and $\boldsymbol{C}'\boldsymbol{C}$ are Toeplitz $\Longrightarrow$ blur is approximately shift-invariant.

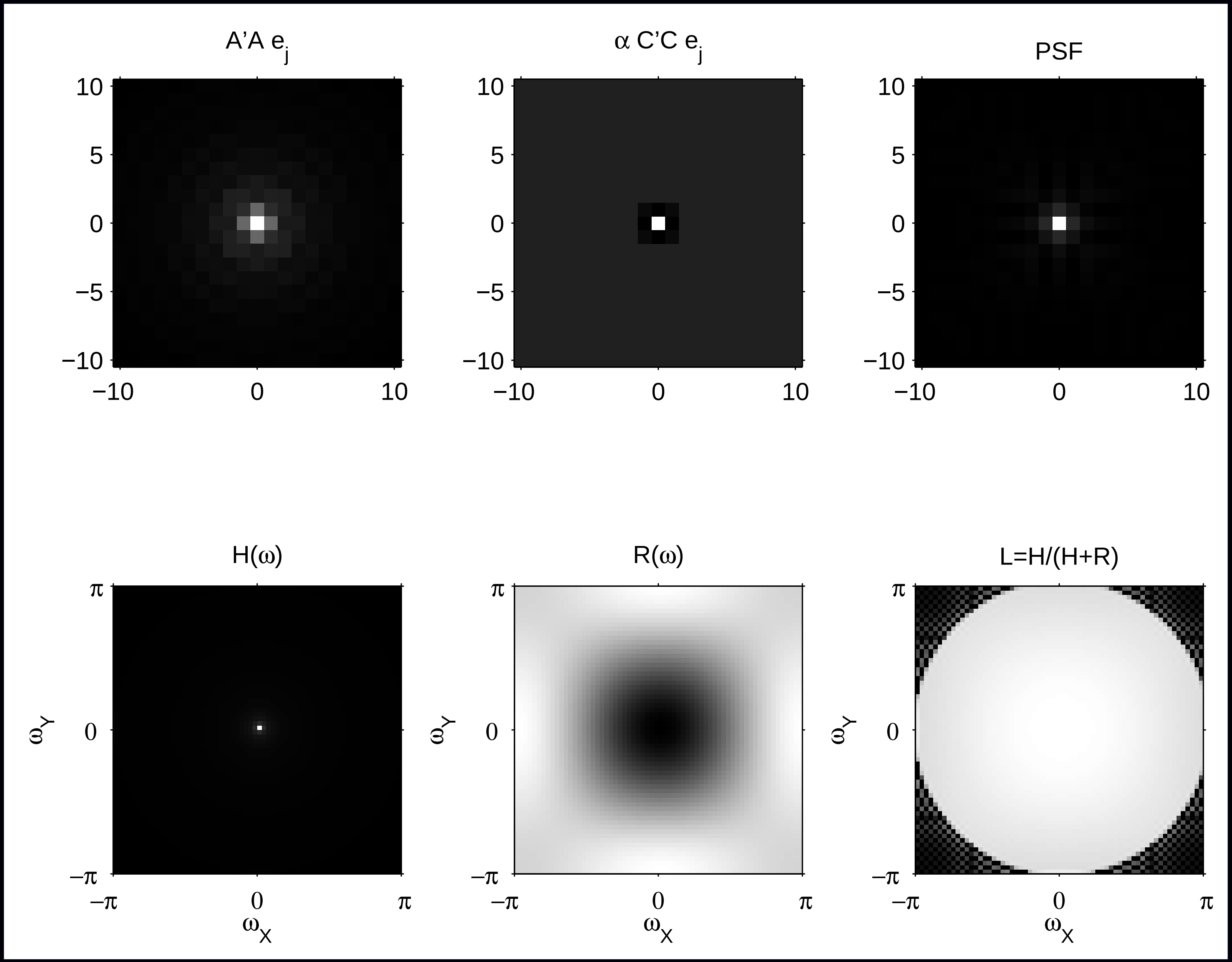
Frequency response of blur:

$$L(\omega) = \frac{H(\omega)}{H(\omega) + \alpha R(\omega)}$$

- $H(\omega_k) = \mathrm{FFT}(\boldsymbol{A}'\boldsymbol{A}\, e_j)$ (lowpass)
- $R(\omega_k) = \mathrm{FFT}(\boldsymbol{C}'\boldsymbol{C}\, e_j)$ (highpass)

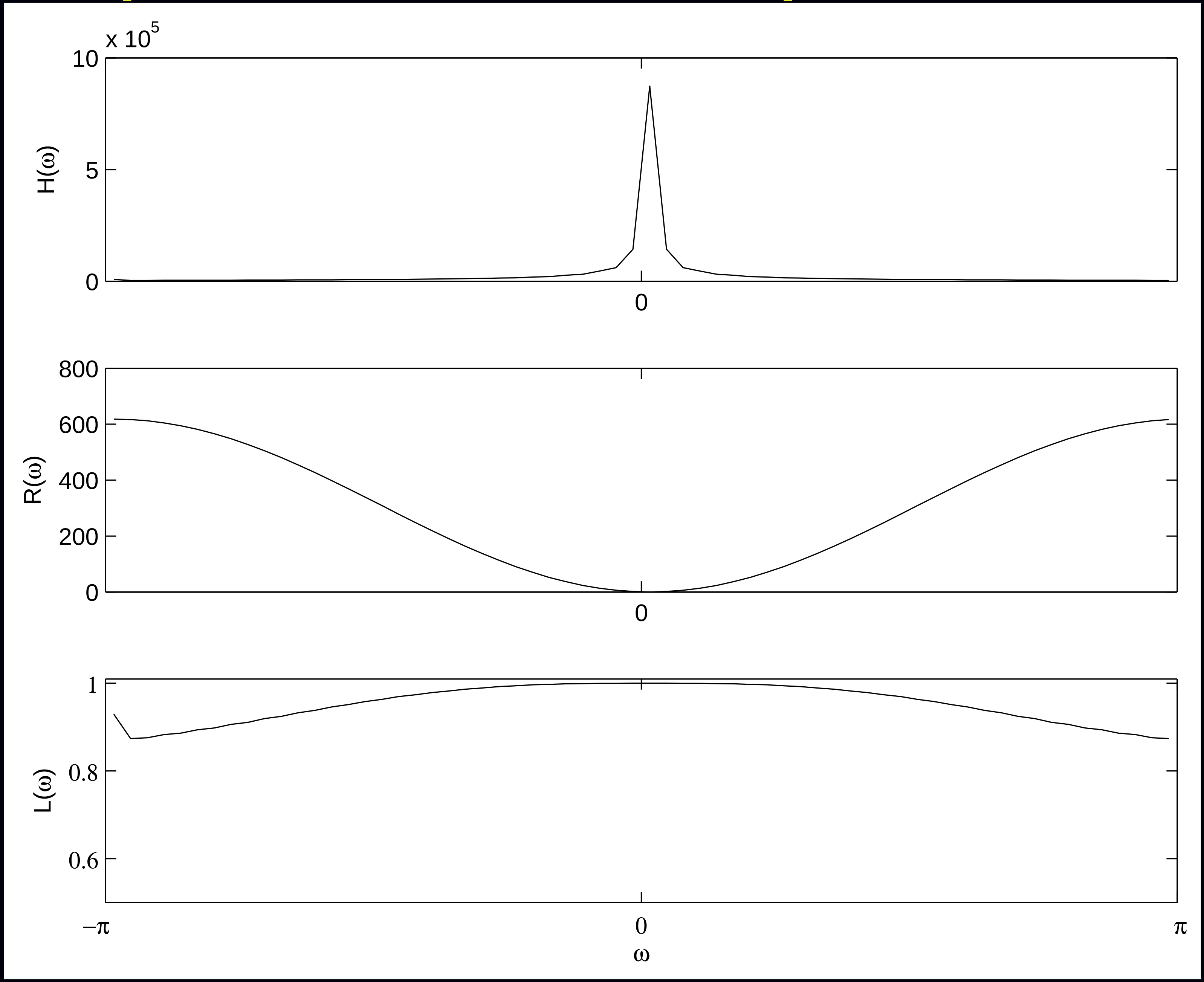
Adjust α to achieve desired spatial resolution.

Spatial Resolution Example



Radial k-space trajectory, FWHM of PSF is 1.2 pixels

Spatial Resolution Example: Profiles



Resolution/noise tradeoffs

Noise analysis:

$$\mathsf{Cov}\left\{\hat{\boldsymbol{f}}\right\} = \left[\boldsymbol{A}'\boldsymbol{A} + \alpha\boldsymbol{C}'\boldsymbol{C}\right]^{-1}\boldsymbol{A}'\,\mathsf{Cov}\{\boldsymbol{y}\}\,\boldsymbol{A}\left[\boldsymbol{A}'\boldsymbol{A} + \alpha\boldsymbol{C}'\boldsymbol{C}\right]^{-1}$$

Using circulant approximations to $\boldsymbol{A}'\boldsymbol{A}$ and $\boldsymbol{C}'\boldsymbol{C}$ yields:

