Fast variance image predictions for quadratically regularized statistical image reconstruction in fan-beam tomography

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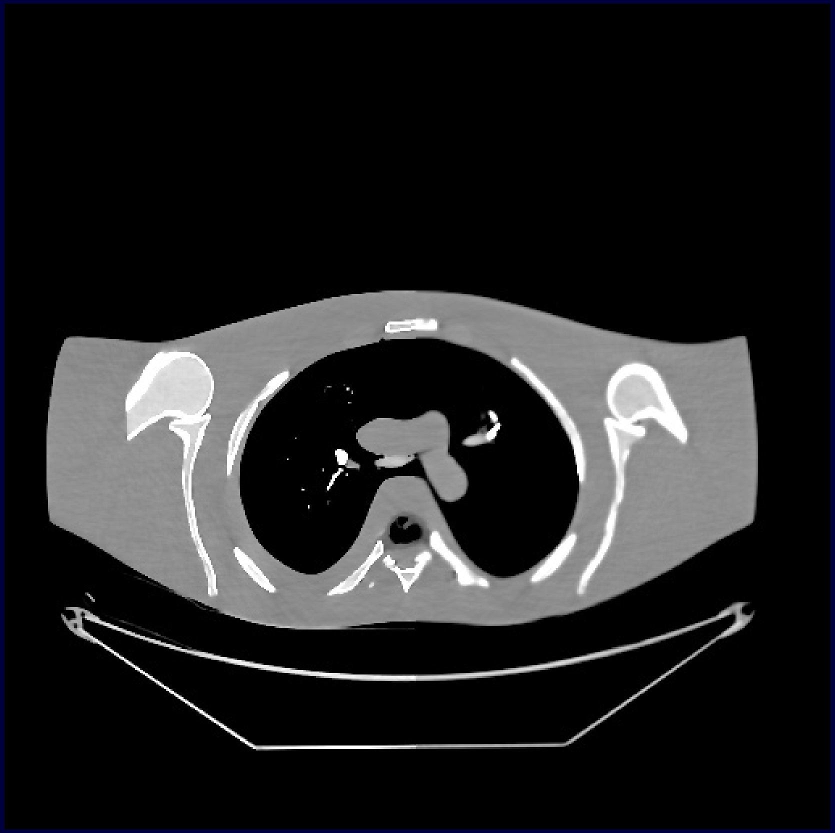
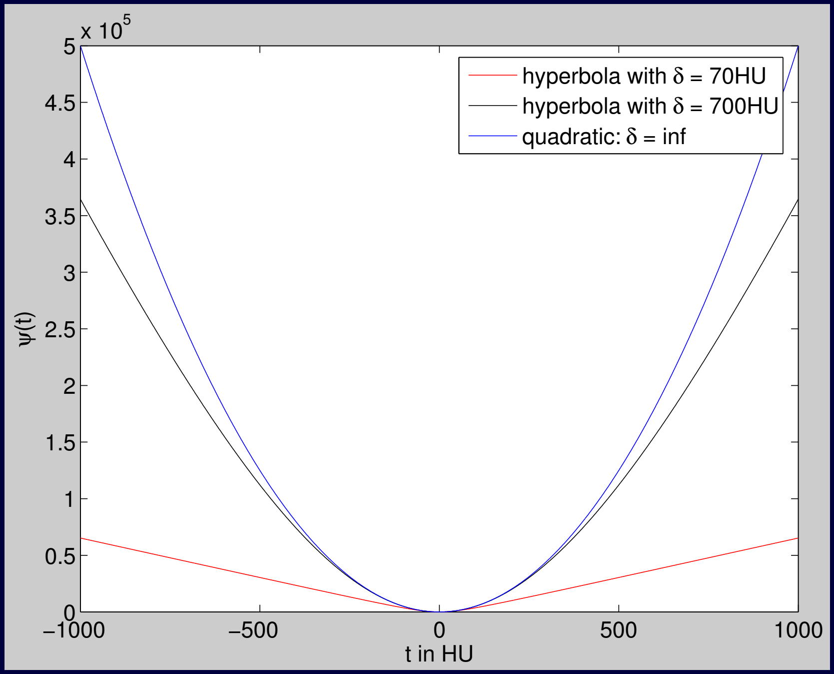
Statistical X-ray CT image reconstruction

$$\hat{\mu} = \arg\max_{\mu \geq 0} \Phi(\mathbf{y}, \mu) = \arg\max_{\mu \geq 0} \mathbf{L}(\mathbf{y}, \mu) - \beta \mathbf{R}(\mu), \quad \mathbf{R}(\mu) = \sum_{\mathbf{k}} \psi([\mathbf{C}\mu]_{\mathbf{k}}).$$

