Model-based MR image reconstruction with compensation for through-plane field inhomogeneity

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Introduction

- Magnetic field inhomogeneity (ΔB_0) can cause blur/distortion in MR images if uncorrected
- Existing correction methods have drawbacks
  - Assume field map constant within each voxel

    Sutton, Noll, Fessler, IEEE T-MI, Feb. 2003

    Fessler, *et al.*, IEEE T-SP, Sep. 2005
  - Apply only to particular trajectories, *e.g.*, spirals

    Noll, Fessler, Sutton, IEEE T-MI, Mar. 2005
  - Use slow iterative method based on piecewise-linear model

    Sutton, Noll, Fessler, ISMRM, 2004

Goal:

- account for within-voxel field inhomogeneity (piecewise-linear model)
- allow arbitrary trajectories
- provide accelerated (iterative) algorithm

Review of "conventional" MR signal model

Baseband MR signal:

$$s(t) = \iint f(x,y,z_0)\, \mathrm{e}^{-\imath\, \omega(x,y,z_0)\, t}\, \mathrm{e}^{-\imath 2\pi [k_X(t) x + k_Y(t) y]}\, \mathrm{d}x\, \mathrm{d}y$$

- z_0: axial center of the slice.
- $f(x,y,z)$: transverse magnetization of object (unknown).
- $\omega(x,y,z_0)$: off-resonance frequency map for slice z_0.
  Field map: $\omega = \gamma \Delta B_0$, assumed known.
- $(k_X(t), k_Y(t))$: k-space trajectory of the (2D) scan.

Goal: estimate $f(x,y,z_0)$ from M noisy signal samples:

$$y_i = s(t_i) + \varepsilon_i, \qquad i = 1, \ldots, M,$$

$\boldsymbol{\varepsilon}_i$: zero-mean, complex, white gaussian noise.

Conventional discretization

Finite-series expansions of object and field map using rect functions:

$$f(x,y,z_0) = \sum_{j=1}^{N} f_j \, b(x - x_j, y - y_j)$$

$$\omega(x,y,z_0) = \sum_{j=1}^{N} \omega_j \, b(x - x_j, y - y_j)$$

- $b(x,y) = \mathrm{rect}_2\left(\frac{x}{\Delta}, \frac{y}{\Delta}\right)$ denotes the object basis function (square pixels of dimension Δ)
- (x_j, y_j): center of the jth basis function translate
- f_j: object pixel values
- ω_j: field map values (assumed constant within each voxel)
- N: number of parameters (pixels)

Conventional discretized signal model

Combining

$$s(t) = \iint f(x,y,z_0)\, e^{-\imath\omega(x,y,z_0)\,t}\, e^{-\imath 2\pi[k_X(t)x + k_Y(t)y]}\, dx\, dy$$

$$f(x,y,z_0) = \sum_{j=1}^{N} f_j\, b(x-x_j, y-y_j)$$

$$\omega(x,y,z_0) = \sum_{j=1}^{N} \omega_j\, b(x-x_j, y-y_j)$$

$$y_i = s(t_i) + \varepsilon_i, \qquad i = 1,\ldots,M$$

$\Longrightarrow$

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{f} + \boldsymbol{\varepsilon}$$

where $M \times N$ system matrix $\boldsymbol{A}$ has elements

$$a_{ij} = \mathrm{sinc}_2(k_X(t_i)\Delta, k_Y(t_i)\Delta) \qquad e^{-\imath\omega_j t_i} \qquad e^{-\imath 2\pi(k_X(t_i)x_j + k_Y(t_i)y_j)}$$

basis spectrum | field inhomog. | NUFFT

Conventional regularized LS reconstruction

$$\hat{f} = \arg\min_{f \in \mathbb{C}^N} \Psi(f), \qquad \Psi(f) = \frac{1}{2} \underbrace{\|y - Af\|^2}_{\text{data fit}} + \underbrace{\beta \mathsf{R}(f)}_{\text{regularize}}$$

Conjugate gradient (CG) iterative algorithm, uses gradient:

$$\nabla \Psi(f) = A'(Af - y) + \beta \nabla \mathsf{R}(f) .$$

Computing Af requires

$$\sum_{j=1}^{N} a_{ij} f_j = \mathrm{sinc}_2(k_X(t_i)\Delta, k_Y(t_i)\Delta) \left[\sum_{j=1}^{N} e^{-\imath \omega_j t_i} e^{-\imath 2\pi (k_X(t_i) x_j + k_Y(t_i) y_j)} f_j \right] .$$