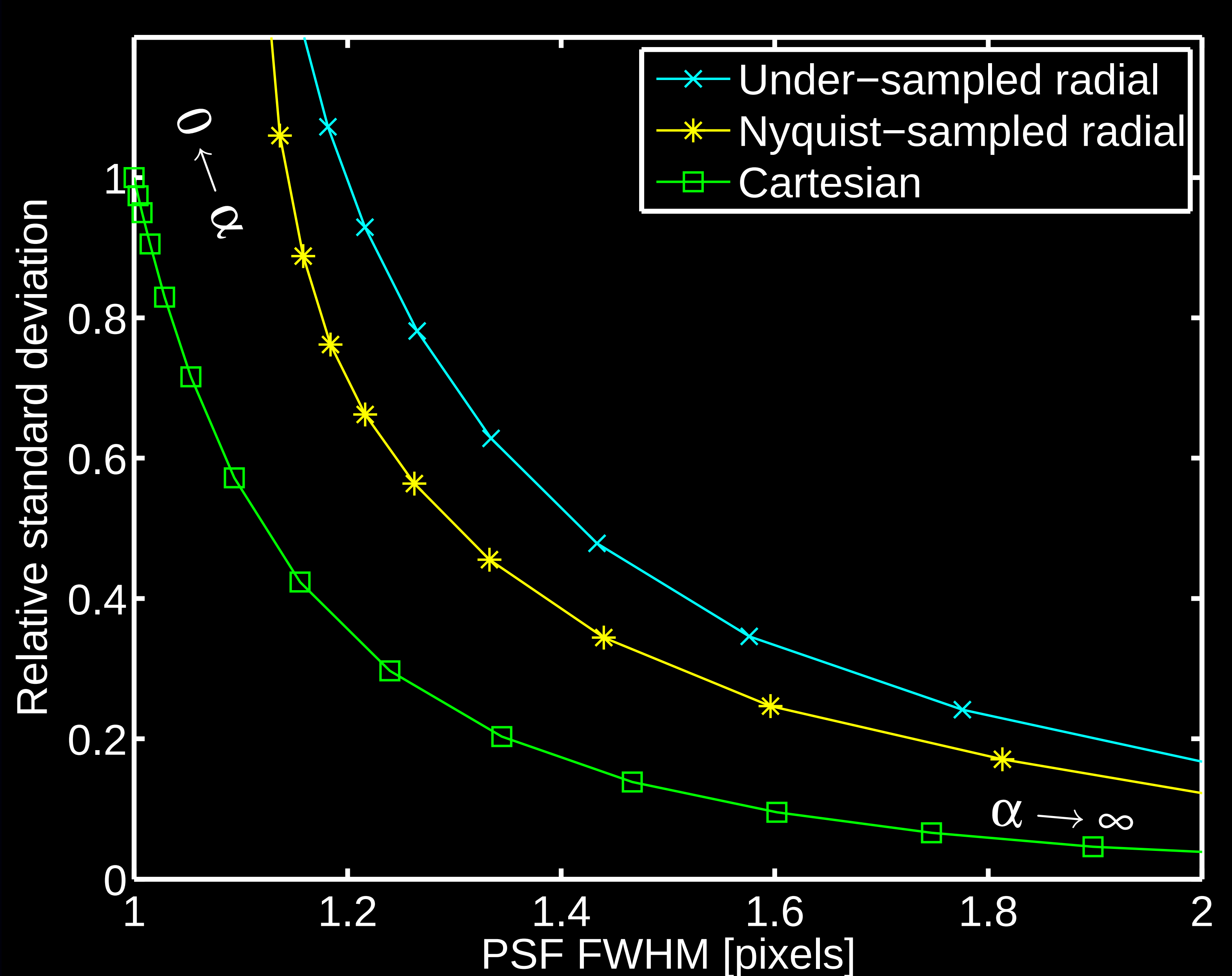
$$\mathsf{Var}\left\{\hat{f}_j\right\} \approx \sigma_\varepsilon^2 \sum_k \frac{H(\omega_k)}{(H(\omega_k) + \alpha R(\omega_k))^2}$$

- $H(\omega_k) = \mathrm{FFT}(\boldsymbol{A}'\boldsymbol{A}\,e_j)$ (lowpass)
- $R(\omega_k) = \mathrm{FFT}(\boldsymbol{C}'\boldsymbol{C}\,e_j)$ (highpass)

$\Longrightarrow$ Predicting reconstructed image noise requires just 2 FFTs.
(*cf.* gridding approach?)

Adjust α to achieve desired spatial resolution / noise tradeoff.

Resolution/Noise Tradeoff Example



In short: one can choose α rapidly and predictably for quadratic regularization.

Iterative Minimization by Conjugate Gradients

Choose initial guess $\boldsymbol{f}^{(0)}$ (*e.g.*, fast conjugate phase / gridding).
Iteration (unregularized):

$$\boldsymbol{g}^{(n)} = \nabla\Psi\left(\boldsymbol{f}^{(n)}\right) = \boldsymbol{A}'(\boldsymbol{A}\boldsymbol{f}^{(n)} - \boldsymbol{y}) \quad \text{gradient}$$

$$\boldsymbol{p}^{(n)} = \boldsymbol{P}\boldsymbol{g}^{(n)} \quad \text{precondition}$$

$$\gamma_n = \begin{cases} 0, & n = 0 \\ \dfrac{\langle \boldsymbol{g}^{(n)}, \boldsymbol{p}^{(n)} \rangle}{\langle \boldsymbol{g}^{(n-1)}, \boldsymbol{p}^{(n-1)} \rangle}, & n > 0 \end{cases}$$

$$\boldsymbol{d}^{(n)} = -\boldsymbol{p}^{(n)} + \gamma_n \boldsymbol{d}^{(n-1)} \quad \text{search direction}$$

$$\boldsymbol{v}^{(n)} = \boldsymbol{A}\boldsymbol{d}^{(n)}$$

$$\alpha_n = \langle \boldsymbol{d}^{(n)}, -\boldsymbol{g}^{(n)} \rangle / \|\boldsymbol{v}^{(n)}\|^2 \quad \text{step size}$$

$$\boldsymbol{f}^{(n+1)} = \boldsymbol{f}^{(n)} + \alpha_n \boldsymbol{d}^{(n)} \quad \text{update}$$

Bottlenecks: computing $\boldsymbol{A}\boldsymbol{f}^{(n)}$ and $\boldsymbol{A}'\boldsymbol{r}$.

- $\boldsymbol{A}$ is too large to store explicitly (not sparse)
- Even if $\boldsymbol{A}$ were stored, directly computing $\boldsymbol{A}\boldsymbol{f}$ is $O(MN)$ per iteration, whereas FFT is only $O(M \log M)$.

Computing $\boldsymbol{A f}$ Rapidly

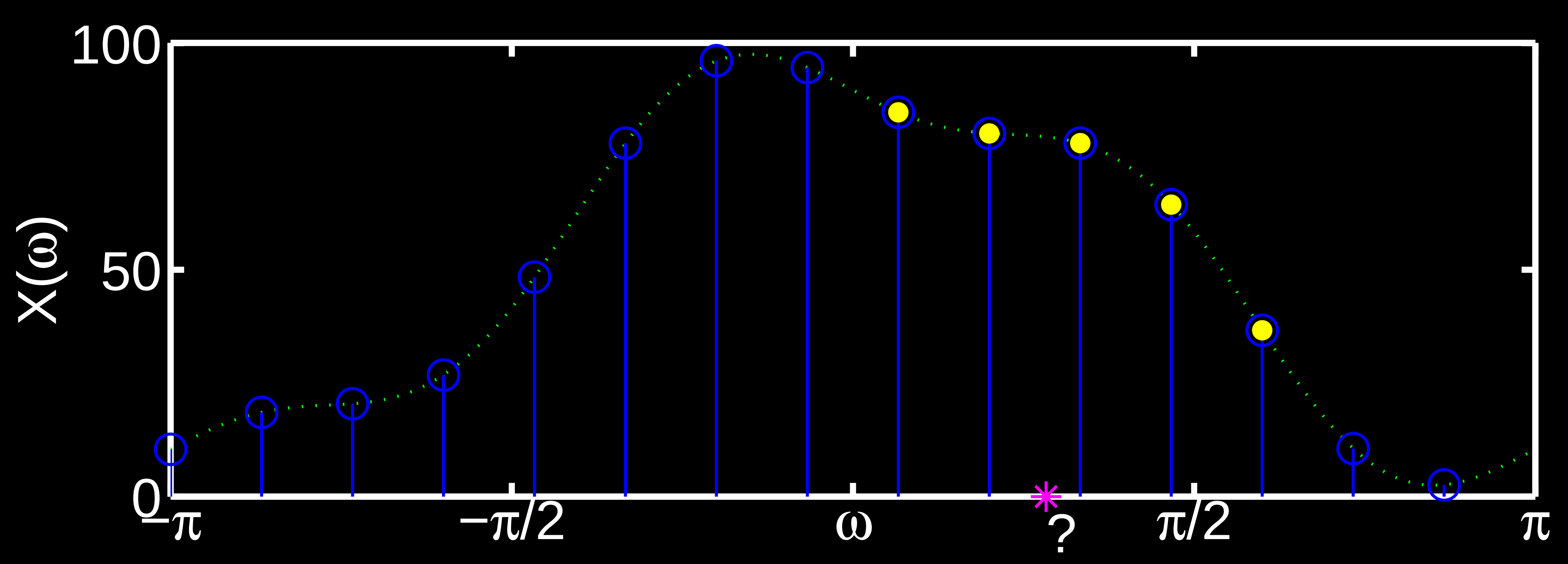
$$[\boldsymbol{A}\boldsymbol{f}]_i = \sum_{j=1}^{N} a_{ij} f_j = P(\vec{\kappa}_i) \sum_{j=1}^{N} \mathrm{e}^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}_j} f_j, \qquad i = 1, \ldots, M$$

- Pixel locations $\{\vec{r}_j\}$ are uniformly spaced
- k-space locations $\{\vec{\kappa}_i\}$ are unequally spaced