- Edge-preserving penalty functions, such as "hyperbola" penalty: $\psi(t) = \delta^2(\sqrt{1 + (t/\delta)^2} - 1)$.
- How to choose the regularization parameter δ? based on the noise level!
  - Too small δ: preserve noise!
  - Too large δ: smooth out the details!
- A statistical reconstruction example with same β but different δ values.





Covariance approximation: the matrix method

- For tomography, the measurements $\mathbf{y} = [y_1, \ldots, y_n]'$ have independent Poisson distributions.
- An accurate covariance approximation has been derived in (Fessler, IEEE T-IP, 1996) for penalized likelihood estimators.

$$\mathrm{Cov}\{\hat{\mu}\} \approx (\boldsymbol{A}'\boldsymbol{W}\boldsymbol{A} + \beta\boldsymbol{R})^{-1}\boldsymbol{A}'\boldsymbol{W}\boldsymbol{A}(\boldsymbol{A}'\boldsymbol{W}\boldsymbol{A} + \beta\boldsymbol{R})^{-1}, \quad (1)$$

- $\boldsymbol{A}$: the system matrix
- $\boldsymbol{W} = \mathrm{diag}(\bar{\mathbf{y}})$
- $\boldsymbol{R}$: the Hessian matrix of roughness penalty

Variance approximation: the FFT method

- The matrix method described in the previous slide has been used in various applications, (Qi 2001, Stayman 2004).

- Circulant approximation and FFTs are usually used in practical computation for shift-invariant imaging systems.

$$\mathsf{Var}\{\mu_j\} \approx \sum_k \frac{\mathcal{F}(\boldsymbol{A}'\boldsymbol{W}\boldsymbol{A}e_j)_k}{[\mathcal{F}(\boldsymbol{A}'\boldsymbol{W}\boldsymbol{A}e_j)_k + \mathcal{F}(\boldsymbol{R}e_j)_k]^2}, \qquad (2)$$

  where $\mathcal{F}$ is a Fourier Transform and e_j is the jth unit vector.

- Convenient for evaluating the variance at a few image locations of interest.

Drawbacks of the FFT method

- The FFT method provides accurate variance/standard deviation prediction at some image location interested.
- The computation of this FFT approximation is expensive for realistic image size when the variance must be computed for all pixels, particularly for shift-variant systems like fan-beam tomography.
- It needs one FFT for each pixel.
- Goal: faster variance approximation without losing accuracy.

Continuous-space covariance approximation

- Go back to continuous space from discrete space! With the same philosophy in (Fessler,1996), one can derive the continuous-space covariance operator $\mathcal{K}_{\hat{\mu}}$,

$$\mathcal{K}_{\hat{\mu}} = \mathrm{Cov}\{\hat{\mu}\} \approx (\mathcal{A}^*\mathcal{W}\mathcal{A} + \mathcal{R})^{-1}\mathcal{A}^*\mathcal{W}\mathcal{A}(\mathcal{A}^*\mathcal{W}\mathcal{A} + \mathcal{R})^{-1},$$

- $\mathcal{A}$: the projection operator
- $\mathcal{W}$: the fan-beam weighting operator, $(\mathcal{W}p)(s,\beta) = w(s,\beta)p(s,\beta)$
- $\mathcal{R}$: the regularization operator

Fourier covariance approximation

- Consider an impulse object $\delta_j(x,y) = \delta(x - x_j, y - y_j)$. Using local Fourier-domain analysis, the local covariance operator can be expressed as

$$\mathcal{K}_{\hat{u}} = \mathcal{F}^{-1}\left(\frac{H_j(\rho,\Phi)}{[H_j(\rho,\Phi) + R_j(\rho,\Phi)]^2}\right)\mathcal{F}, \qquad (3)$$

  with respect to some image location (x_j, y_j).

