# Iterative Reconstruction in MRI Using Iterative Methods

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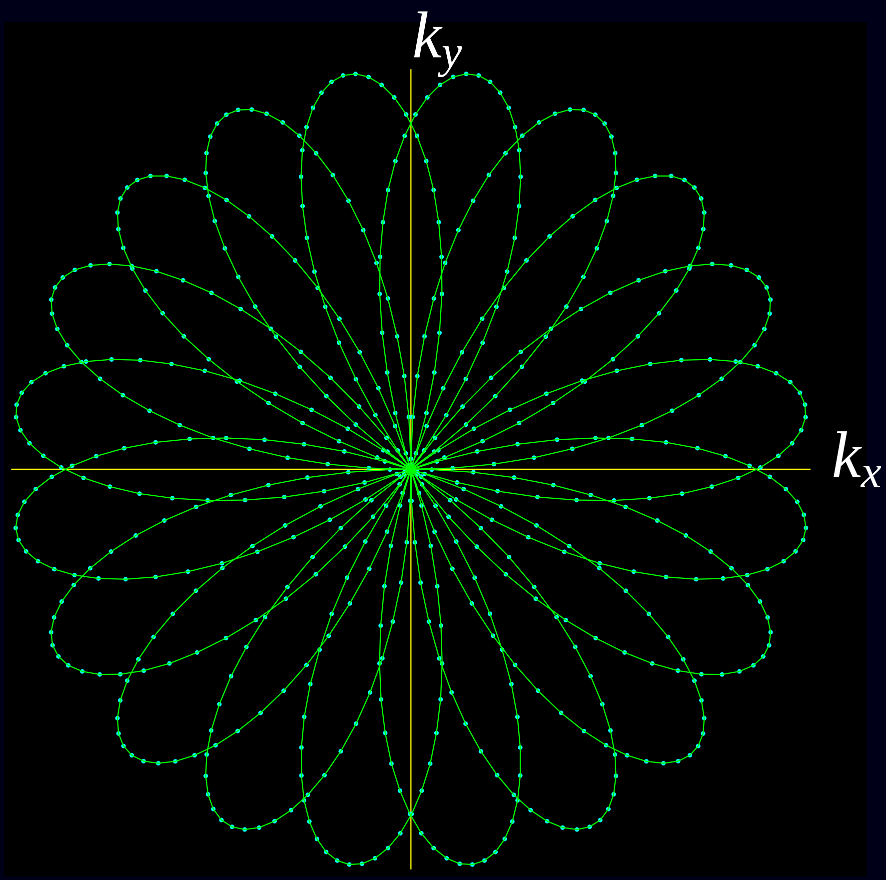
# Outline

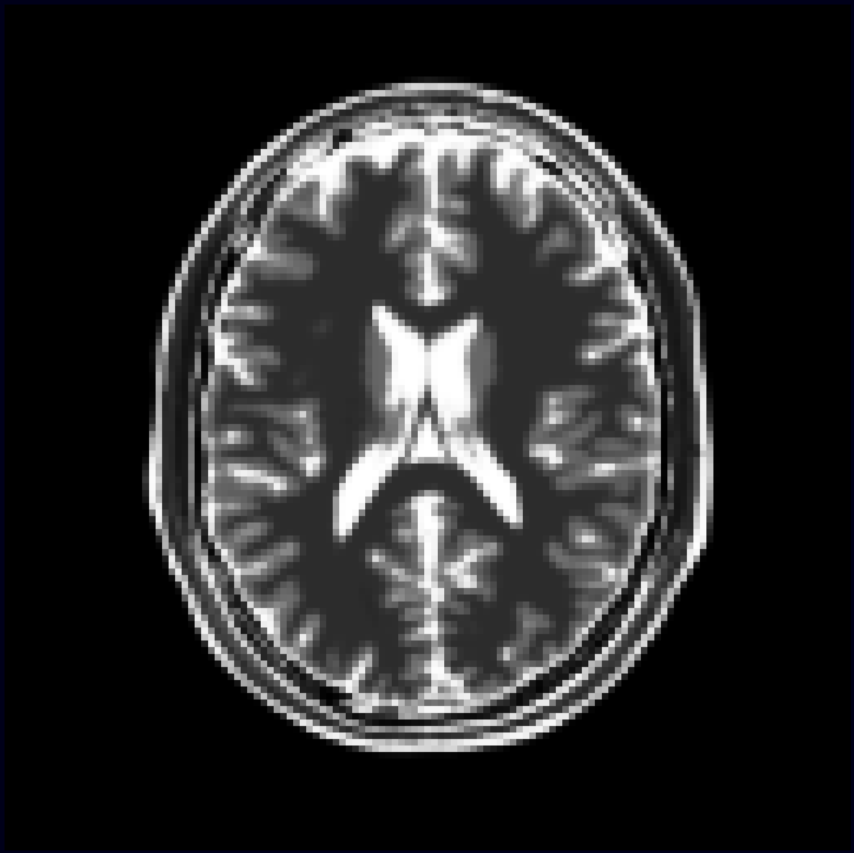
- MR image reconstruction
- Model-based reconstruction
- Iterations and computation (NUFFT etc.)
- New regularization approach

# MR Image Reconstruction

“k-space” image




$\Longrightarrow$



# Textbook MRI Measurement Model

Ignoring *lots* of things:

$$y_i = s(t_i) + \text{noise}_i, \qquad i = 1, \ldots, N$$

$$s(t) = \int f(\vec{r})\, e^{-\imath 2\pi \vec{k}(t) \cdot \vec{r}}\, d\vec{r},$$

where $\vec{r}$ denotes spatial position, and
$\vec{k}(t)$ denotes the "k-space trajectory" of the MR pulse sequence, determined by user-controllable magnetic field gradients.

$e^{-\imath 2\pi \vec{k}(t) \cdot \vec{r}}$ provides spatial information $\Longrightarrow$ Nobel Prize

- MRI measurements are (roughly) samples of the Fourier transform of the object's transverse magnetization $f(\vec{r})$.
- Basic image reconstruction problem:
  recover $f(\vec{r})$ from measurements $\{y_i\}_{i=1}^{N}$.

Inherently under-determined (ill posed) problem
$\Longrightarrow$ no canonical solution.

# Image Reconstruction Strategies

The unknown object $f(\vec{r})$ is a continuous-space function,
but the recorded measurements $\boldsymbol{y} = (y_1, \ldots, y_N)$ are finite.

Options?

- Continuous-discrete formulation using many-to-one linear model:

$$\boldsymbol{y} = \mathcal{A} f + \boldsymbol{\varepsilon}.$$

  Minimum norm solution (*cf.* "natural pixels"):

$$\min_{\hat{f}} \left\| \hat{f} \right\| \text{ subject to } \boldsymbol{y} = \mathcal{A} \hat{f}$$

$$\hat{f} = \mathcal{A}^* (\mathcal{A}\mathcal{A}^*)^{-1} \boldsymbol{y} = \sum_{i=1}^N c_i \, \mathrm{e}^{-\imath 2\pi \vec{k}(t) \cdot \vec{r}}, \text{ where } \mathcal{A}\mathcal{A}^* \boldsymbol{c} = \boldsymbol{y}.$$

- Discrete-discrete formulation
  Assume parametric model for object:

$$f(\vec{r}) = \sum_{j=1}^M f_j \, b_j(\vec{r}) \, .$$

- Continuous-continuous formulation
  Pretend that a continuum of measurements are available:

$$F(\vec{k}) = \int f(\vec{r})\, e^{-\imath 2\pi \vec{k}\cdot\vec{r}}\, d\vec{r},$$

  vs samples $y_i = F(\vec{k}_i) + \varepsilon_i$.
  The "solution" is an inverse Fourier transform:

$$f(\vec{r}) = \int F(\vec{k})\, e^{\imath 2\pi \vec{k}\cdot\vec{r}}\, d\vec{k}.$$