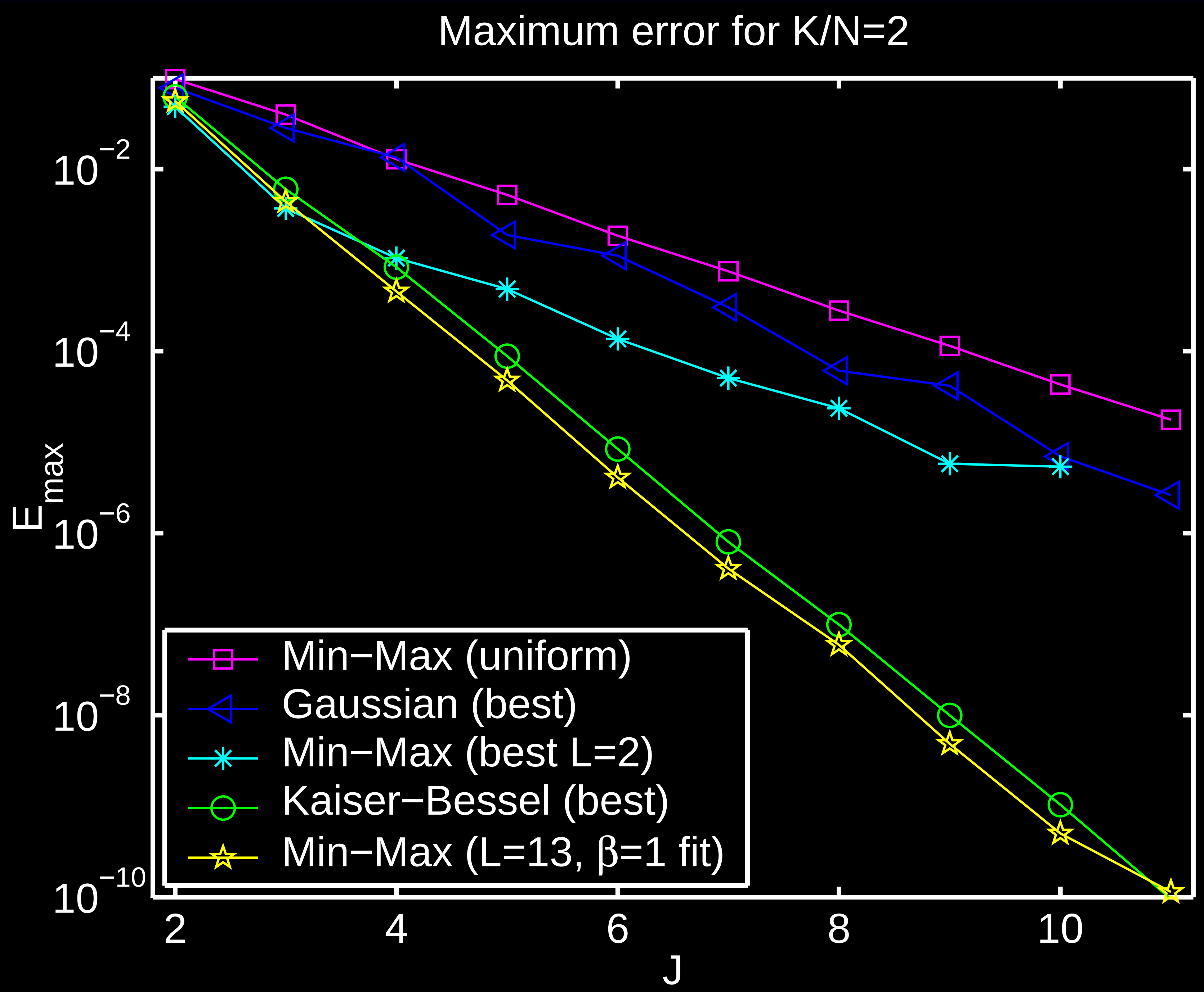
$\Longrightarrow$ needs nonuniform fast Fourier transform (NUFFT)

NUFFT (Type 2)

- Compute over-sampled FFT of equally-spaced signal samples
- Interpolate onto desired unequally-spaced frequency locations
- Dutt & Rokhlin, SIAM JSC, 1993, Gaussian bell interpolator
- Fessler & Sutton, IEEE T-SP, 2003, min-max interpolator
  and min-max optimized Kaiser-Bessel interpolator.
  NUFFT toolbox: `http://www.eecs.umich.edu/~fessler/code`



Worst-Case NUFFT Interpolation Error



Further Acceleration using Toeplitz Matrices

Cost-function gradient:

$$\begin{aligned} \boldsymbol{g}^{(n)} &= \boldsymbol{A}'(\boldsymbol{A}\boldsymbol{f}^{(n)} - \boldsymbol{y}) \\ &= \boldsymbol{T}\boldsymbol{f}^{(n)} - \boldsymbol{b}, \end{aligned}$$

where

$$\boldsymbol{T} \triangleq \boldsymbol{A}'\boldsymbol{A}, \qquad \boldsymbol{b} \triangleq \boldsymbol{A}'\boldsymbol{y}.$$

In the absence of field inhomogeneity, the Gram matrix $\boldsymbol{T}$ is Toeplitz:

$$[\boldsymbol{A}'\boldsymbol{A}]_{jk} = \sum_{i=1}^{M} |P(\vec{\kappa}_i)|^2 \, \mathrm{e}^{-\imath 2\pi \vec{\kappa}_i \cdot (\vec{r}_j - \vec{r}_k)}.$$

Computing $\boldsymbol{T}\boldsymbol{f}^{(n)}$ requires an ordinary ($2\times$ over-sampled) FFT.

(Chan & Ng, SIAM Review, 1996)

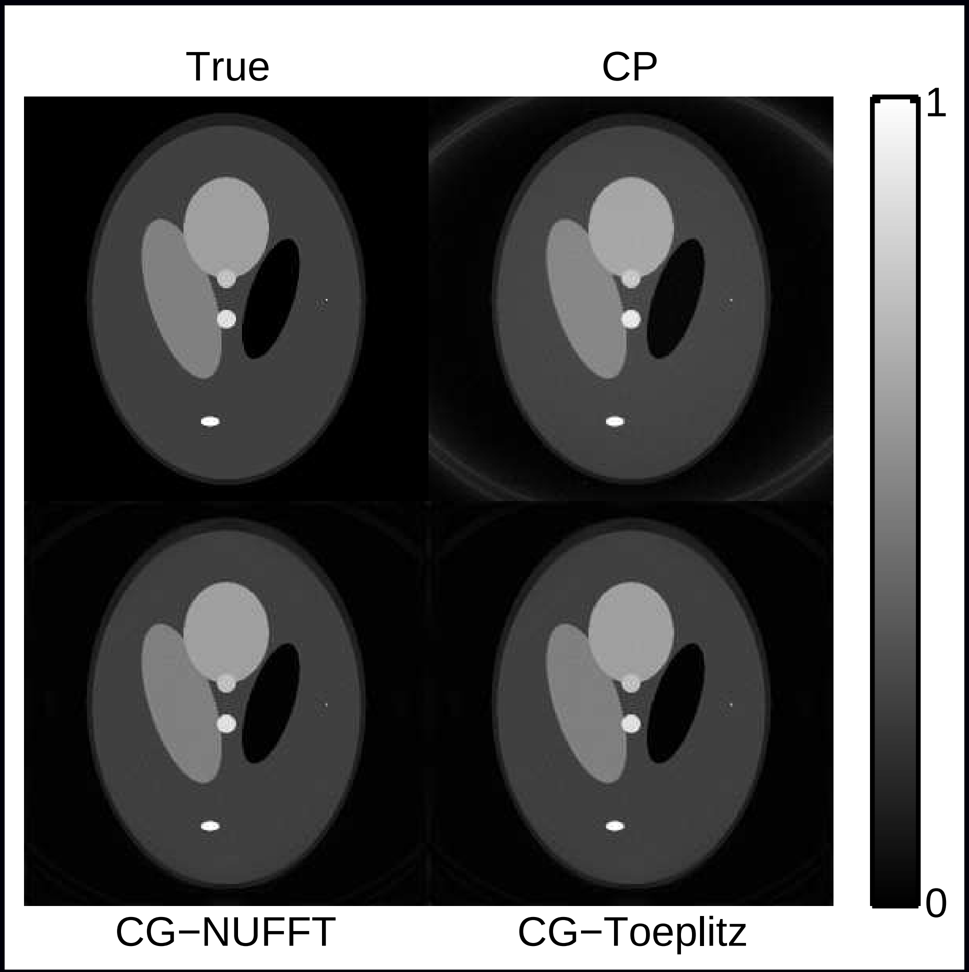
In 2D: block Toeplitz with Toeplitz blocks (BTTB).

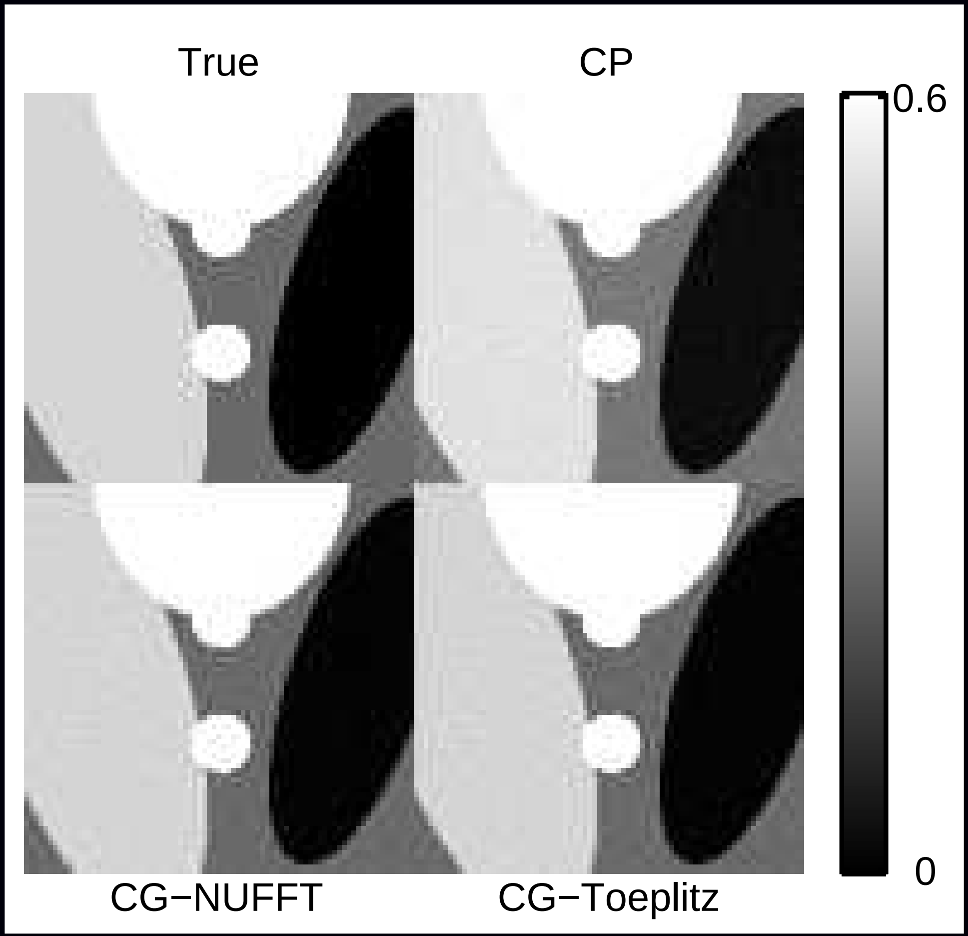
Precomputing the first column of $\boldsymbol{T}$ and $\boldsymbol{b}$ requires a couple NUFFTs.
(Wajer, ISMRM 2001, Eggers ISMRM 2002, Liu ISMRM 2005)

This formulation seems ideal for “hardware” FFT systems.