- $\mathcal{A}^*\mathcal{W}\mathcal{A}$: the Gram operator

- $\mathcal{F}$: the Fourier operator

- $H_j(\rho,\Phi)$: the local frequency response of the Gram operator $\mathcal{A}^*\mathcal{W}\mathcal{A}\delta_j$

- $R_j(\rho,\Phi)$: the local frequency response of $\mathcal{R}\delta_j$

Continuous-space variance approximation

- The variance at location (x_j, y_j) can then be expressed as an integral in the frequency domain,

$$\mathsf{Var}\{\hat{\mu}_j\} = \int_0^{2\pi}\int_0^{\infty} \frac{H_j(\rho,\Phi)}{[H_j(\rho,\Phi)+R(\rho,\Phi)]^2}\rho\, d\rho\, d\Phi .$$

- The local frequency response of the Gram operator can be found by taking local Fourier transform of $\mathcal{A}^*\mathcal{W}\mathcal{A}\delta_j$:

$$H_j(\rho,\Phi) \triangleq H(\rho,\Phi;x_j,y_j) = \frac{1}{|\rho|} w_j(\Phi).$$

- $w_j(\varphi) \triangleq w(\varphi;x_j,y_j) = w(s',\beta')J(s')\Big|_{\varphi'=\varphi} + w(s',\beta')J(s')\Big|_{\varphi'=\varphi-\pi}$: the fan-beam angular dependent weighting function
- $w(s',\beta')$: the data statistics
- $J(s)$: the determinant of the Jacobian matrix of transforming from the fan-beam coordinates to parallel-beam coordinates

Fourier domain variance integral

- Using "local Fourier analysis", the variance of $\hat{\mu}_j$ at location (x_j, y_j) can be approximated analytically as

$$\mathsf{Var}\{\hat{\mu}_j\} \approx \int_0^{2\pi} \int_0^{\infty} \frac{w_j(\Phi)/|\rho|}{(w_j(\Phi)/|\rho| + \beta R_j(\rho, \Phi))^2} \rho \, d\rho \, d\Phi,$$

- The parallel-beam geometry is just a special case with the angular weighting function only consisting of the data statistics.
- Discretize this integral and evaluate it for a variance map!

Quadratic $R(\rho,\Phi)$ is approximately separable

- Consider quadratic penalty, whose $R(\rho,\Phi)$ is approximately separable in ρ and Φ,
$$R_j(\rho,\Phi) \approx (2\pi\rho)^2 \tilde{R}_j(\Phi).$$

- The variance approximation on previous slide becomes

$$\begin{aligned} \mathrm{Var}\{\hat{\mu}_j\} &\approx \int_0^{2\pi}\int_0^{\rho_{\max}} \frac{\frac{w_j(\Phi)}{|\rho|}}{\left(\frac{w_j(\Phi)}{|\rho|} + \beta(2\pi\rho)^2\tilde{R}_j(\Phi)\right)^2}\rho\,d\rho \qquad (4) \\ &= \frac{\rho_{\max}^3}{3}\int_0^{2\pi}\frac{1}{\left[w_j(\Phi)+\beta 4\pi^2\rho_{\max}^3\tilde{R}_j(\Phi)\right]^2}\,d\Phi, \end{aligned}$$

for a quadratic penalty function.

Computation of analytical variance estimation

- The computation of $w_j(\Phi)$ for all pixels only requires the same computation time as one backprojection.
- The variance prediction integral can be evaluated by a finite summation with correctly chosen $\rho_{\max}$.
- The analytical prediction requires much less computation than the FFT method and thus is practical for realistic tomography image size.