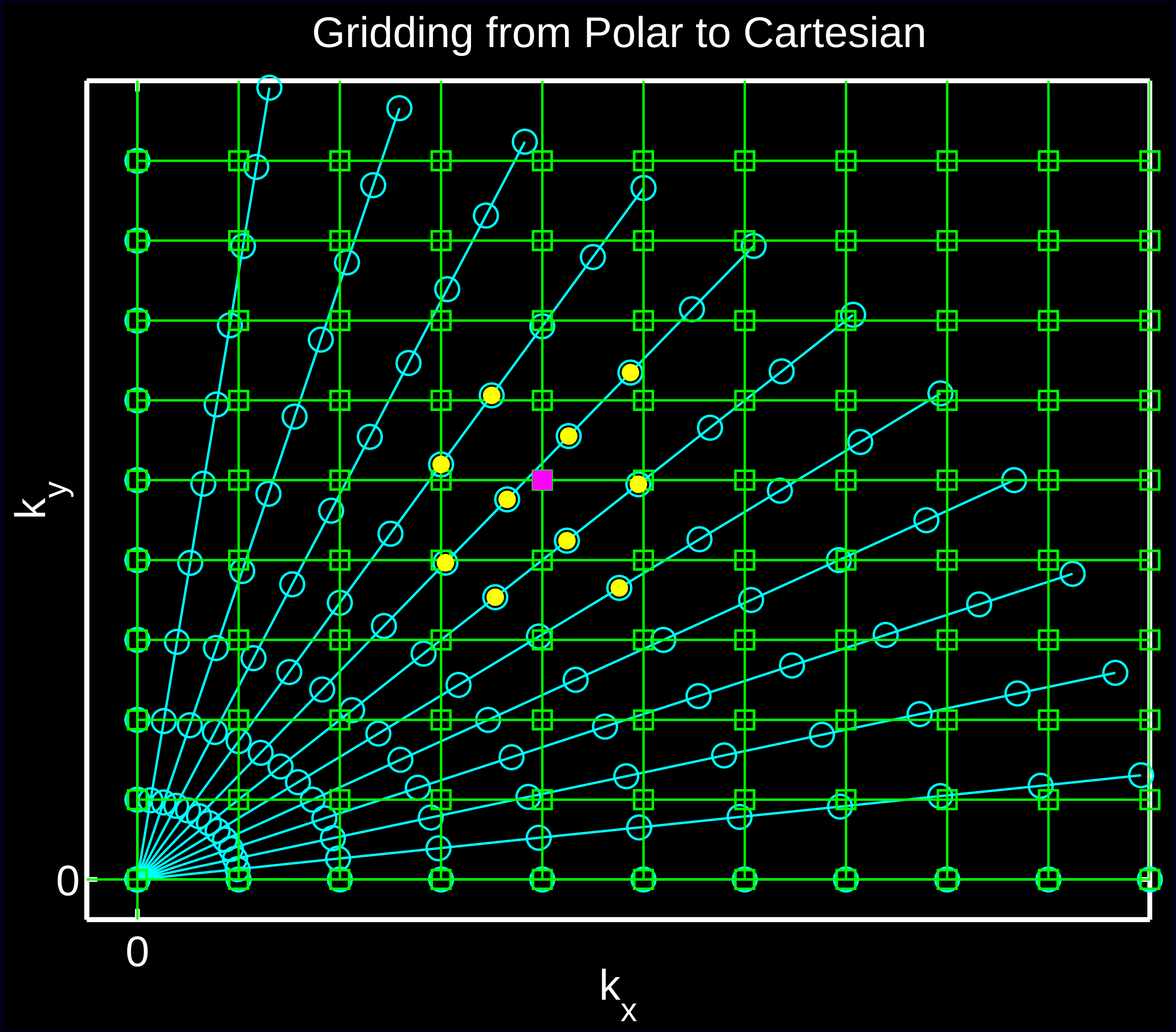
  Now discretize the integral solution:

$$\hat{f}(\vec{r}) = \sum_{i=1}^{N} F(\vec{k}_i)\, e^{\imath 2\pi \vec{k}\cdot\vec{r}}\, w_i,$$

  where $w_i$ values are "sampling density compensation factors."
  Numerous methods for choosing $w_i$ value in the literature.

# Conventional MR Image Reconstruction

1. Interpolate measurements onto rectilinear grid ("gridding")
2. Apply inverse FFT to estimate samples of $f(\vec{r})$



# Limitations of Gridding Reconstruction

1. Artifacts/inaccuracies due to interpolation
2. Contention about sample density "weighting"
3. Oversimplifications of Fourier transform signal model:
   - Magnetic field inhomogeneity
   - Magnetization decay ($T_2$)
   - Eddy currents
   - ...
4. Sensitivity encoding ?
5. ...

# Model-Based Image Reconstruction

MR signal equation with more complete physics:

$$s(t) = \int f(\vec{r})\, s_{\text{coil}}(\vec{r})\, e^{-\imath\,\omega(\vec{r})\,t}\, e^{-R_2^*(\vec{r})\,t}\, e^{-\imath 2\pi \vec{k}(t)\cdot\vec{r}}\, d\vec{r}$$

$$y_i = s(t_i) + \text{noise}_i, \qquad i = 1, \ldots, N$$

- $s_{\text{coil}}(\vec{r})$ Receive-coil sensitivity pattern(s) (for SENSE)
- $\omega(\vec{r})$ Off-resonance frequency map
  (due to field inhomogeneity / magnetic susceptibility)
- $R_2^*(\vec{r})$ Relaxation map

Other factors (?)

- Eddy current effects; in $\vec{k}(t)$
- Concomitant gradient terms
- Chemical shift
- Motion

Goal?

(it depends)

# Field Inhomogeneity-Corrected Reconstruction

$$s(t) = \int f(\vec{r})\, s_{\text{coil}}(\vec{r})\, e^{-\imath \omega(\vec{r}) t}\, e^{-R_2^*(\vec{r})\, t}\, e^{-\imath 2\pi \vec{k}(t) \cdot \vec{r}}\, d\vec{r}$$

Goal: reconstruct $f(\vec{r})$ given field map $\omega(\vec{r})$.
(Assume all other terms are known or unimportant.)

## Motivation

Essential for functional MRI of brain regions near sinus cavities!

(Sutton *et al.*, ISMRM 2001; T-MI 2003)

# Sensitivity-Encoded (SENSE) Reconstruction

$$s(t) = \int f(\vec{r}) s_{\text{coil}}(\vec{r}) \, e^{-\imath \omega(\vec{r}) t} \, e^{-R_2^*(\vec{r}) t} \, e^{-\imath 2\pi \vec{k}(t) \cdot \vec{r}} \, d\vec{r}$$

Goal: reconstruct $f(\vec{r})$ given sensitivity maps $s_{\text{coil}}(\vec{r})$.
(Assume all other terms are known or unimportant.)

Can combine SENSE with field inhomogeneity correction “easily.”

(Sutton *et al.*, ISMRM 2001)

# Joint Estimation of Image and Field-Map

$$s(t) = \int f(\vec{r})\, s_{\text{coil}}(\vec{r})\, \mathrm{e}^{-\imath\omega(\vec{r})t}\, \mathrm{e}^{-R_2^*(\vec{r})\,t}\, \mathrm{e}^{-\imath 2\pi \vec{k}(t)\cdot\vec{r}}\, \mathrm{d}\vec{r}$$

Goal: estimate *both* the image $f(\vec{r})$ and the field map $\omega(\vec{r})$
(Assume all other terms are known or unimportant.)

Analogy:
joint estimation of emission image and attenuation map in PET.