Toeplitz Acceleration

Example: 256^2 image. radial trajectory, $2\times$ angular under-sampling.





(Iterative provides reduced aliasing energy.)

Toeplitz Acceleration

Method	$A'Dy$	$b = A'y$	T	20 iter	Total Time	NRMS (50dB)
Conj. Phase	0.3				0.3	7.8%
CG-NUFFT				12.5	12.5	4.1%
CG-Toeplitz		0.3	0.8	3.5	4.6	4.1%

- Toeplitz aproach reduces CPU time by more than $2\times$ on conventional workstation (Xeon 3.4GHz)
- Eliminates k-space interpolations $\Longrightarrow$ ideal for FFT hardware
- No SNR compromise
- CG reduces NRMS error relative to CP, but $15\times$ slower... (More dramatic improvements seen in fMRI when correcting for field inhomogeneity.)

NUFFT with Field Inhomogeneity?

Combine NUFFT with min-max temporal interpolator
(Sutton *et al.*, IEEE T-MI, 2003)
(forward version of “time segmentation”, Noll, T-MI, 1991)

Recall signal model including field inhomogeneity:

$$s(t) = \int f(\vec{r})\, e^{-\imath \omega(\vec{r}) t}\, e^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}}\, d\vec{r}.$$

Temporal interpolation approximation (aka “time segmentation”):

$$e^{-\imath \omega(\vec{r}) t} \approx \sum_{l=1}^{L} a_l(t)\, e^{-\imath \omega(\vec{r}) \tau_l}$$

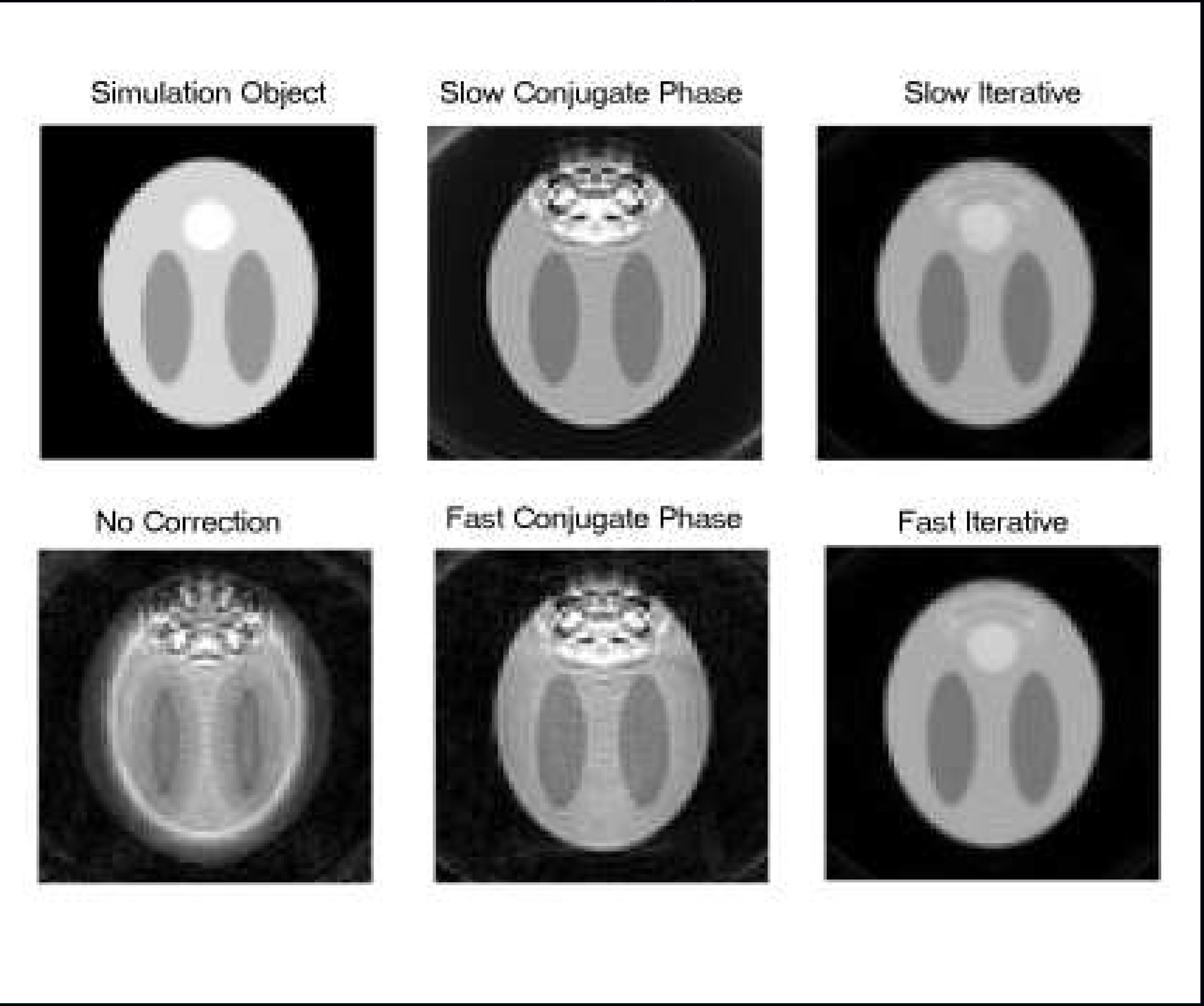
for min-max optimized temporal interpolation functions $\{a_l(\cdot)\}_{l=1}^{L}$.

$$s(t) \approx \sum_{l=1}^{L} a_l(t) \int \left[f(\vec{r})\, e^{-\imath \omega(\vec{r}) \tau_l} \right] e^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}}\, d\vec{r}$$

Linear combination of L NUFFT calls.

Field Corrected Reconstruction Example

Simulation using known field map $\omega(\vec{r})$.

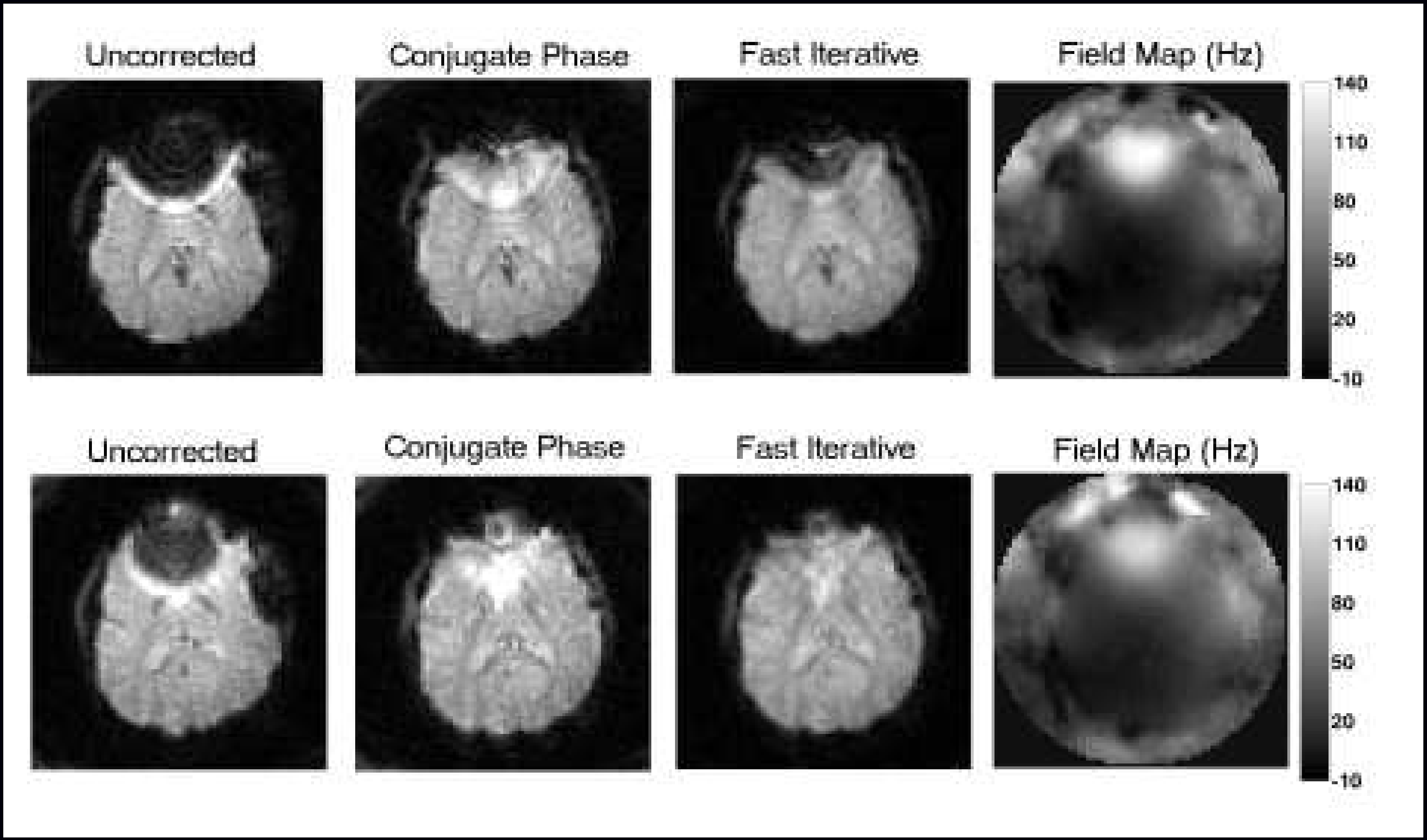


Simulation Quantitative Comparison

- Computation time?
- NRMSE between $\hat{f}$ and f^{true}?