Example: standard quadratic penalty

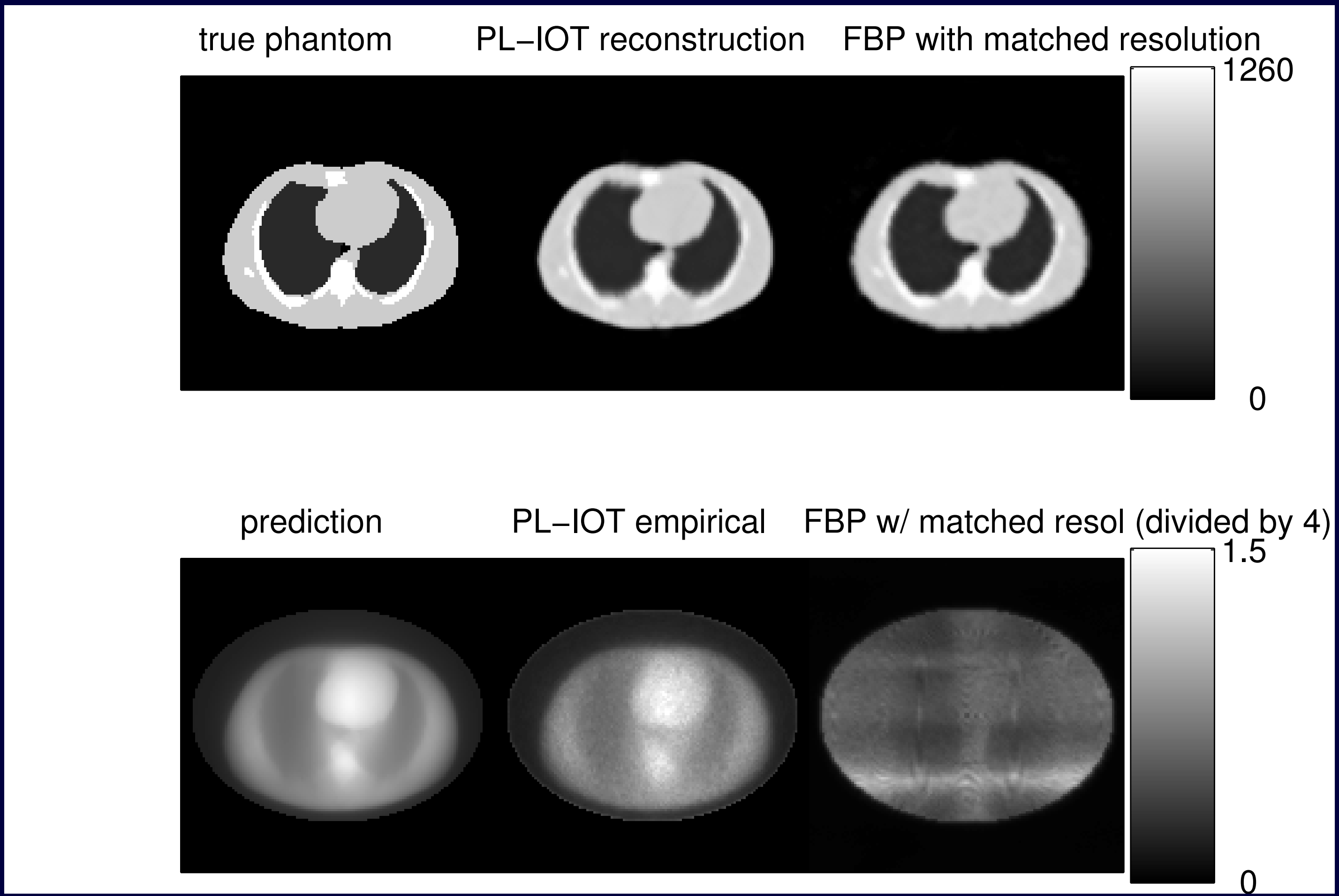
- Consider a standard quadratic penalty s.t. $\tilde{R}_j(\Phi) = \tilde{R}_j$ is independent of Φ. R_j is chosen to match the resolution of PULS (penalized unweighted least square) reconstruction with the same β.