(Sutton *et al.*, ISMRM Workshop, 2001; ISBI 2002; ISMRM 2002; ISMRM 2003; MRM 2004)

# The Kitchen Sink

$$s(t) = \int f(\vec{r})\, s_{\text{coil}}(\vec{r})\, \mathrm{e}^{-\imath\omega(\vec{r})t}\, \mathrm{e}^{-R_2^*(\vec{r})\,t}\, \mathrm{e}^{-\imath 2\pi \vec{k}(t)\cdot\vec{r}}\, \mathrm{d}\vec{r}$$

Goal: estimate image $f(\vec{r})$, field map $\omega(\vec{r})$, *and* relaxation map $R_2^*(\vec{r})$

Requires "suitable" k-space trajectory.

(Sutton *et al.*, ISMRM 2002; Twieg, MRM, 2003)

# Estimation of Dynamic Maps

$$s(t) = \int f(\vec{r})\, s_{\text{coil}}(\vec{r})\, \mathrm{e}^{-\imath \omega(\vec{r}) t}\, \mathrm{e}^{-R_2^*(\vec{r}) t}\, \mathrm{e}^{-\imath 2\pi \vec{k}(t) \cdot \vec{r}}\, \mathrm{d}\vec{r}$$

Goal: estimate dynamic field map $\omega(\vec{r})$ and “BOLD effect” $R_2^*(\vec{r})$ given baseline image $f(\vec{r})$ in fMRI.

Motion...

# Back to Basic Signal Model

$$s(t) = \int f(\vec{r})\, e^{-\imath 2\pi \vec{k}(t) \cdot \vec{r}}\, d\vec{r}$$

Goal: reconstruct $f(\vec{r})$ from $\boldsymbol{y} = (y_1, \ldots, y_N)$, where $y_i = s(t_i) + \varepsilon_i$.

Series expansion of unknown object:

$$f(\vec{r}) \approx \sum_{j=1}^{M} f_j\, b(\vec{r} - \vec{r}_j) \longleftarrow \text{usually 2D rect functions.}$$

$$y_i \approx \int \left[ \sum_{j=1}^{M} f_j\, b(\vec{r} - \vec{r}_j) \right] e^{-\imath 2\pi \vec{k}(t_i) \cdot \vec{r}}\, d\vec{r} = \sum_{j=1}^{M} \left[ \int b(\vec{r} - \vec{r}_j)\, e^{-\imath 2\pi \vec{k}(t_i) \cdot \vec{r}}\, d\vec{r} \right] f_j$$

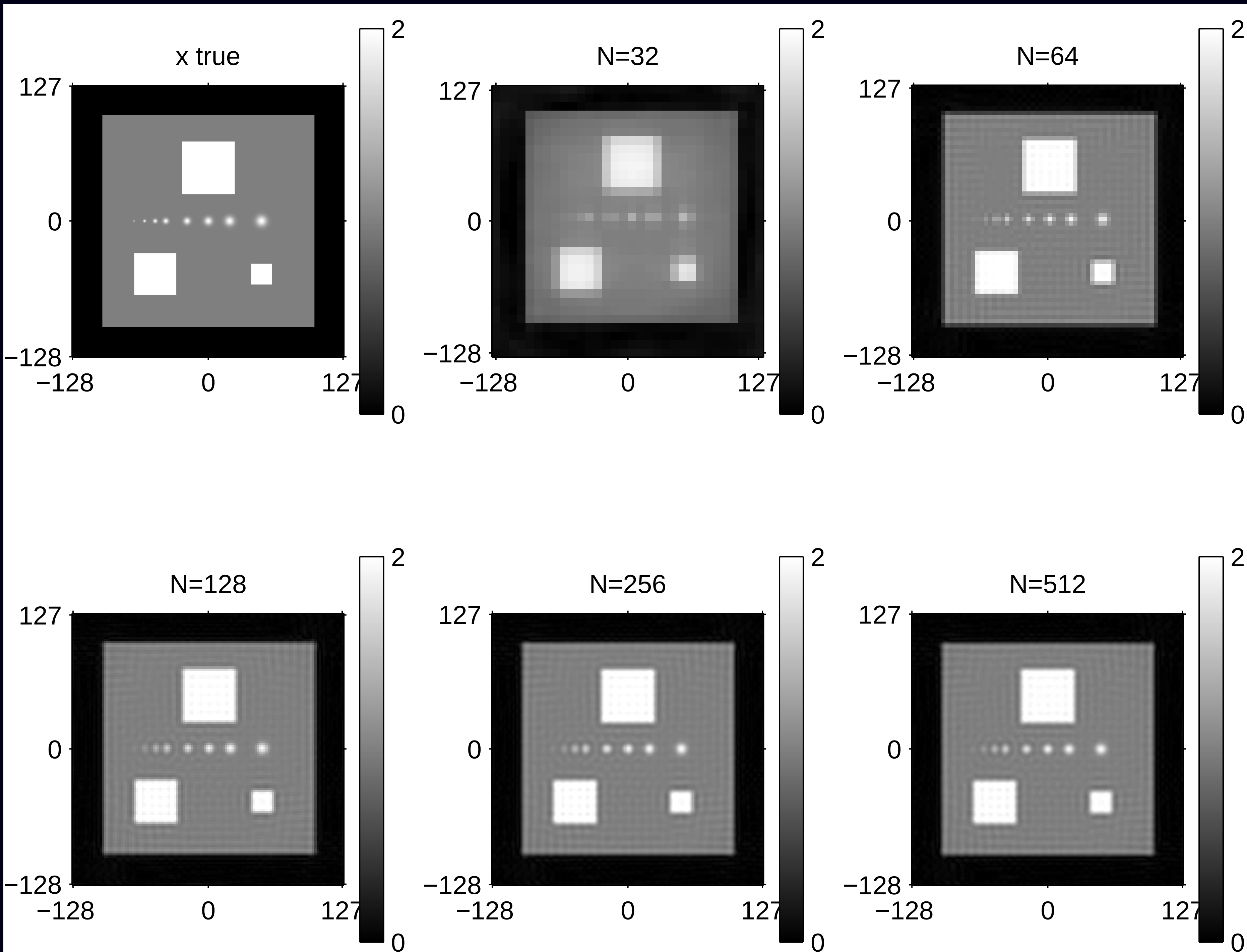
$$= \sum_{j=1}^{M} a_{ij} f_j, \qquad a_{ij} = B(\vec{k}(t_i))\, e^{-\imath 2\pi \vec{k}(t_i) \cdot \vec{r}_j}, \qquad b(\vec{r}) \overset{\text{FT}}{\Longleftrightarrow} B(\vec{k}).$$

Discrete-discrete measurement model with system matrix $\boldsymbol{A} = \{a_{ij}\}$:

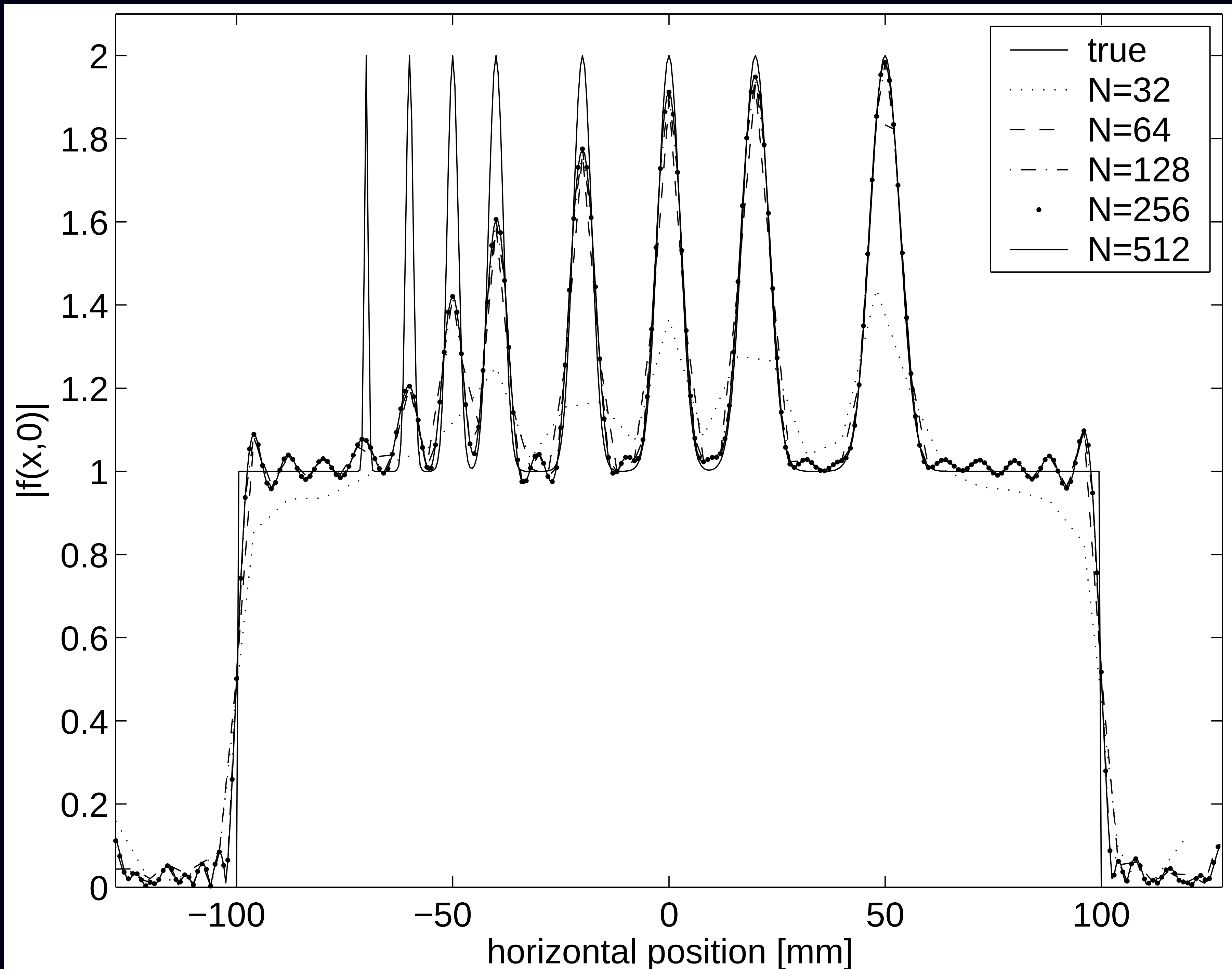
$$\boldsymbol{y} = \boldsymbol{A} \boldsymbol{f} + \boldsymbol{\varepsilon}.$$

Goal: estimate coefficients (pixel values) $\boldsymbol{f} = (f_1, \ldots, f_M)$ from $\boldsymbol{y}$.

# Small Pixel Size Does Not Matter



# Profiles



true
N=32
N=64
N=128
N=256
N=512
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
|f(x,0)|
−100
−50
0
50
100
horizontal position [mm]

# Regularized Least-Squares Estimation

$$\hat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f} \in \mathbb{C}^M} \Psi(\boldsymbol{f}), \qquad \Psi(\boldsymbol{f}) = \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{f}\|^2 + \alpha R(\boldsymbol{f})$$

- data fit term $\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{f}\|^2$
  corresponds to negative log-likelihood of Gaussian distribution
- regularizing roughness penalty term $R(\boldsymbol{f})$ controls noise

$$R(\boldsymbol{f}) \approx \int \|\nabla f\|^2 \, \mathrm{d}\vec{r}$$

- regularization parameter $\alpha > 0$
  controls tradeoff between spatial resolution and noise
  (Fessler & Rogers, IEEE T-IP, 1996)
- Equivalent to Bayesian MAP estimation with prior $\propto \mathrm{e}^{-\alpha R(\boldsymbol{f})}$

Quadratic regularization $R(\boldsymbol{f}) = \|\boldsymbol{C}\boldsymbol{f}\|^2$ leads to closed-form solution:

$$\hat{\boldsymbol{f}} = \left[\boldsymbol{A}'\boldsymbol{A} + \alpha \boldsymbol{C}'\boldsymbol{C}\right]^{-1} \boldsymbol{A}'\boldsymbol{y}$$

(a formula of limited practical use)

# Iterative Minimization by Conjugate Gradients

Choose initial guess $\boldsymbol{f}^{(0)}$ (*e.g.*, fast conjugate phase / gridding).
Iteration (unregularized):

$$\boldsymbol{g}^{(n)} = \nabla\Psi\left(\boldsymbol{f}^{(n)}\right) = \boldsymbol{A}'(\boldsymbol{A}\boldsymbol{f}^{(n)} - \boldsymbol{y}) \quad \text{gradient}$$

$$\boldsymbol{p}^{(n)} = \boldsymbol{P}\boldsymbol{g}^{(n)} \quad \text{precondition}$$

$$\gamma_n = \begin{cases} 0, & n = 0 \\ \dfrac{\langle \boldsymbol{g}^{(n)}, \boldsymbol{p}^{(n)} \rangle}{\langle \boldsymbol{g}^{(n-1)}, \boldsymbol{p}^{(n-1)} \rangle}, & n > 0 \end{cases}$$

$$\boldsymbol{d}^{(n)} = -\boldsymbol{p}^{(n)} + \gamma_n \boldsymbol{d}^{(n-1)} \quad \text{search direction}$$

$$\boldsymbol{v}^{(n)} = \boldsymbol{A}\boldsymbol{d}^{(n)}$$

$$\alpha_n = \langle \boldsymbol{d}^{(n)}, -\boldsymbol{g}^{(n)} \rangle / \langle \boldsymbol{A}\boldsymbol{f}^{(n)}, \boldsymbol{A}\boldsymbol{f}^{(n)} \rangle \quad \text{step size}$$

$$\boldsymbol{f}^{(n+1)} = \boldsymbol{f}^{(n)} + \alpha_n \boldsymbol{d}^{(n)} \quad \text{update}$$

Bottlenecks: computing $\boldsymbol{A}\boldsymbol{f}$ and $\boldsymbol{A}'\boldsymbol{y}$.

- $\boldsymbol{A}$ is too large to store explicitly (not sparse)
- Even if $\boldsymbol{A}$ were stored, directly computing $\boldsymbol{A}\boldsymbol{f}$ is $O(NM)$ per iteration, whereas FFT is only $O(N \log N)$.