Approximation (time segmentation, frequency segmentation, etc.):

$$e^{-\imath \omega_j t_i} \approx \sum_{l=1}^{L} b_{il} c_{lj} .$$

Inner sum becomes:

$$\left[\sum_{l=1}^{L} b_{il} \underbrace{\sum_{j=1}^{N} e^{-\imath 2\pi (k_X(t_i) x_j + k_Y(t_i) y_j)} (c_{lj} f_j)}_{\text{NUFFT}} \right] .$$

Extended signal model

Extension: account for non-ideal slice profile:

$$s(t) = \iiint h(z - z_0)\, f(x,y,z)\, e^{-\imath\omega(x,y,z)t}\, e^{-\imath 2\pi (k_X(t)x + k_Y(t)y)}\, dx\, dy\, dz.$$

Natural series expansion of object magnetization:

$$f(x,y,z) = \sum_{j=1}^{N} f_j\, b(x - x_j, y - y_j), \text{ for } z \approx z_0,$$

which treats the object magnetization as a constant across the slice, leading to usual (unavoidable?) partial volume effects.

Substituting and simplifying yields this "inconvenient" signal model:

$$s(t) = \sum_{j=1}^{N} f_j \iint b(x - x_j, y - y_j)\, e^{-\imath 2\pi (k_X(t)x + k_Y(t)y)} \cdot \underbrace{\left[\int h(z - z_0)\, e^{-\imath\omega(x,y,z)t}\, dz\right]}_{\text{Not a FT}} dx\, dy.$$

Field map series expansion

Piece-wise linear model for field map, with through-plane gradients:

$$\omega(x,y,z) = \sum_{j=1}^{N} \text{rect}_2\left(\frac{x-x_j}{\Delta}, \frac{y-y_j}{\Delta}\right)\left(\omega_j + 2\pi g_j\,(z-z_0)\right),$$

- (x_j, y_j) : in-plane center coordinates of the jth voxel
- ω_j: off-resonance frequency at center of the jth voxel [rad/s]
- g_j: field map through-plane gradient within the jth voxel [Hz / cm]
- Can generalize to include in-plane field-map gradients.

Determine $\{\omega_j\}$ and $\{g_j\}$ using regularized field map estimates (Fessler *et al.*, ISBI 2006) and central differences.

Extended signal model continued

Substituting field map series expansion into preceding signal model and simplifying yields:

$$s(t) = \iint \sum_{j=1}^{N} H(tg_j)\, f_j \operatorname{rect}_2\left(\frac{x-x_j}{\Delta}, \frac{y-y_j}{\Delta}\right) e^{-\imath\omega_j t}\, e^{-\imath 2\pi(k_X(t)x+k_Y(t)y)}\, dx\, dy$$

$$= \operatorname{sinc}_2(k_X(t)\Delta, k_Y(t)\Delta) \cdot \sum_{j=1}^{N} H(tg_j)\, e^{-\imath\omega_j t}\, e^{-\imath 2\pi(k_X(t)x_j+k_Y(t)y_j)}\, f_j,$$

where $h \stackrel{\text{FT}}{\longleftrightarrow} H$.

- $H(tg_j)$ describes signal loss due to through-plane dephasing.
- $e^{-\imath\omega_j t}$ describes phase accumulation due to off-resonance.
- Simplifies to "conventional" signal model if no through-plane field gradients, *i.e.*, if $g_j = 0$.
- Presence of $H(tg_j)$ prohibits direct use of previous fast methods.
- How to form a fast algorithm?

Proposed signal model

Proposed approximation:

$$H(t_i g_j)\, e^{-\imath \omega_j t_i} \approx \sum_{l=1}^{L} b_{il} c_{lj}, \qquad \begin{array}{l} i = 1, \ldots, M \\ j = 1, \ldots, N \end{array}.$$

Substituting into "inconvenient" signal model and simplifying yields:

$$s(t_i) \approx \sum_{l=1}^{L} b_{il}\, \mathrm{sinc}_2(k_x(t_i)\Delta, k_y(t_i)\Delta) \underbrace{\sum_{j=1}^{N} e^{-\imath 2\pi (k_x(t_i) x_j + k_y(t_i) y_j)} (c_{lj} f_j)}_{\text{NUFFT (or FFT for EPI)}}.$$

With this form, multiplication with $\boldsymbol{A}$ or $\boldsymbol{A}'$ requires L NUFFT calls.

Thus CG-NUFFT requires $O(LN \log N)$ flops per iteration.
(Same compute time per iteration as CG with off-resonance only.)

Key approximation

$$H(t_i g_j)\, e^{-\imath \omega_j t_i} \approx \sum_{l=1}^{L} b_{il} c_{lj}, \quad \textit{i.e.}, \quad \boldsymbol{H} \approx \boldsymbol{B}\boldsymbol{C}$$

How to choose basis signals $\boldsymbol{B}$ and coefficients $\boldsymbol{C}$?