- The variance approximation in this case is of a very simple form:

$$\mathrm{Var}\{\hat{\mu}_j\} \approx \frac{\rho_{\max}^3}{3} \int_0^{2\pi} \frac{1}{\left[w_j(\Phi) + \beta 4\pi^2 \rho_{\max}^3 \tilde{R}_j\right]^2} \mathrm{d}\Phi \,. \tag{5}$$

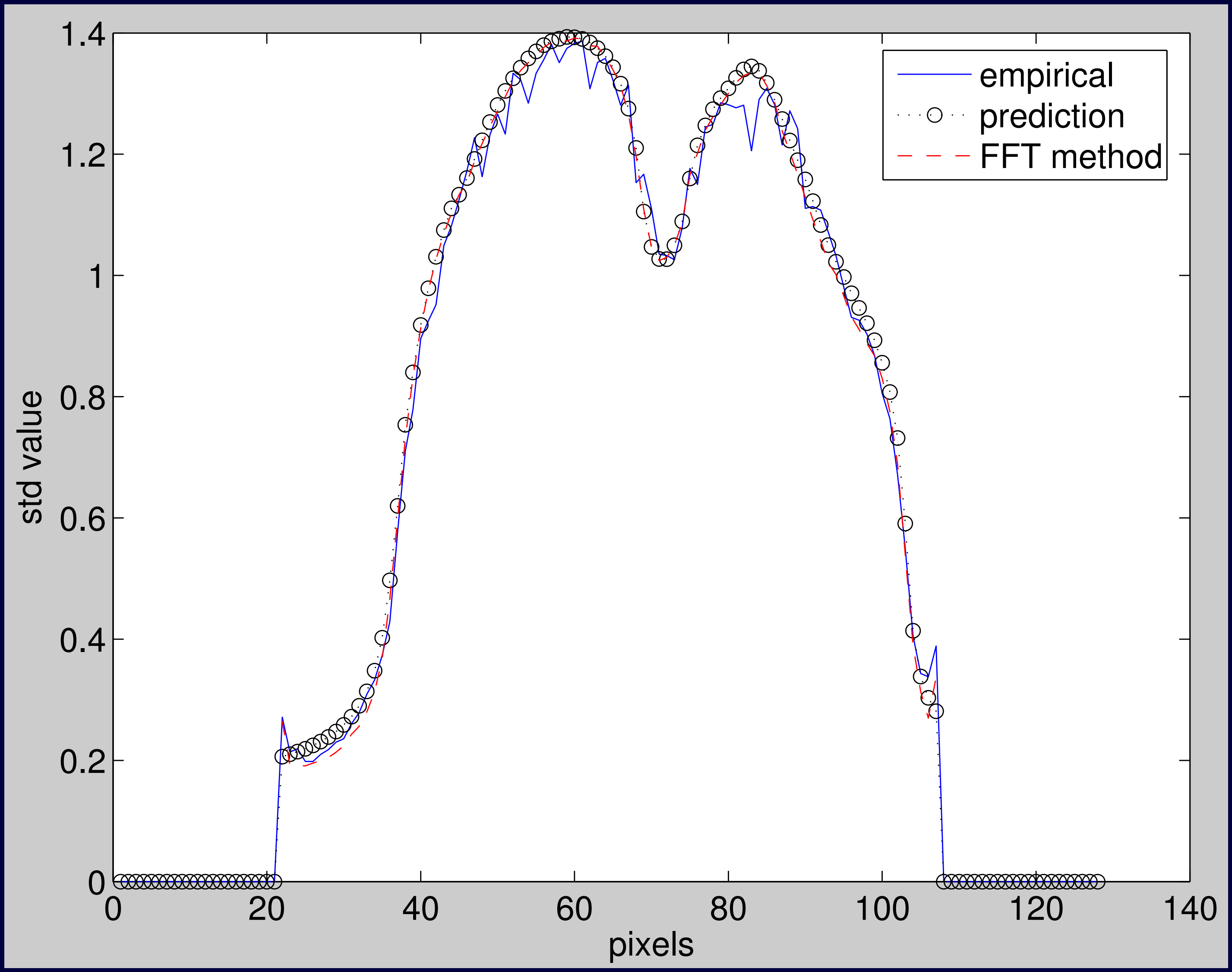
QPL reconstruction simulation

- 3rd-generation GE CT scanner.
- 128x128 Zubal phantom, 400 iterations of PL-IOT (incremental optimization transfer algorithm, Ahn 2004), 450 realizations.
- FBP and PL-IOT reconstruction($\beta = 2^{12}$) have matched resolution: FWHM = 1.76 pixels i.e., 6.0mm