# Computing $Af$ Rapidly

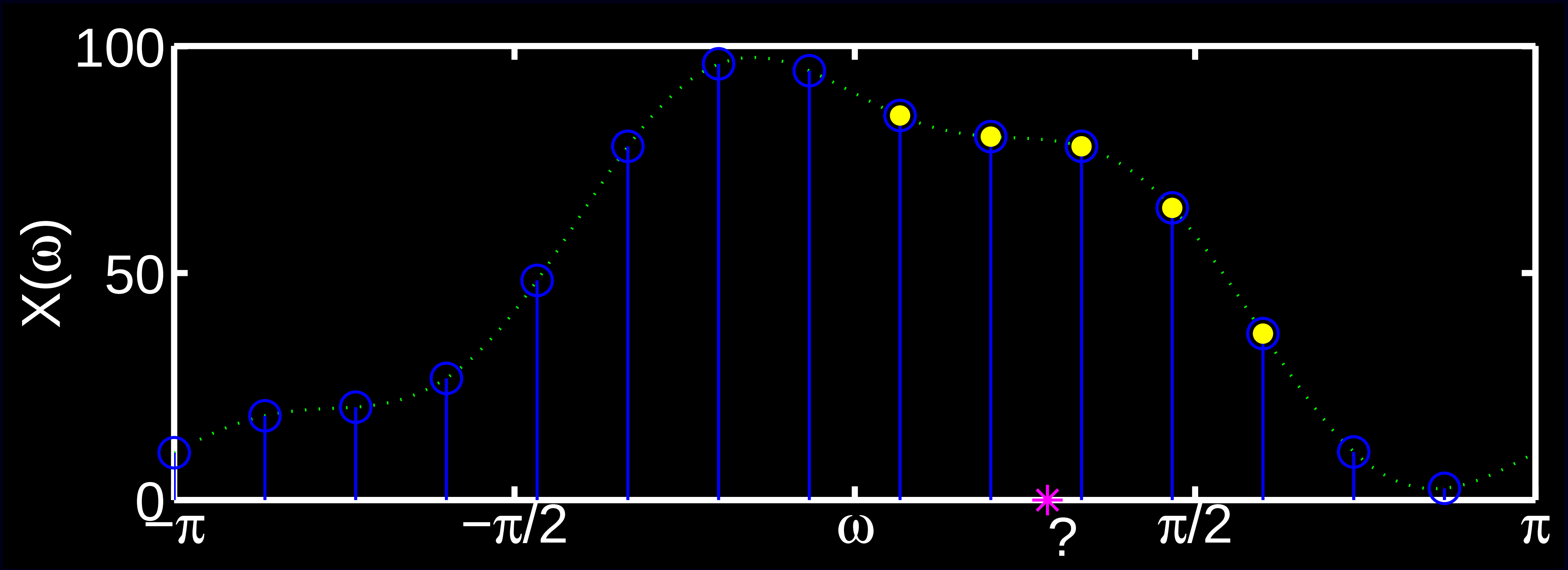
$$[\boldsymbol{A}\boldsymbol{f}]_i = \sum_{j=1}^{M} a_{ij} f_j = B(\vec{k}(t_i)) \sum_{j=1}^{M} \mathrm{e}^{-\imath 2\pi \vec{k}(t_i) \cdot \vec{r}_j} f_j, \qquad i = 1, \ldots, N$$

- Pixel locations $\{\vec{r}_j\}$ are uniformly spaced
- k-space locations $\left\{\vec{k}(t_i)\right\}$ are unequally spaced

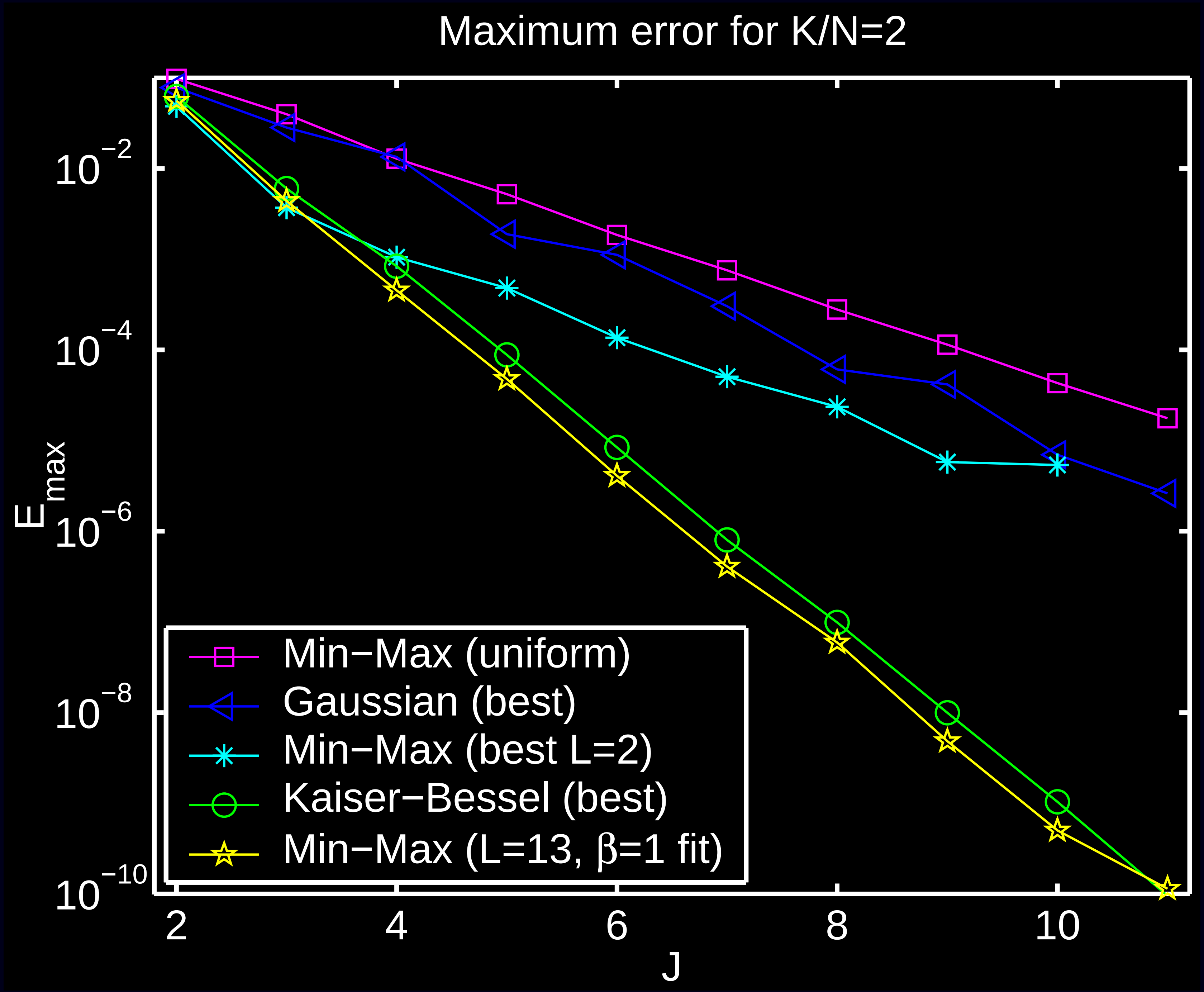
$\Longrightarrow$ needs nonuniform fast Fourier transform (NUFFT)

# NUFFT (Type 2)

- Compute over-sampled FFT of equally-spaced signal samples
- Interpolate onto desired unequally-spaced frequency locations
- Dutt & Rokhlin, SIAM JSC, 1993, Gaussian bell interpolator
- Fessler & Sutton, IEEE T-SP, 2003, min-max interpolator
  and min-max optimized Kaiser-Bessel interpolator.
  NUFFT toolbox: `http://www.eecs.umich.edu/~fessler/code`



# Worst-Case NUFFT Interpolation Error



# Field inhomogeneity?

Combine NUFFT with min-max temporal interpolator
(Sutton *et al.*, IEEE T-MI, 2003)
(forward version of "time segmentation", Noll, T-MI, 1991)

Recall:

$$s(t) = \int f(\vec{r})\, e^{-\imath \omega(\vec{r}) t}\, e^{-\imath 2\pi \vec{k}(t) \cdot \vec{r}}\, d\vec{r}$$

Temporal interpolation approximation (aka "time segmentation"):

$$e^{-\imath \omega(\vec{r}) t} \approx \sum_{l=1}^{L} a_l(t)\, e^{-\imath \omega(\vec{r}) \tau_l}$$