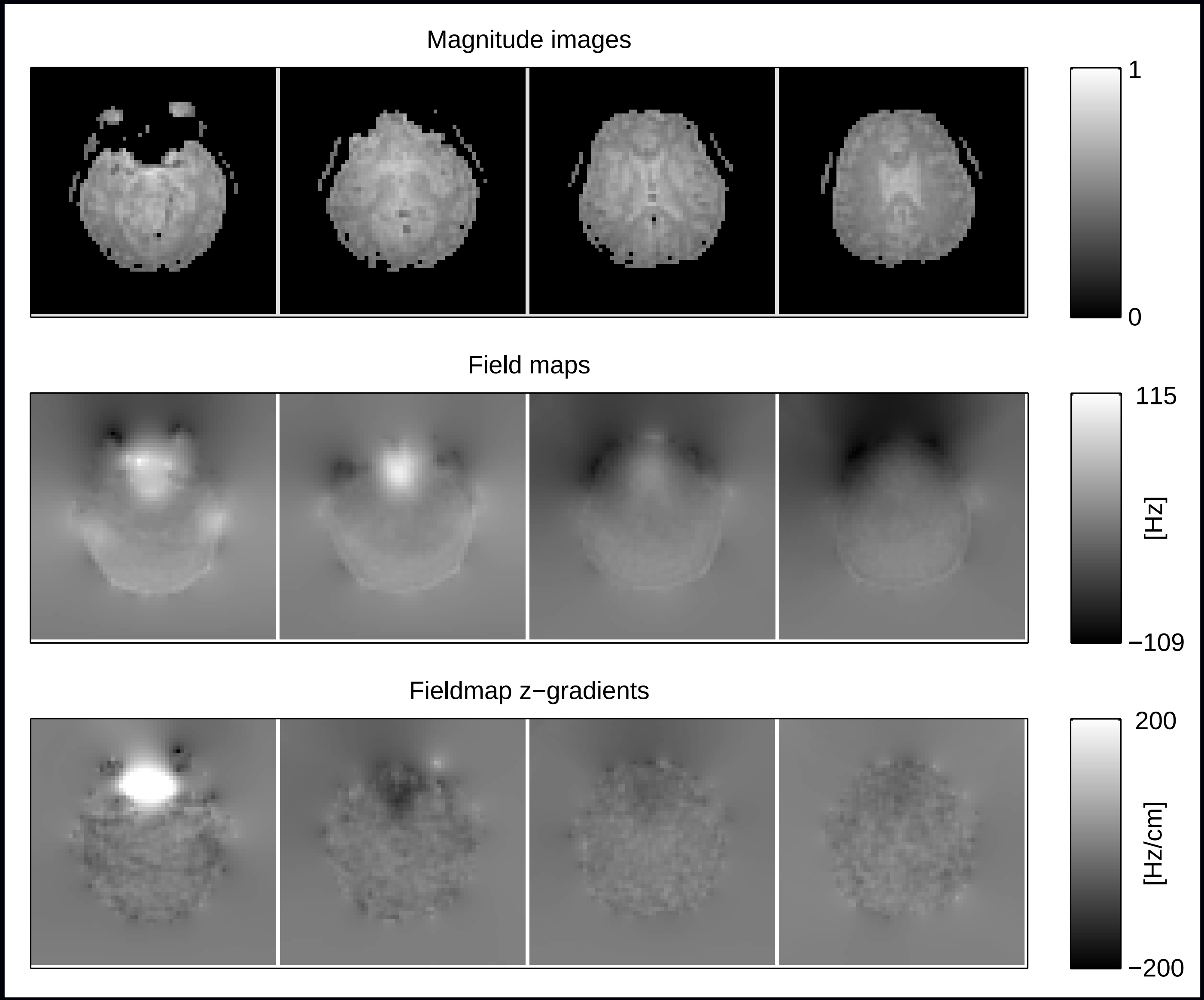
(Alternate view: $\boldsymbol{C}$ is basis images and $\boldsymbol{B}$ is temporal interpolation.)

Brute force: principle components analysis (PCA) via SVD.
Drawback: N and M huge.

Here, each (ω_j, g_j) pair corresponds to a signal $H(t_i g_j)\, e^{-\imath \omega_j t_i}$.