Reconstruction Method	Time (s)	NRMSE complex	NRMSE magnitude
No Correction	0.06	1.35	0.22
Full Conjugate Phase	4.07	0.31	0.19
Fast Conjugate Phase	0.33	0.32	0.19
Fast Iterative (10 iters)	2.20	0.04	0.04
Exact Iterative (10 iters)	128.16	0.04	0.04

Human Data: Field Correction



Acceleration using Toeplitz Approximations

In the presence of field inhomogeneity, the system matrix is:

$$a_{ij} = P(\vec{\kappa}_i)\, e^{-\imath \omega(\vec{r}_j) t_i}\, e^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}_j}$$

The Gram matrix $\boldsymbol{T} = \boldsymbol{A}'\boldsymbol{A}$ is *not* Toeplitz:

$$[\boldsymbol{A}'\boldsymbol{A}]_{jk} = \sum_{i=1}^{M} |P(\vec{\kappa}_i)|^2\, e^{-\imath 2\pi \vec{\kappa}_i \cdot (\vec{r}_j - \vec{r}_k)}\, e^{-\imath \left(\omega(\vec{r}_j) - \omega(\vec{r}_k)\right) t_i}.$$

Approximation (“time segmentation”):

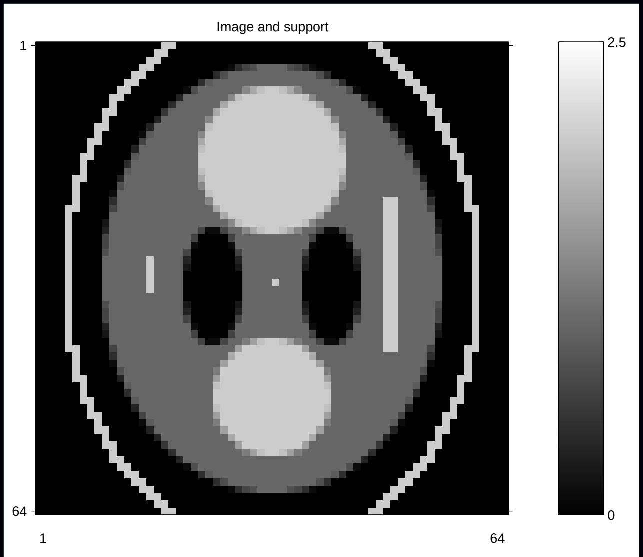
$$e^{-\imath \left(\omega(\vec{r}_j) - \omega(\vec{r}_k)\right) t_i} \approx \sum_{l=1}^{L} b_{il}\, e^{-\imath \left(\omega(\vec{r}_j) - \omega(\vec{r}_k)\right) \tau_l}$$

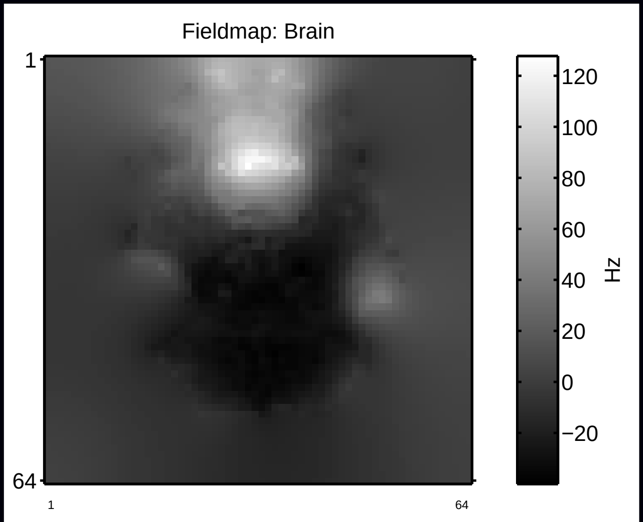
$$\boldsymbol{T} = \boldsymbol{A}'\boldsymbol{A} \approx \sum_{l=1}^{L} \boldsymbol{D}_l' \boldsymbol{T}_l \boldsymbol{D}_l, \qquad \begin{aligned} \boldsymbol{D}_l &\triangleq \mathrm{diag}\left\{ e^{-\imath \omega(\vec{r}_j) \tau_l} \right\} \\ [\boldsymbol{T}_l]_{jk} &\triangleq \textstyle\sum_{i=1}^{M} |P(\vec{\kappa}_i)|^2\, b_{il}\, e^{-\imath 2\pi \vec{\kappa}_i \cdot (\vec{r}_j - \vec{r}_k)}. \end{aligned}$$

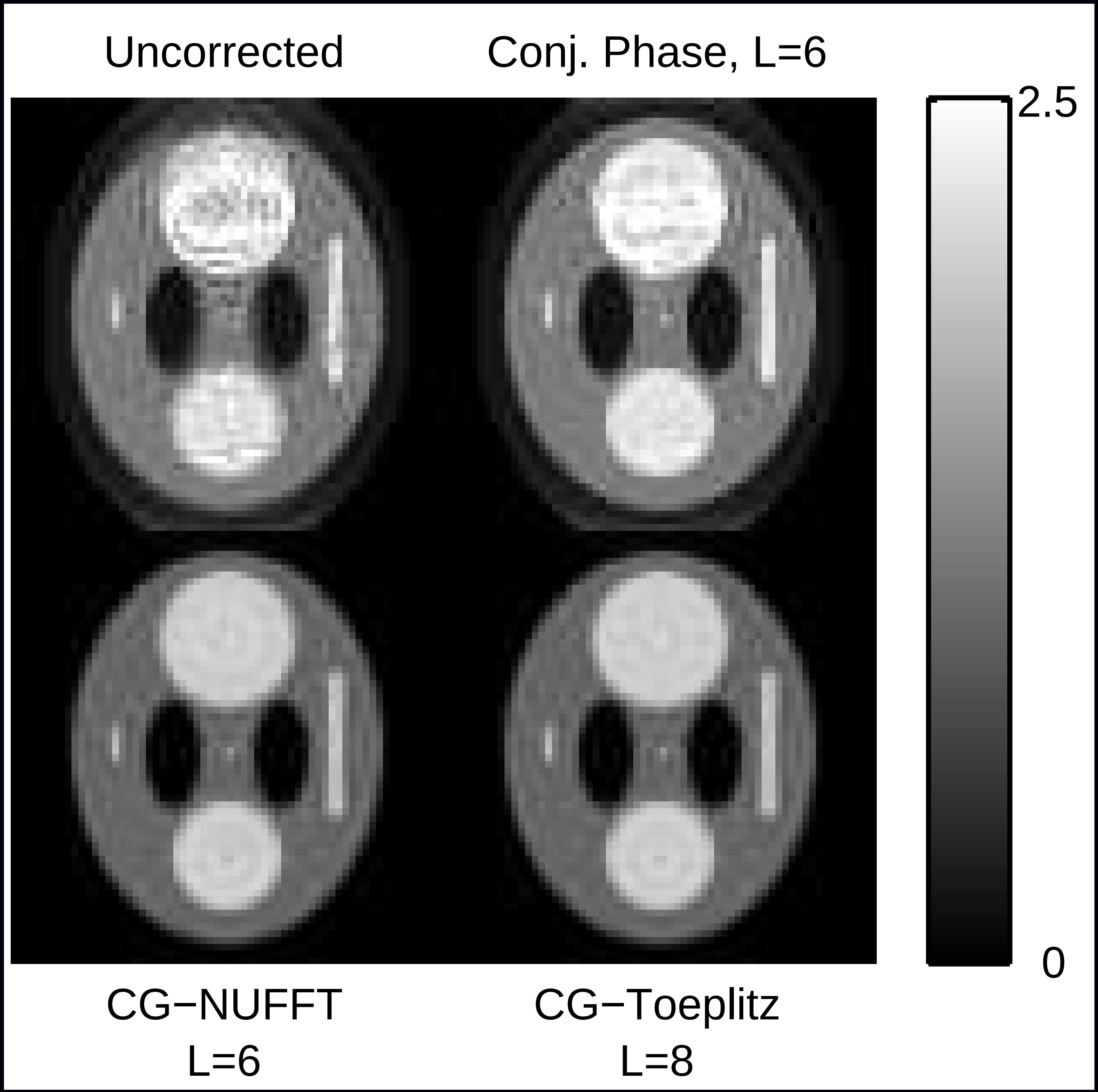
Each $\boldsymbol{T}_l$ is Toeplitz $\Longrightarrow$ $\boldsymbol{T}\boldsymbol{f}$ using L pairs of FFTs.