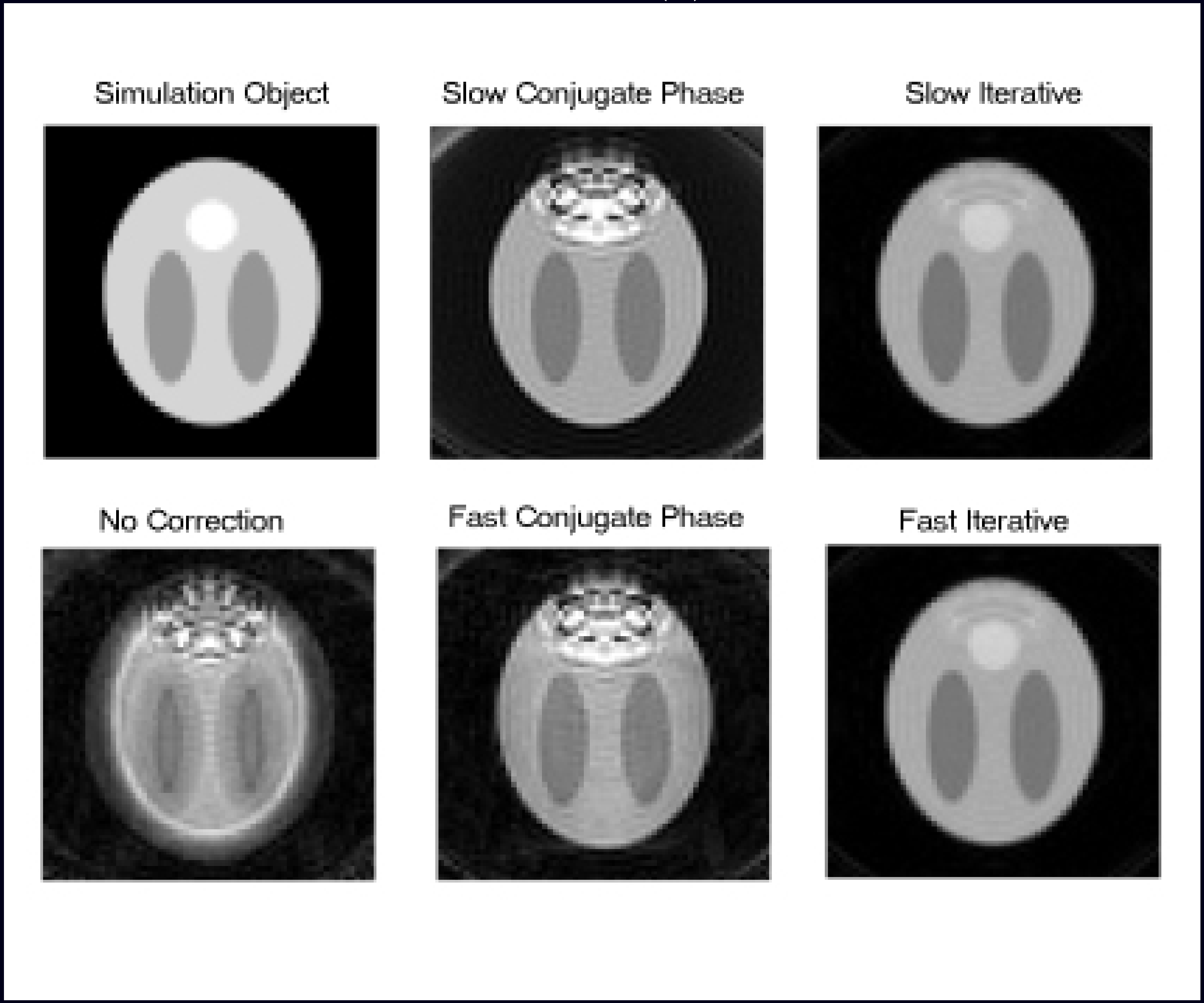
for min-max optimized temporal interpolation functions $\{a_l(\cdot)\}_{l=1}^{L}$.

$$s(t) \approx \sum_{l=1}^{L} a_l(t) \int \left[ f(\vec{r})\, e^{-\imath \omega(\vec{r}) \tau_l} \right] e^{-\imath 2\pi \vec{k}(t) \cdot \vec{r}}\, d\vec{r}$$

Linear combination of $L$ NUFFT calls.

# Field Corrected Reconstruction Example

Simulation using known field map $\omega(\vec{r})$.

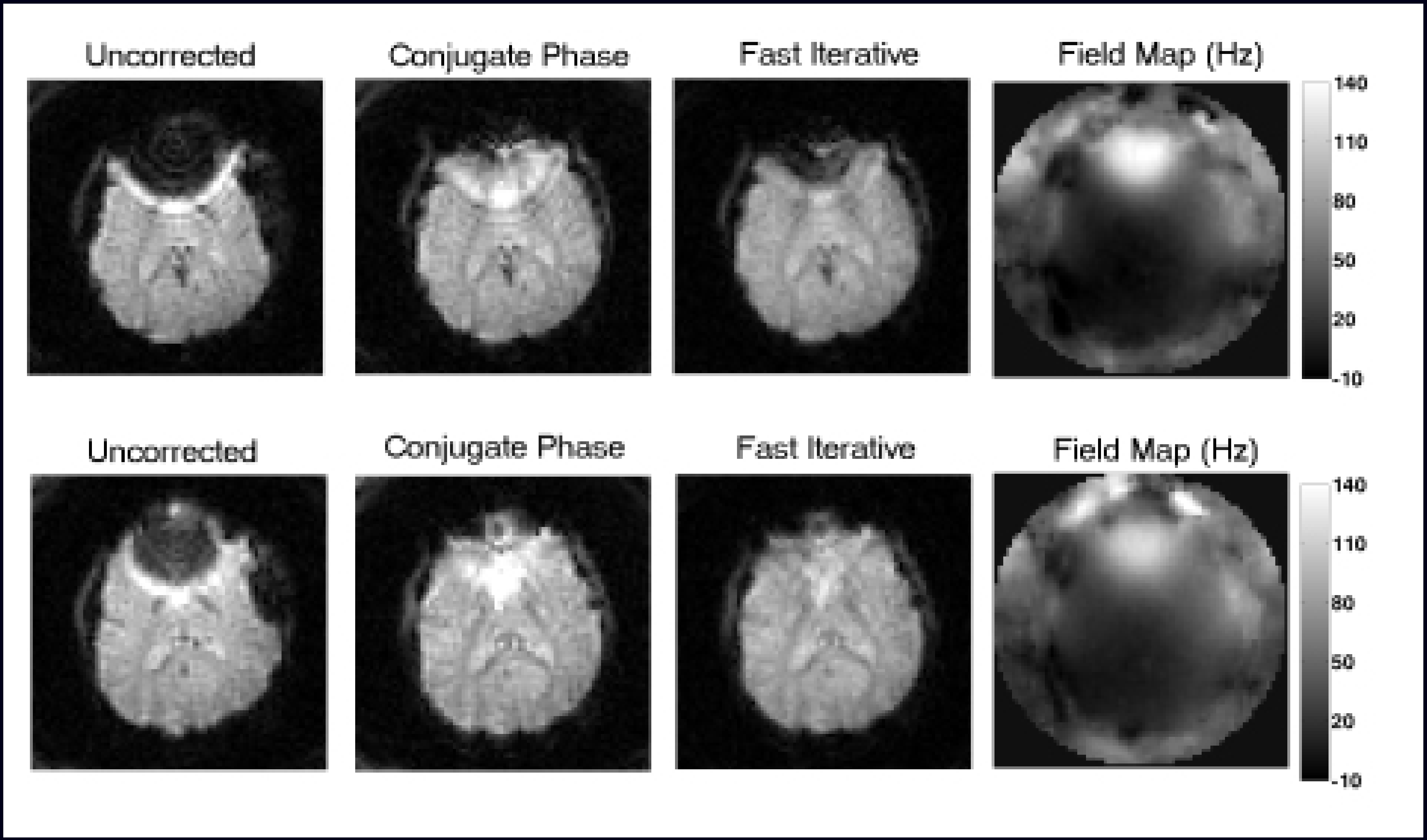


# Simulation Quantitative Comparison

- Computation time?
- NRMSE between $\hat{f}$ and $f^{\text{true}}$?

| Reconstruction Method | Time (s) | NRMSE complex | NRMSE magnitude |
|---|---|---|---|
| No Correction | 0.06 | 1.35 | 0.22 |
| Full Conjugate Phase | 4.07 | 0.31 | 0.19 |
| Fast Conjugate Phase | 0.33 | 0.32 | 0.19 |
| Fast Iterative (10 iters) | 2.20 | 0.04 | 0.04 |
| Exact Iterative (10 iters) | 128.16 | 0.04 | 0.04 |