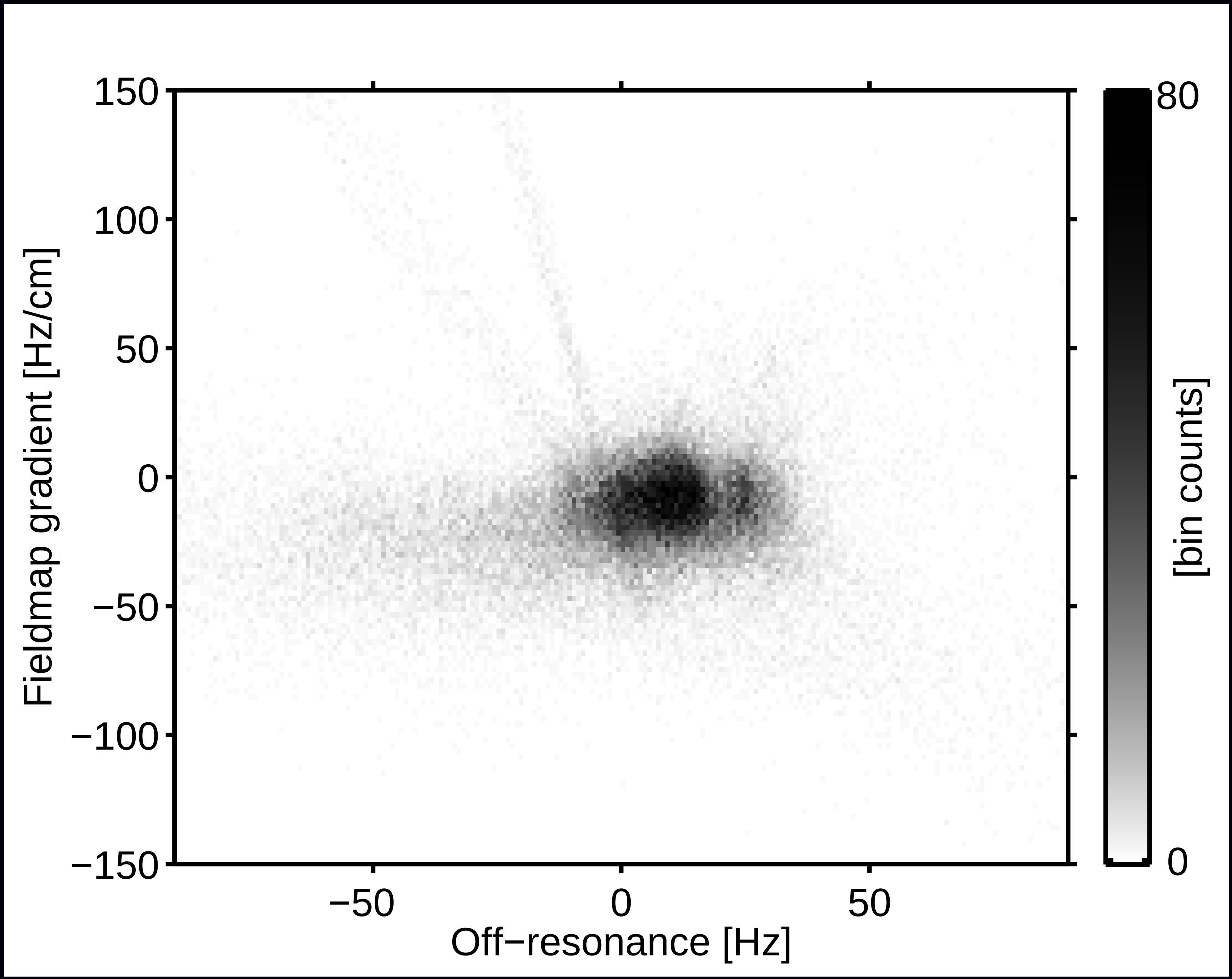
Solution: “parametric PCA:” apply PCA to a representative subset.

MR field map data



- Human brain MR field maps: Yip, Fessler, Noll, MRM, Nov. 2006.
- $64 \times 64 \times 40$, 24 cm transaxial FOV, $\Delta_Z = 1$ mm

Histogram of (ω_j,g_j) pairs



(within the brain voxels exceeding 1% of the maximum magnitude value)

Histogram-based parametric PCA

Coarsely sampled bin centers $(\tilde{\omega}_k, \tilde{g}_k), \quad k = 1, \ldots, K \ll N$.
These bin centers parameterize the "representative signals."

- For now, chosen to uniformly sample (ω, g) parameter space.
- Interesting question: how to optimize sampling?

Let w_k denote number of (ω_j, g_j) pairs in the kth bin.