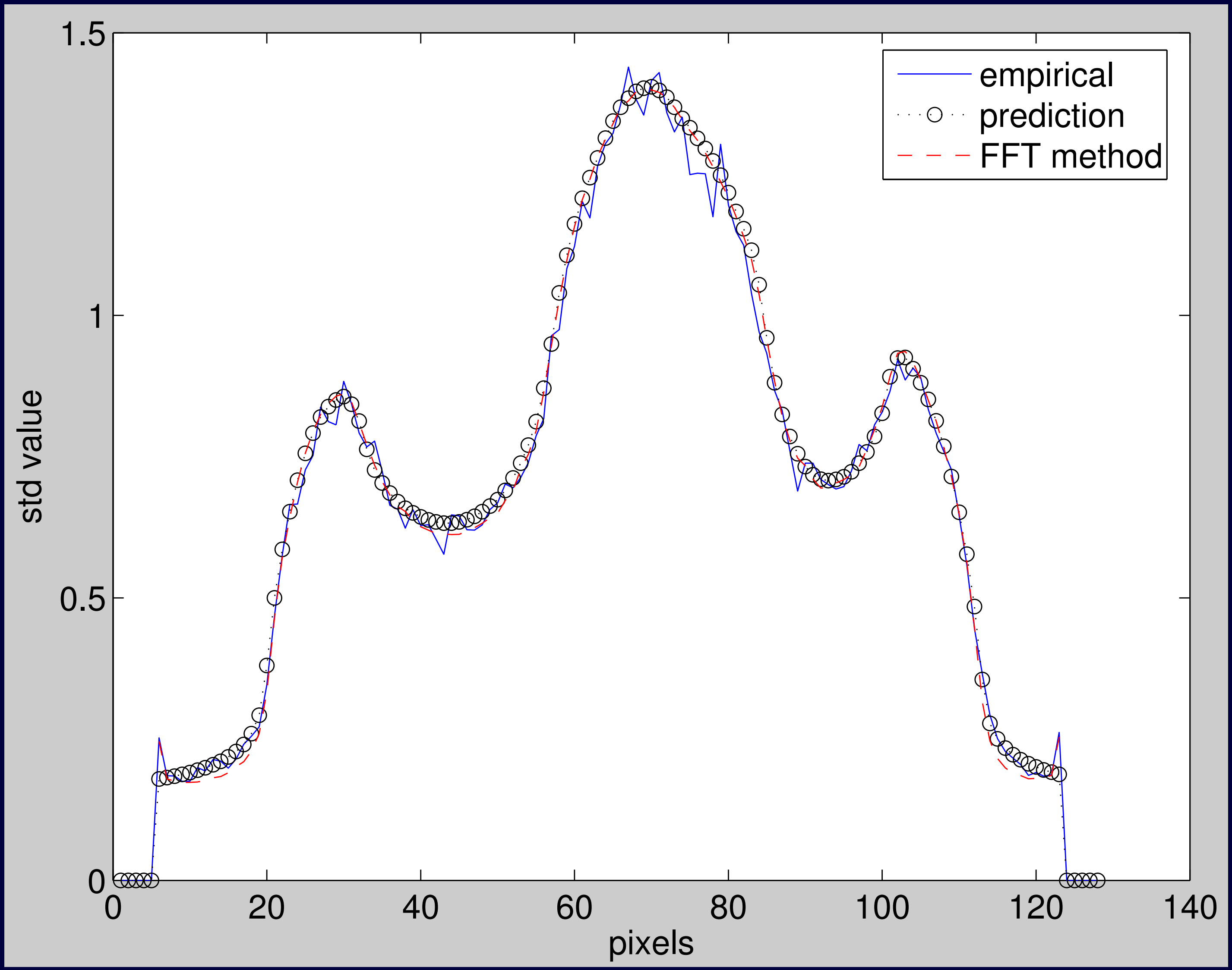
Standard deviation image prediction results

- Vertical Profiles



Standard deviation image prediction results

- Horizontal profiles



Future work

- Evaluate the performance of the proposed method on the modified quadratic penalty which leads to nearly uniform and isotropic spatial resolution (Shi, 2005).
- Investigate how to apply this prediction in choosing the regularization parameter, possibly a locally-varied δ in edge-preserving regularization.
- Investigate how well the proposed method can perform in covariance matrix prediction.
- Generalize the method to 3D cone beam CT.