(Fessler *et al.*, IEEE T-SP, Sep. 2005, brain imaging special issue)

Toeplitz Results







Toeplitz Acceleration

		Precomputation						NRMS % vs SNR				
Method	L	$\boldsymbol{B},\boldsymbol{C}$	$\boldsymbol{A}'\boldsymbol{D}\boldsymbol{y}$	$\boldsymbol{b} = \boldsymbol{A}'\boldsymbol{y}$	$\boldsymbol{T}_l$	15 iter	Total Time	∞	50 dB	40 dB	30 dB	20 dB
Conj. Phase	6	0.4	0.2				0.6	30.7	37.3	46.5	65.3	99.9
CG-NUFFT	6	0.4				5.0	5.4	5.6	16.7	26.5	43.0	70.4
CG-Toeplitz	8	0.4		0.2	0.6	1.3	2.5	5.5	16.7	26.4	42.9	70.4

- Reduces CPU time by $2\times$ on conventional workstation (Mac G5)
- No SNR compromise
- Eliminates k-space interpolations $\Longrightarrow$ ideal for FFT hardware

Joint Field-Map / Image Reconstruction

Signal model:

$$y_i = s(t_i) + \varepsilon_i, \qquad s(t) = \int f(\vec{r})\, \mathrm{e}^{-\imath \omega(\vec{r}) t}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}}\, \mathrm{d}\vec{r}\,.$$

After discretization:

$$\boldsymbol{y} = \boldsymbol{A}(\boldsymbol{\omega})\, \boldsymbol{f} + \boldsymbol{\varepsilon}, \qquad a_{ij}(\boldsymbol{\omega}) = P(\vec{\kappa}_i)\, \mathrm{e}^{-\imath \omega_j t_i}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}_j}\,.$$

Joint estimation via regularized (nonlinear) least-squares:

$$(\hat{\boldsymbol{f}}, \hat{\boldsymbol{\omega}}) = \underset{\boldsymbol{f} \in \mathbb{C}^N, \boldsymbol{\omega} \in \mathbb{R}^N}{\arg\min} \left\| \boldsymbol{y} - \boldsymbol{A}(\boldsymbol{\omega})\, \boldsymbol{f} \right\|^2 + \beta_1 R_1(\boldsymbol{f}) + \beta_2 R_2(\boldsymbol{\omega}).$$

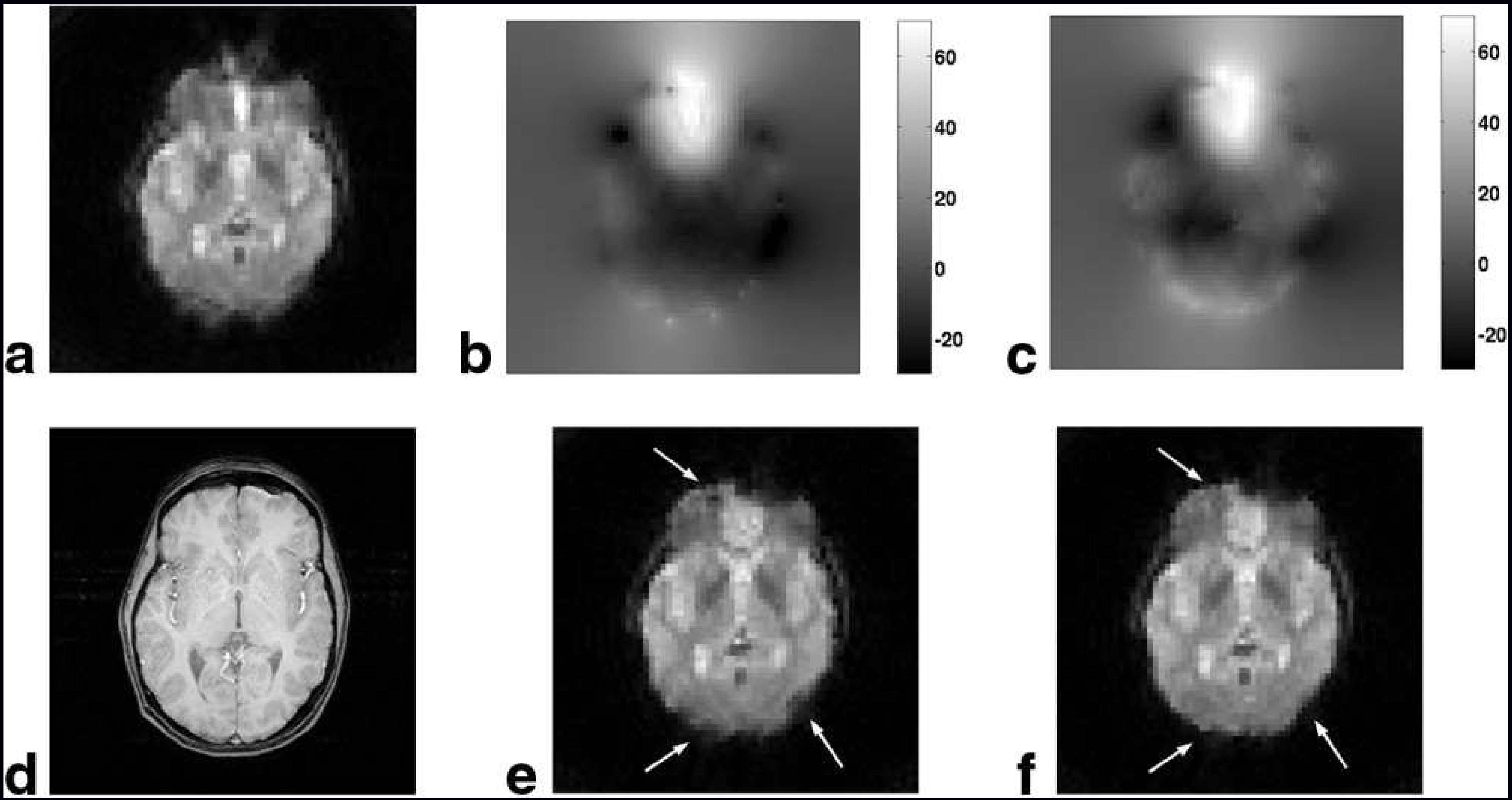
Alternating minimization:

- Using current estimate of fieldmap $\hat{\boldsymbol{\omega}}$, update $\hat{\boldsymbol{f}}$ using CG algorithm.
- Using current estimate $\hat{\boldsymbol{f}}$ of image, update fieldmap $\hat{\boldsymbol{\omega}}$ using gradient descent.

Use spiral-in / spiral-out sequence or “racetrack” EPI.

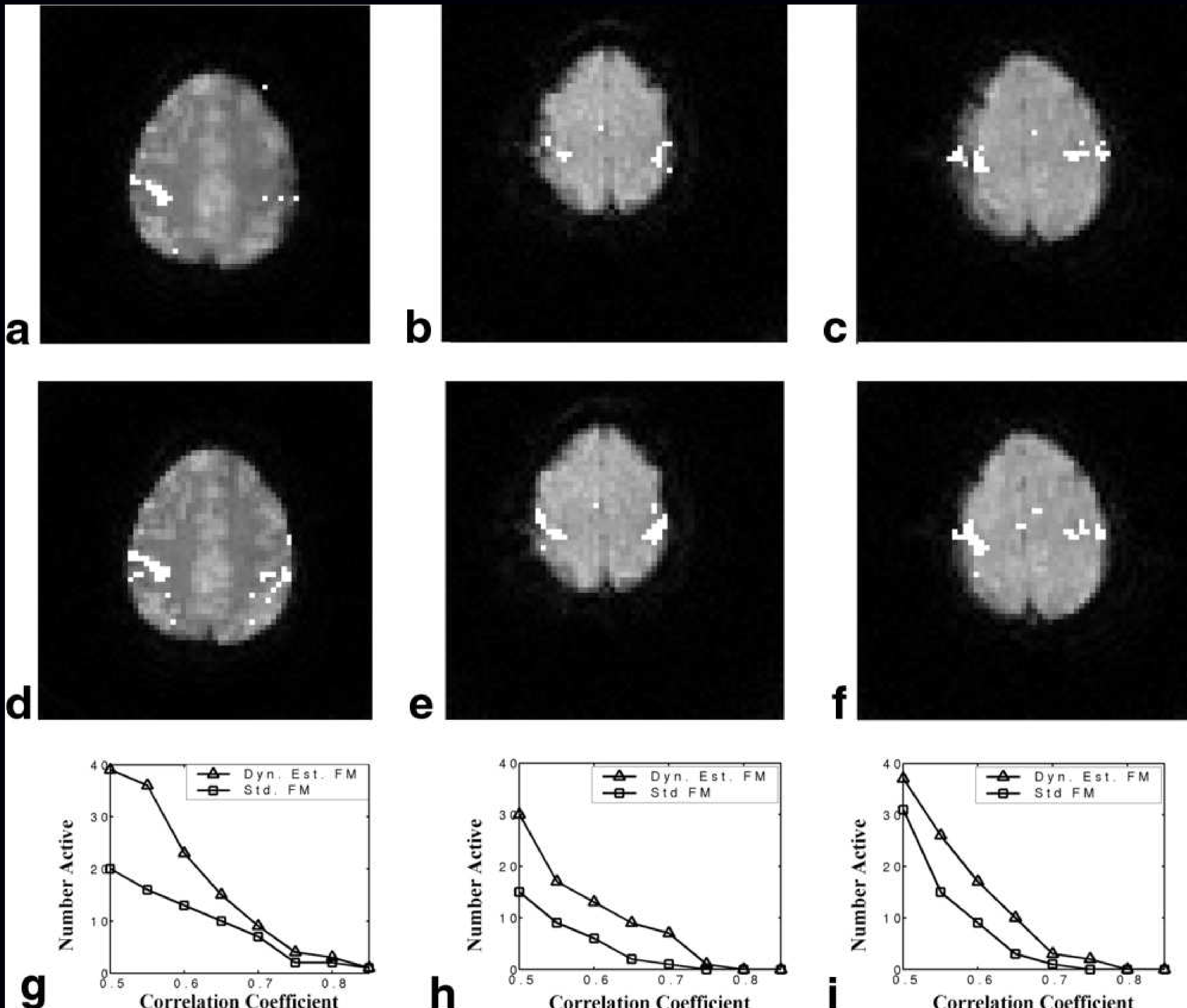
(Sutton *et al.*, MRM, 2004)

Joint Estimation Example



(a) uncorr., (b) std. map, (c) joint map, (d) T1 ref, (e) using std, (f) using joint.

Activation Results: Static vs Dynamic Field Maps



Functional results for the two reconstructions for 3 human subjects.