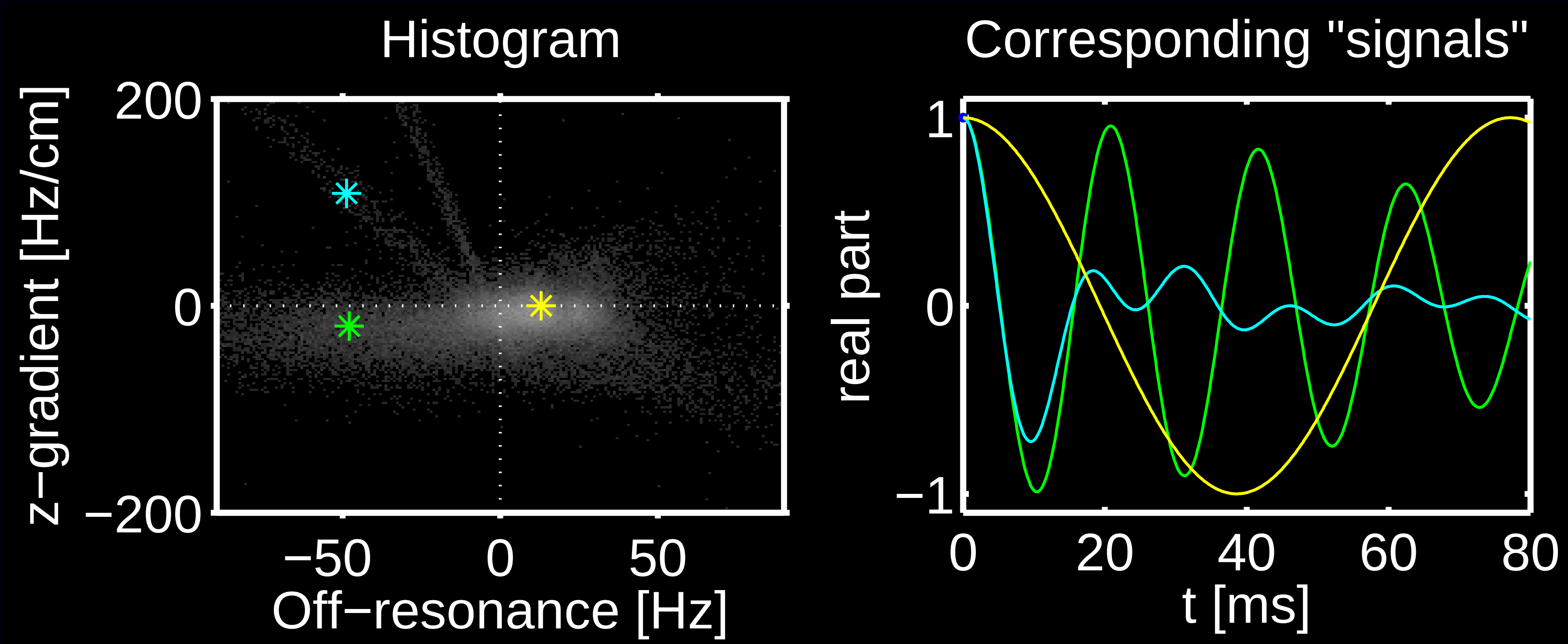
Choose basis $\boldsymbol{B}$ via this weighted PCA problem:

$$\min_{\boldsymbol{B}, \tilde{\boldsymbol{C}}} \sum_{k=1}^{K} w_k \sum_{i=1}^{M} \left| H(t_i \tilde{g}_k) \, \mathrm{e}^{-\imath \tilde{\omega}_k t_i} - \sum_{l=1}^{L} b_{il} \tilde{c}_{lj} \right|^2$$

Solution for basis signals $\boldsymbol{B}$ uses SVD of $M \times K$ matrix, not $M \times N$.