# Human Data: Field Correction



# Regularization

- Conventional regularization for MRI uses a roughness penalty for the *complex* voxel values:

$$R(\boldsymbol{f}) \approx \sum_{j=1}^{M} |f_j - f_{j-1}|^2 \qquad \text{(in 1D).}$$

- Regularizes the real and imaginary image components equally.
- In some MR studies, including BOLD fMRI:
  - magnitude of $f_j$ carries the information of interest,
  - phase of $f_j$ should be spatially smooth.
  - This *a priori* information is ignored by $R(\boldsymbol{f})$.
- Alternatives to $R(\boldsymbol{f})$:
  - Constrain $\boldsymbol{f}$ to be real?
    (Unrealistic: RF phase inhomogeneity, eddy currents, ...)
  - Determine phase of $\boldsymbol{f}$ "somehow," then estimate its magnitude.
    - Non-iteratively (Noll, Nishimura, Macovski, IEEE T-MI, 1991)
    - Iteratively (Lee, Pauly, Nishimura, ISMRM, 2003)

# Separate Magnitude/Phase Regularization

Decompose $\boldsymbol{f}$ into its “magnitude” $\boldsymbol{m}$ and phase $\boldsymbol{x}$:

$$f_j(\boldsymbol{m}, \boldsymbol{x}) = m_j e^{\imath x_j}, \qquad m_j \in \mathbb{R}, \qquad x_j \in \mathbb{R}, \qquad j = 1, \ldots, M.$$

(Allow “magnitude” $m_j$ to be negative.)

Proposed cost function with separate regularization of $\boldsymbol{m}$ and $\boldsymbol{x}$:

$$\Psi(\boldsymbol{m}, \boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{A} \boldsymbol{f}(\boldsymbol{m}, \boldsymbol{x})\|^2 + \gamma R_1(\boldsymbol{m}) + \beta R_2(\boldsymbol{x}).$$

Choose $\beta \gg \gamma$ to strongly smooth phase estimate.

Joint estimation of magnitude and phase via regularized LS:

$$(\hat{\boldsymbol{m}}, \hat{\boldsymbol{x}}) = \underset{\boldsymbol{m} \in \mathbb{R}^M,\ \boldsymbol{x} \in \mathbb{R}^M}{\arg\min} \Psi(\boldsymbol{m}, \boldsymbol{x})$$

$\Psi$ is not convex $\Longrightarrow$ need good initial estimates $(\boldsymbol{m}^{(0)}, \boldsymbol{x}^{(0)})$.

# Alternating Minimization

Magnitude Update:

$$\boldsymbol{m}^{\text{new}} = \arg\min_{\boldsymbol{m}\in\mathbb{R}^M} \Psi\left(\boldsymbol{m}, \boldsymbol{x}^{\text{old}}\right)$$

Phase Update:

$$\boldsymbol{x}^{\text{new}} = \arg\min_{\boldsymbol{x}\in\mathbb{R}^M} \Psi(\boldsymbol{m}^{\text{new}}, \boldsymbol{x}),$$

Since $f_j = m_j e^{\imath x_j}$ is linear in $m_j$, the magnitude update is easy. Apply a few iterations of slightly modified CG algorithm (constrain $\boldsymbol{m}$ to be real)

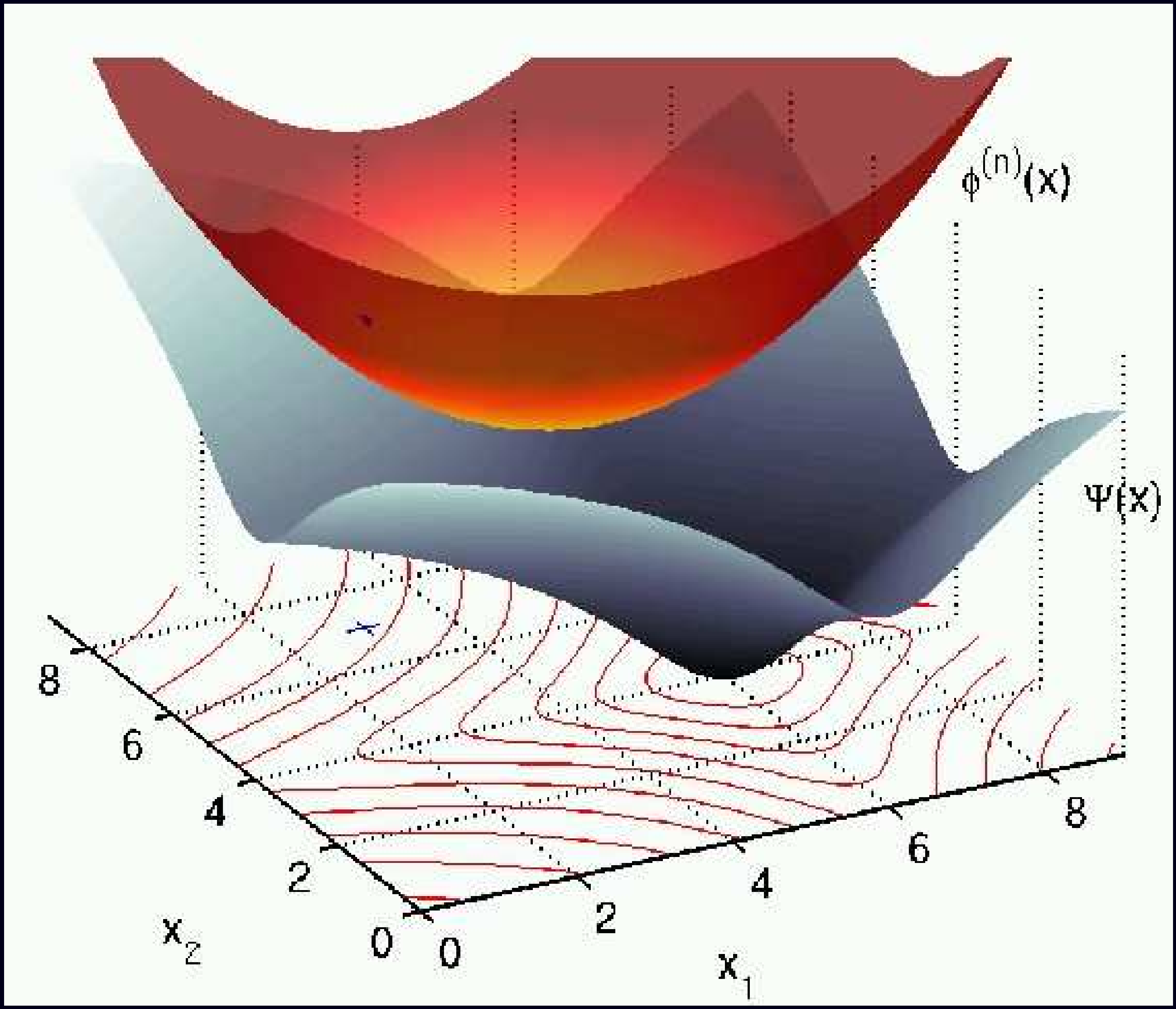
But $f_j = m_j e^{\imath x_j}$ is highly nonlinear in $\boldsymbol{x}$. Complicates “argmin.”

Steepest descent?

$$\boldsymbol{x}^{(n+1)} = \boldsymbol{x}^{(n)} - \lambda \nabla_{\boldsymbol{x}} \Psi\left(\boldsymbol{m}^{\text{old}}, \boldsymbol{x}^{(n)}\right).$$

Choosing the stepsize $\lambda$ is difficult.