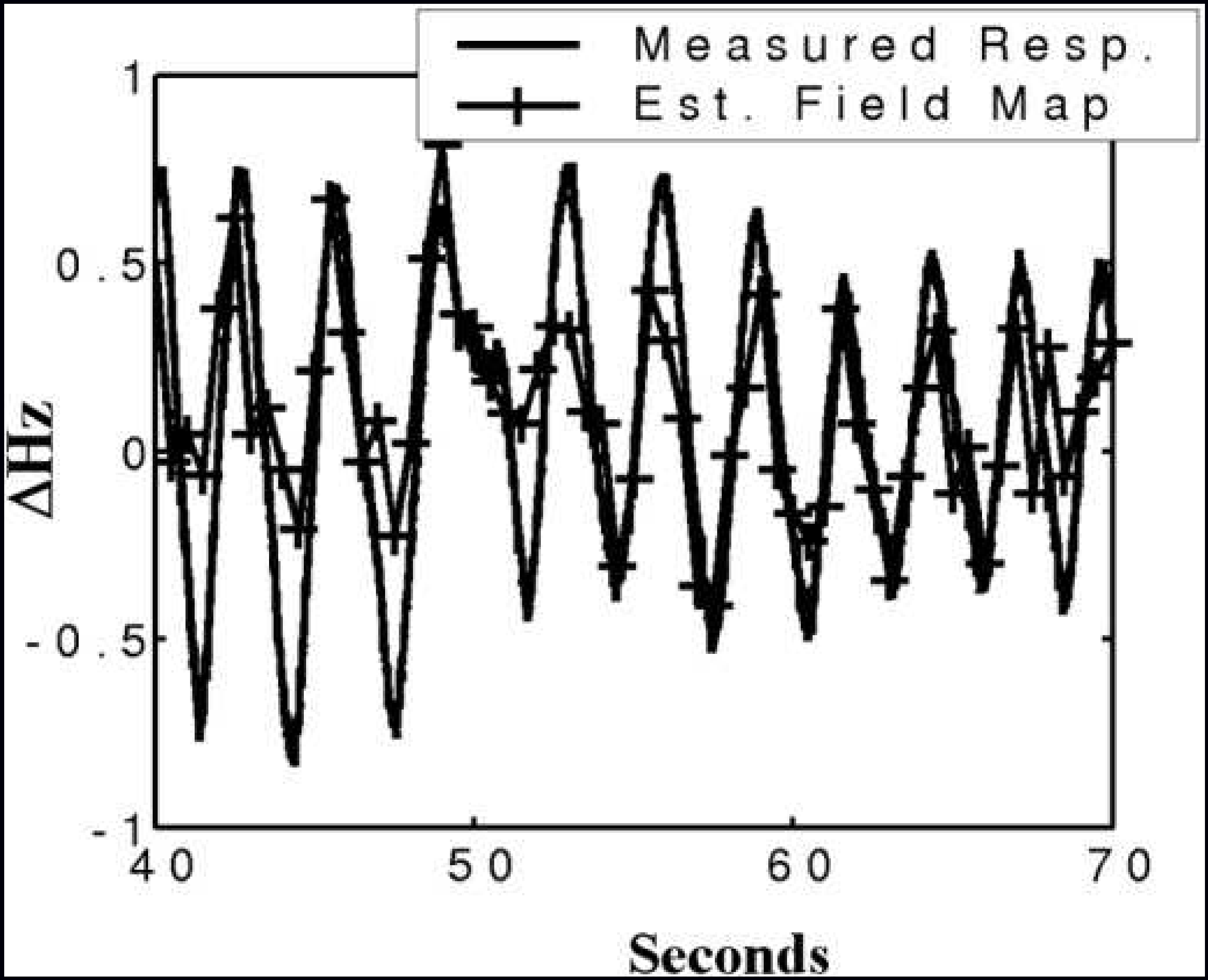
Reconstruction using the standard field map
for (a) subject 1, (b) subject 2, and (c) subject 3.

Reconstruction using the jointly estimated field map
for (d) subject 1, (e) subject 2, and (f) subject 3.

Number of pixels with correlation coefficients higher than thresholds
for (g) subject 1, (h) subject 2, and (i) subject 3.

Take home message: dynamic field mapping is possible, using iterative reconstruction as an essential tool.
(Standard field maps based on echo-time differences work poorly for spiral-in / spiral-out sequences due to phase discrepancies.)

Tracking Respiration-Induced Field Changes



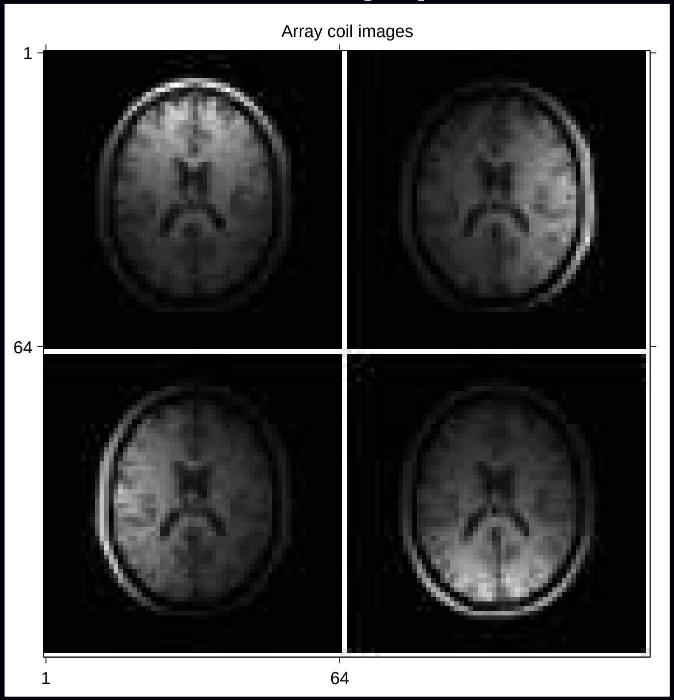
Parallel Imaging

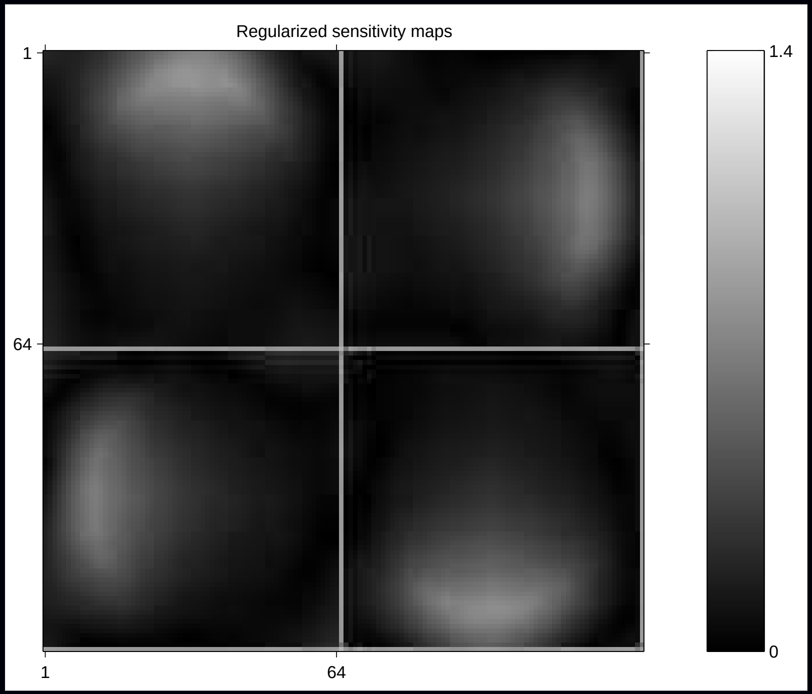
Sensitivity encoded (SENSE) imaging

Use multiple receive coils (requires multiple RF channels).
Exploit spatial localization of sensitivity pattern of each coil.

Note: at 1.5T, RF is about 60MHz.
$\Longrightarrow$ RF wavelength is about $3 \cdot 10^8$m/s$/60 \cdot 10^6$Hz $= 5$ meters

RF coil sensitivity patterns





Pruessmann *et al.*, MRM, 1999

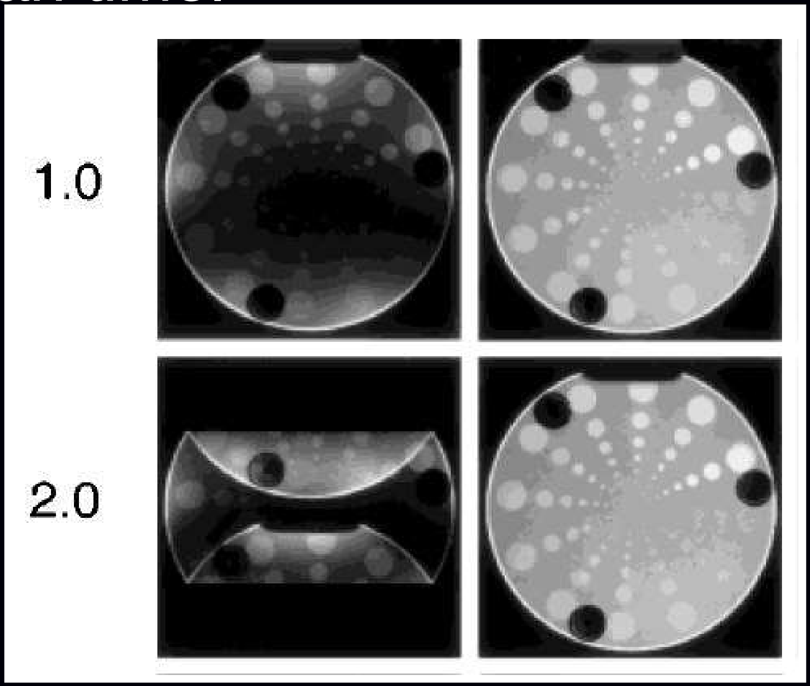
SENSE Model

Multiple coil data:

$$y_{li} = s_l(t_i) + \varepsilon_{li}, \quad s_l(t) = \int f(\vec{r})\, s_l^{\text{coil}}(\vec{r})\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t)\cdot\vec{r}}\, \mathrm{d}\vec{r}, \quad l = 1, \ldots, L = N_{\text{coil}}$$

Goal: reconstruct $f(\vec{r})$ from coil data $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_L$
"given" sensitivity maps $\left\{ s_l^{\text{coil}}(\vec{r}) \right\}_{l=1}^{L}$.

Benefit: reduced scan time.



Left: sum of squares; right: SENSE.

SENSE Reconstruction

Signal model:

$$s_l(t) = \int f(\vec{r})\, s_l^{\text{coil}}(\vec{r})\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}} \, \mathrm{d}\vec{r}$$

Discretized form:

$$\boldsymbol{y}_l = \boldsymbol{A}\boldsymbol{D}_l \boldsymbol{f} + \boldsymbol{\varepsilon}_l, \quad l = 1, \ldots, L,$$

where $\boldsymbol{A}$ is the usual frequency/phase encoding matrix and
$\boldsymbol{D}_l$ contains the sensitivity pattern of the lth coil: $\boldsymbol{D}_l = \mathrm{diag}\left\{ s_l^{\text{coil}}(\vec{r}_j) \right\}$.