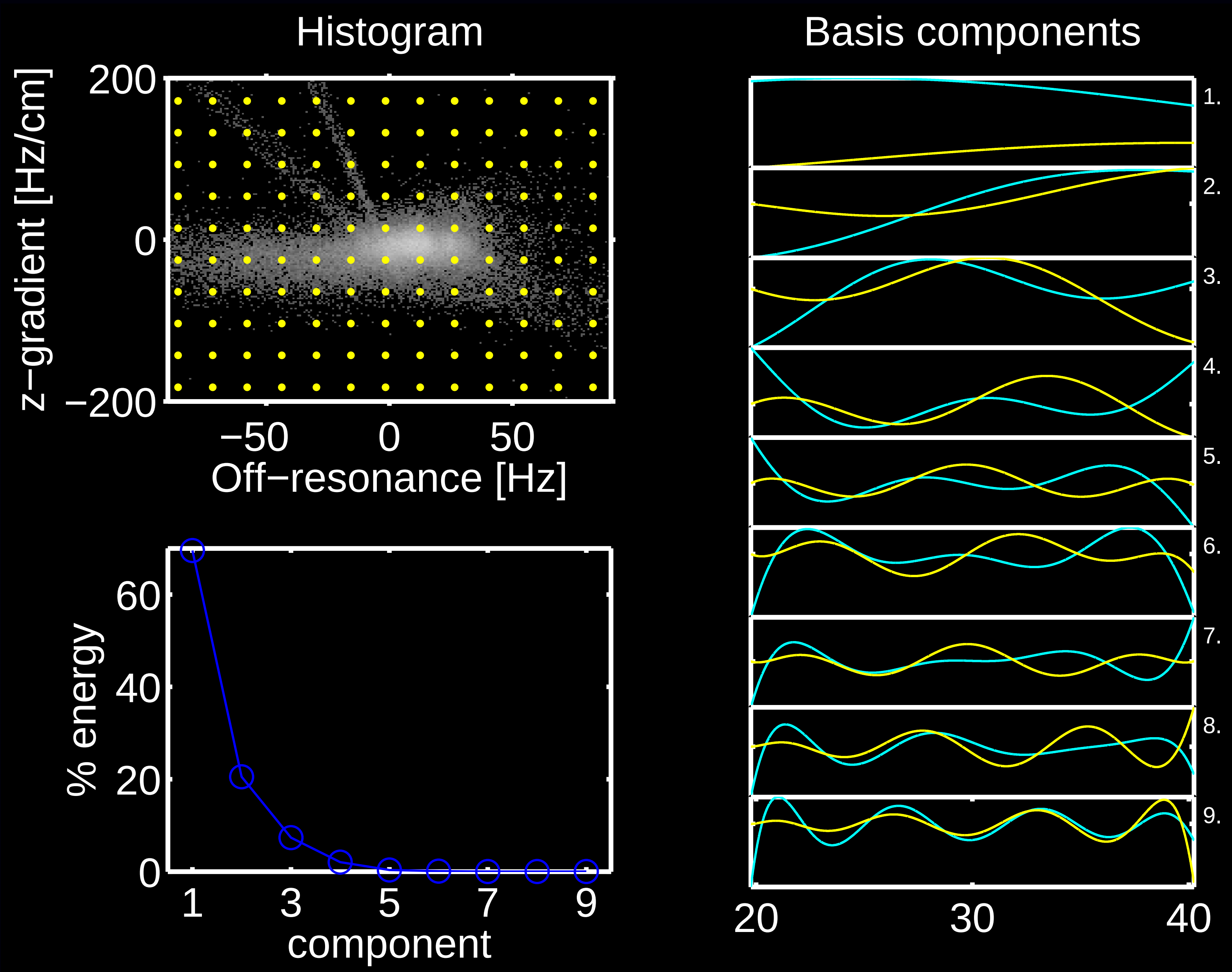
Similar approach for (ω, R_2^*) in Fessler, *et al.*, IEEE T-SP, Sep. 2005.

Example representative signals



- $H(t_i \tilde{g}_k)\, e^{-\imath \tilde{\omega}_k t_i}$
- $h(z) = \mathrm{rect}(z/\Delta_Z) \xleftrightarrow{\mathrm{FT}} H(\nu) = \Delta_Z \,\mathrm{sinc}(\nu \Delta_Z)$
- $\Delta_Z = 4$ mm thick slice
- Typically 20-40 ms readout time for fMRI

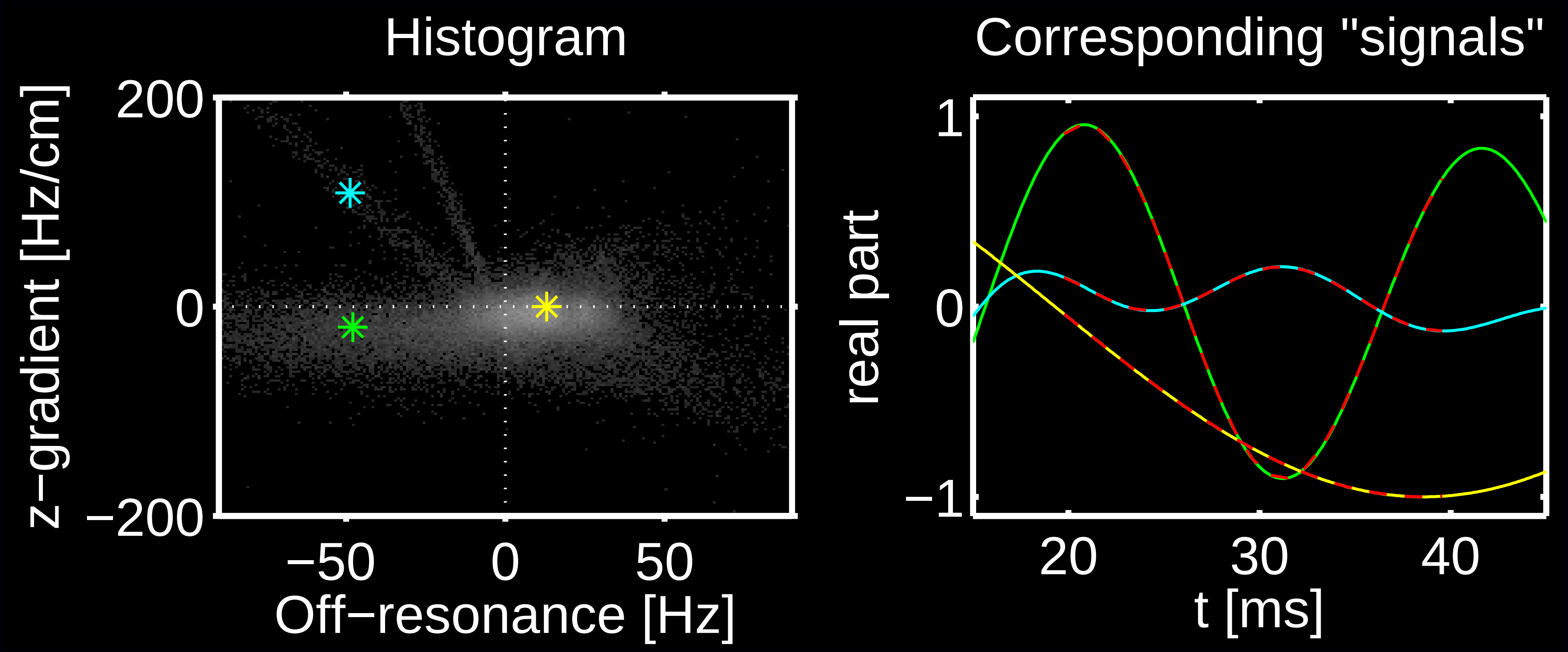
Resulting basis functions



(Weighted SVD design, uniform sampling. Energy computed from *C* for entire 3D volume.)

Signal approximation illustrated

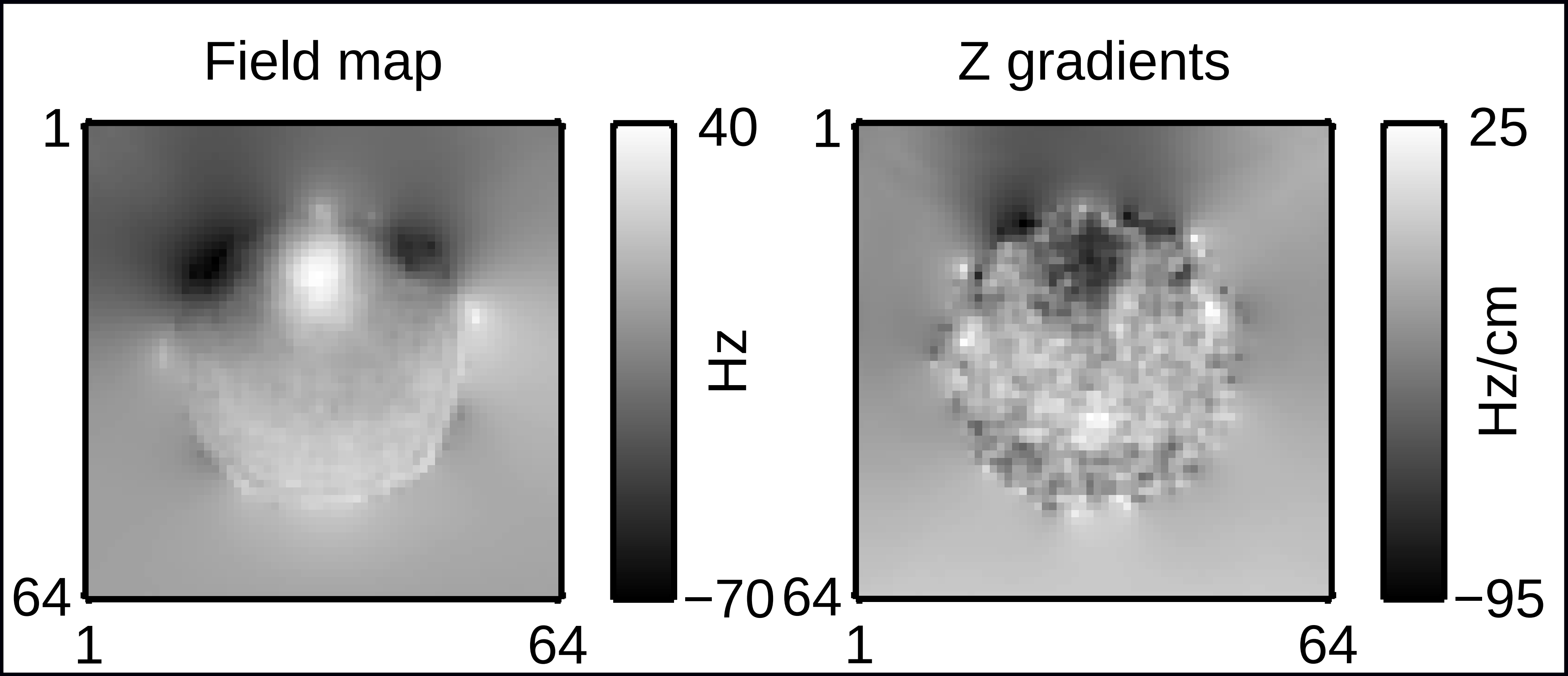
$$H(t_i g_j)\, e^{-\imath \omega_j t_i} \approx \sum_{l=1}^{L} b_{il} c_{lj}, \quad i.e., \quad \boldsymbol{H} \approx \boldsymbol{BC}$$