Regularized least-squares estimation:

$$\hat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \sum_{l=1}^{L} \|\boldsymbol{y}_l - \boldsymbol{A}\boldsymbol{D}_l \boldsymbol{f}\|^2 + \beta R(\boldsymbol{f}).$$

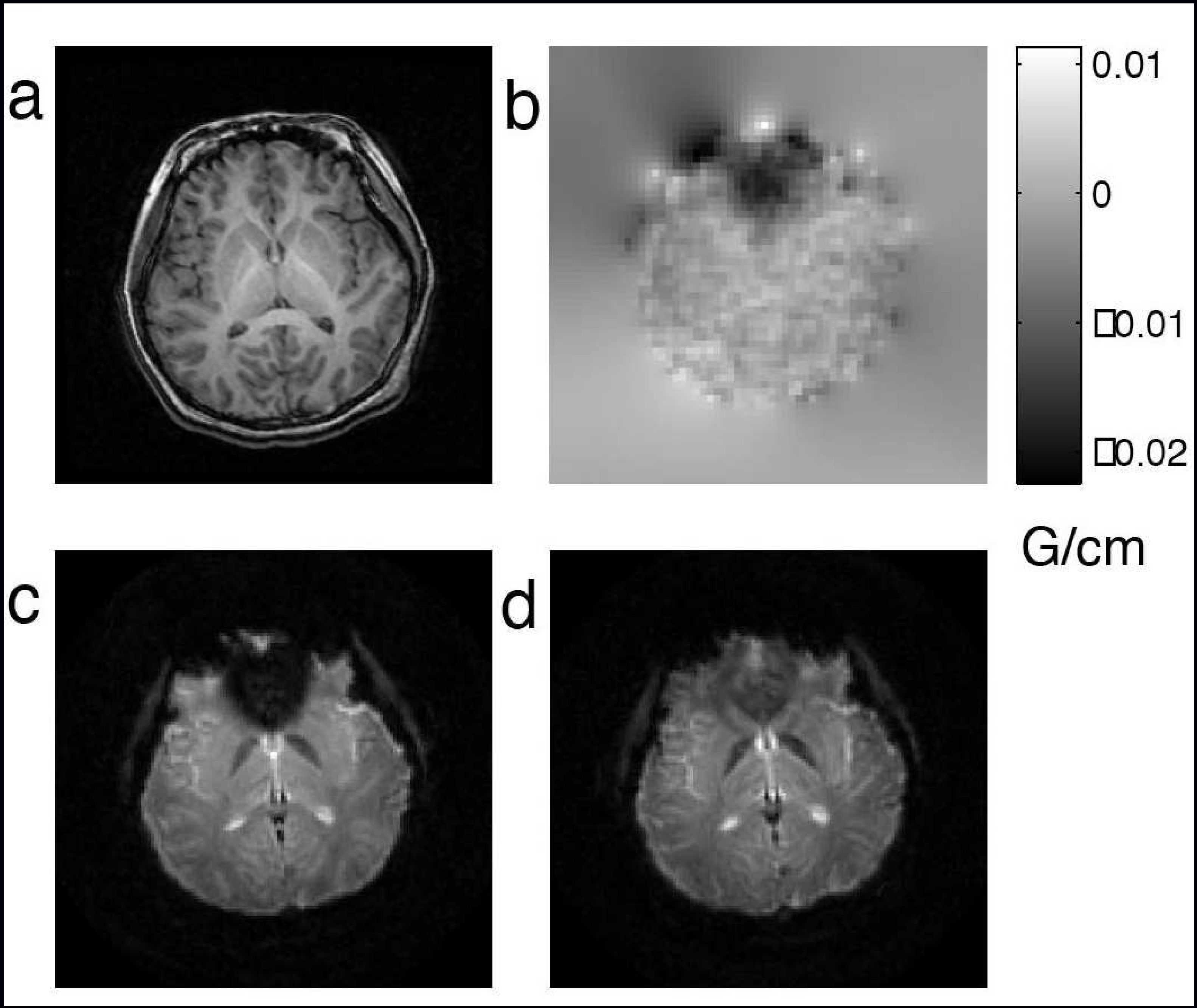
Can generalize to account for noise correlation due to coil coupling.
Easy to apply CG algorithm, including Toeplitz/NUFFT acceleration.

For Cartesian SENSE, iterations are not needed.
(Solve small system of linear equations for each voxel.)

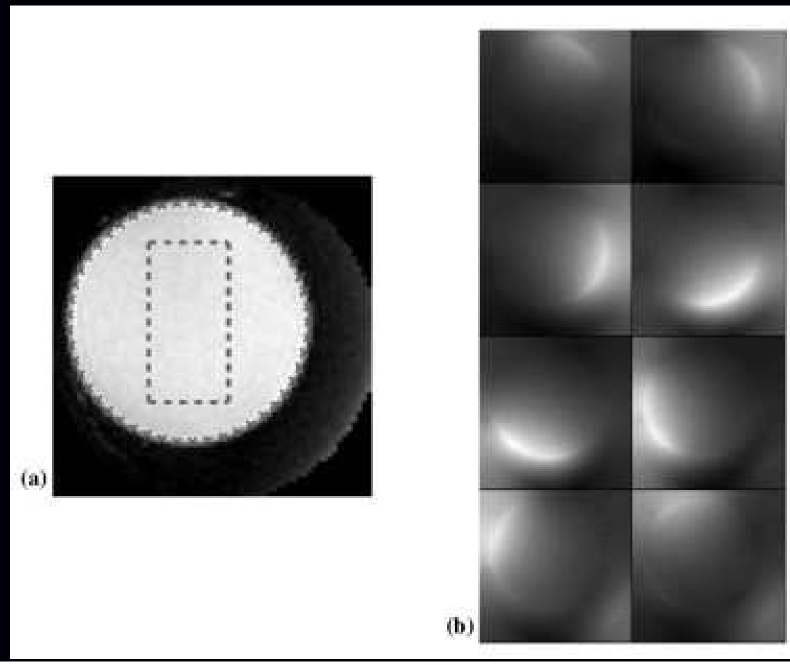
RF Pulse Design

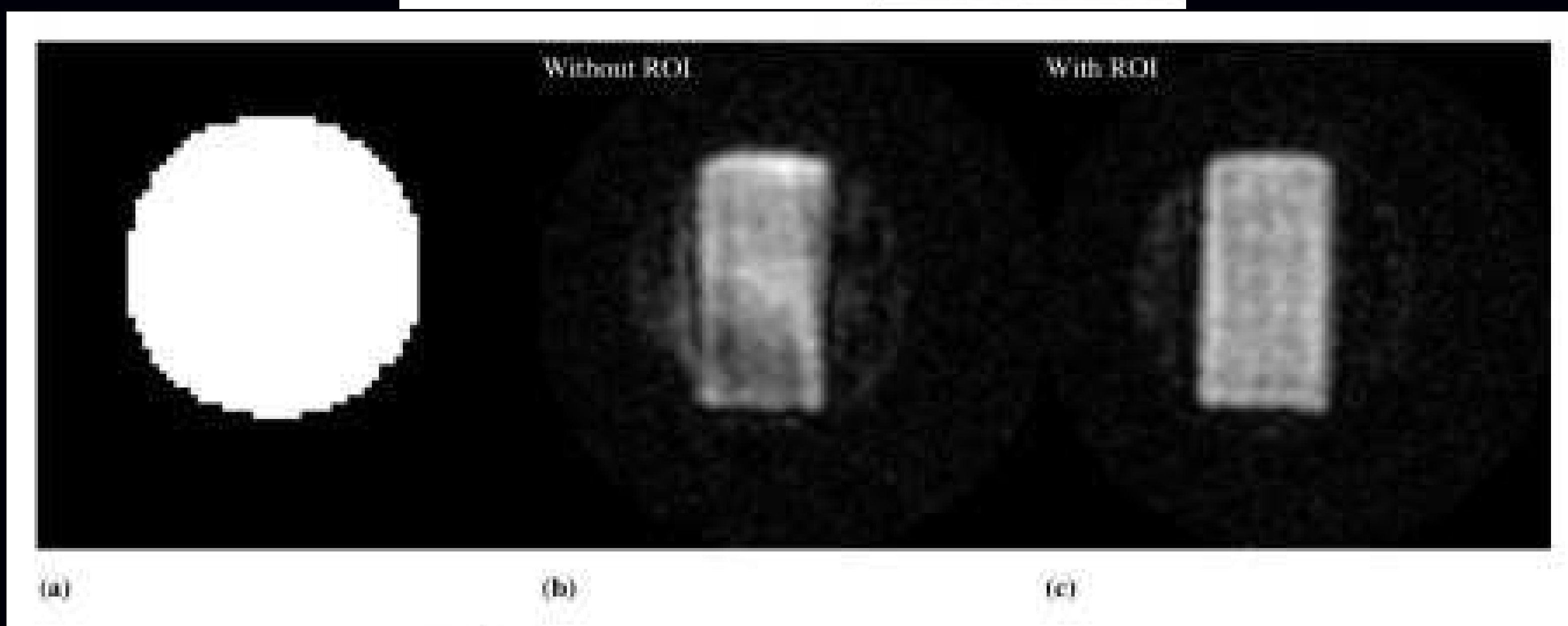
Example: Iterative RF Pulse Design

(3D tailored RF pulses for through-plane dephasing compensation)



Multiple-coil Transmit Imaging Pulses (Mc-TIP)





Summary

- Iterative reconstruction: much potential in MRI
- Quadratic regularization parameter selection is tractable
- Computation: reduced by tools like NUFFT / Toeplitz
- But optimization algorithm design remains important
  (*cf.* Shepp and Vardi, 1982, PET)

Some current challenges

- Sensitivity pattern mapping for SENSE
- Through-voxel field inhomogeneity gradients
- Motion / dynamics / partial k-space data
- Establishing diagnostic efficacy with clinical data...

Image reconstruction toolbox:
`http://www.eecs.umich.edu/~fessler`