(Designed for 20 msec readout with $T_E = 30$ msec.)

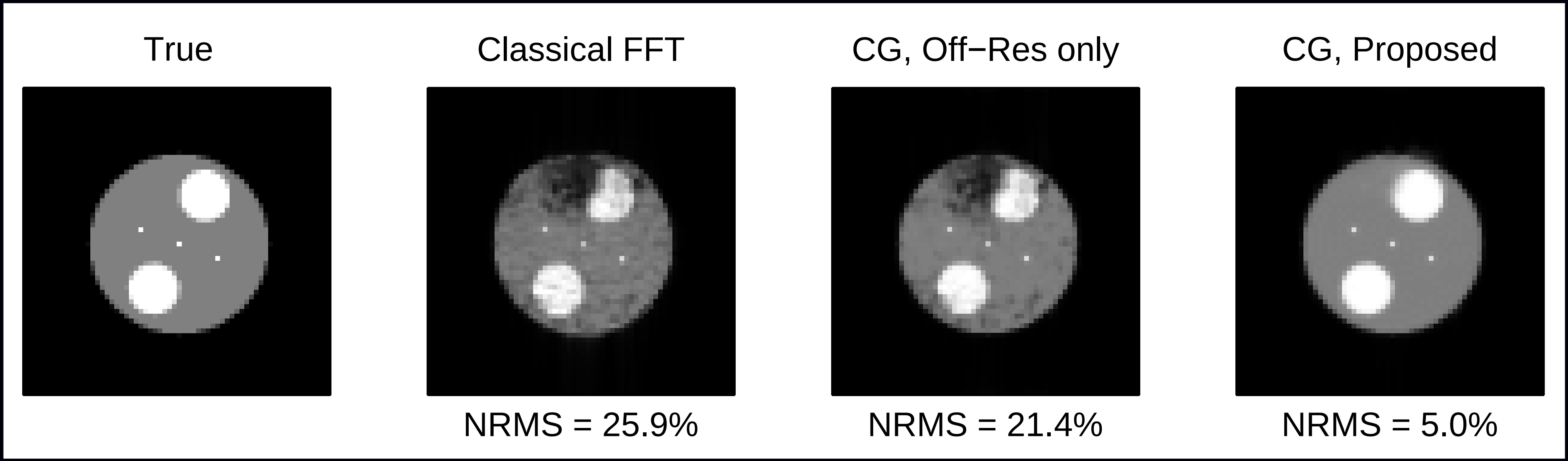
Simulation

- Simulation using (ω_j, g_j) maps shown below.
- EPI trajectory: 20 msec readout, $T_E = 30$ msec
- Data generated with exact signal model (noiseless)



Results

- Reconstructed by conventional inverse FFT.
  CPU time $\ll$ 1 sec
- Reconstructed by CG with $L = 5$, off-resonance model only.
  10 iteration CPU time = 2 sec
- Reconstructed by CG with $L = 5$, as proposed,
  10 iteration CPU time = 2 sec
- Regularizaton parameter β chosen so that FWHM $\approx$ 1.1 pixels



Summary

- Model-based approaches to MR reconstruction
- Account for physical effects such as off-resonance, through-plane susceptibility
- Improves image quality
- Fast algorithm compared to previous model-based approach
- Increased CPU time relative to classical inverse FFT

Future work

- In-plane field gradients
- Toeplitz approximation to $\boldsymbol{A}'\boldsymbol{A}$
- Real data
- Effects of errors in field map and its gradients
- Other k-space trajectories such as spiral-in