# Optimization Transfer



# Surrogate Functions

To minimize a cost function $\Phi(\boldsymbol{x})$, choose surrogate functions $\phi^{(n)}(\boldsymbol{x})$ that satisfy the following *majorization* conditions:

$$\begin{aligned} \phi^{(n)}\left(\boldsymbol{x}^{(n)}\right) &= \Phi(\boldsymbol{x}^{(n)}) \\ \phi^{(n)}(\boldsymbol{x}) &\geq \Phi(\boldsymbol{x}), \qquad \forall \boldsymbol{x} \in \mathbb{R}^M. \end{aligned}$$

Iteratively minimize the surrogates as follows:

$$\boldsymbol{x}^{(n+1)} = \arg\min_{\boldsymbol{x}^{(n)} \in \mathbb{R}^M} \phi^{(n)}(\boldsymbol{x}) \, .$$

This will decrease $\Phi$ monotonically; $\Phi(\boldsymbol{x}^{(n+1)}) \leq \Phi(\boldsymbol{x}^{(n)})$.

The art is in the design of surrogates.
Tradeoffs:

- ◦ complexity
- ◦ computation per iteration
- ◦ convergence rate / number of iterations.

# Surrogate Functions for MR Phase

$$\mathsf{L}(\boldsymbol{x}) \triangleq \|\boldsymbol{y} - \boldsymbol{A} \boldsymbol{f}(\boldsymbol{m}, \boldsymbol{x})\|^2 = \sum_{i=1}^{N} \mathsf{h}_i([\boldsymbol{A} \boldsymbol{f}(\boldsymbol{m}, \boldsymbol{x})]_i),$$

where $\mathsf{h}_i(t) \triangleq |y_i - t|^2$ is convex.

Extending De Pierro (IEEE T-MI, 1995), for $\pi_{ij} \geq 0$ and $\sum_{j=1}^{M} \pi_{ij} = 1$:

$$[\boldsymbol{A} \boldsymbol{f}(\boldsymbol{m}, \boldsymbol{x})]_i = \sum_{j=1}^{M} b_{ij} \, \mathrm{e}^{\imath x_j} = \sum_{j=1}^{M} \pi_{ij} \left[ \frac{b_{ij}}{\pi_{ij}} \left( \mathrm{e}^{\imath x_j} - \mathrm{e}^{\imath x_j^{(n)}} \right) + \bar{\boldsymbol{y}}_i^{(n)} \right],$$

where $b_{ij} \triangleq a_{ij} m_j$, $\bar{\boldsymbol{y}}_i^{(n)} \triangleq [\boldsymbol{A} \boldsymbol{f}(\boldsymbol{m}, \boldsymbol{x}^{(n)})]_i$. Choose $\pi_{ij} \geq 0$ and $\sum_{j=1}^{M} \pi_{ij} = 1$.

Since $\mathsf{h}_i$ is convex:

$$\begin{aligned} \mathsf{h}_i([\boldsymbol{A} \boldsymbol{f}(\boldsymbol{m}, \boldsymbol{x})]_i) &= \mathsf{h}_i \left( \sum_{j=1}^{M} \pi_{ij} \left[ \frac{b_{ij}}{\pi_{ij}} \left( \mathrm{e}^{\imath x_j} - \mathrm{e}^{\imath x_j^{(n)}} \right) + \bar{\boldsymbol{y}}_i^{(n)} \right] \right) \\ &\leq \sum_{j=1}^{M} \pi_{ij} \, \mathsf{h}_i \left( \frac{b_{ij}}{\pi_{ij}} \left( \mathrm{e}^{\imath x_j} - \mathrm{e}^{\imath x_j^{(n)}} \right) + \bar{\boldsymbol{y}}_i^{(n)} \right), \end{aligned}$$

with equality when $\boldsymbol{x} = \boldsymbol{x}^{(n)}$.

# Separable Surrogate Function

$$\mathsf{L}(\boldsymbol{x}) = \sum_{i=1}^{N} \mathsf{h}_i([\boldsymbol{A} \boldsymbol{f}(\boldsymbol{m}, \boldsymbol{x})]_i) \leq \sum_{i=1}^{N} \sum_{j=1}^{M} \pi_{ij} \mathsf{h}_i \left( \frac{b_{ij}}{\pi_{ij}} \left( e^{\imath x_j} - e^{\imath x_j^{(n)}} \right) + \bar{\boldsymbol{y}}_i^{(n)} \right)$$
$$= \sum_{j=1}^{M} \underbrace{\sum_{i=1}^{N} \pi_{ij} \mathsf{h}_i \left( \frac{b_{ij}}{\pi_{ij}} \left( e^{\imath x_j} - e^{\imath x_j^{(n)}} \right) + \bar{\boldsymbol{y}}_i^{(n)} \right)}_{Q_j(x_j; \boldsymbol{x}^{(n)})}.$$

Construct similar surrogates $\{S_j\}$ for (convex) roughness penalty...

$$\text{Surrogate: } \phi^{(n)}(\boldsymbol{x}) = \sum_{j=1}^{M} Q_j(x_j; \boldsymbol{x}^{(n)}) + \beta S_j(x_j; \boldsymbol{x}^{(n)}).$$

Parallelizable (simultaneous) update, with 1D minimizations:

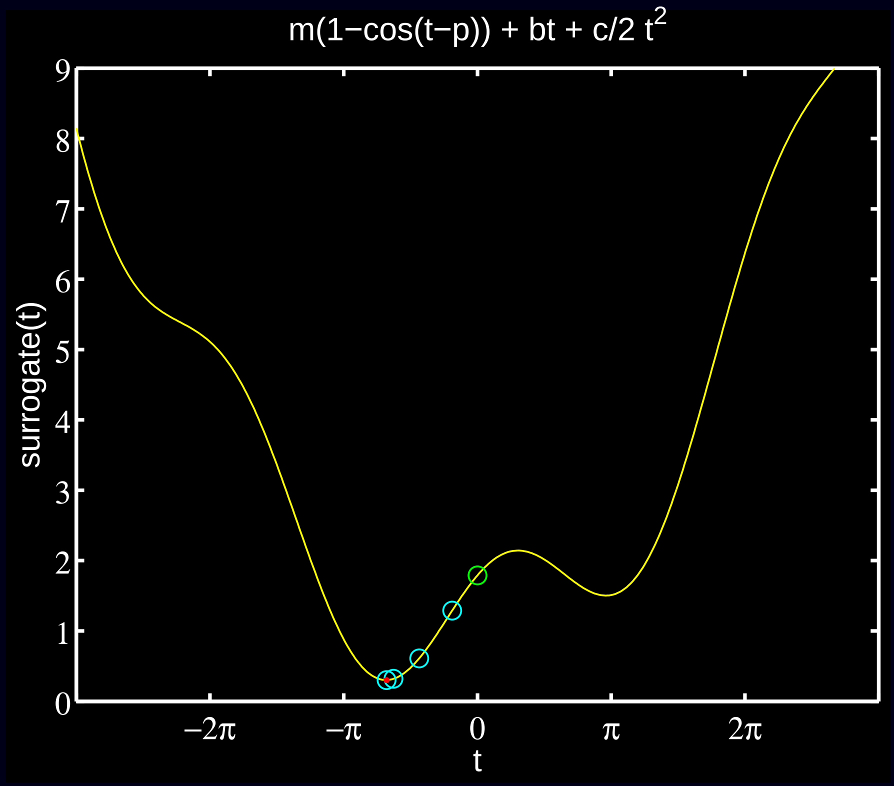
$$\boldsymbol{x}^{(n+1)} = \arg\min_{\boldsymbol{x}^{(n)} \in \mathbb{R}^M} \phi^{(n)}(\boldsymbol{x}) \implies x_j^{(n+1)} = \arg\min_{x_j \in \mathbb{R}} Q_j(x_j; \boldsymbol{x}^{(n)}) + \beta S_j(x_j; \boldsymbol{x}^{(n)}).$$

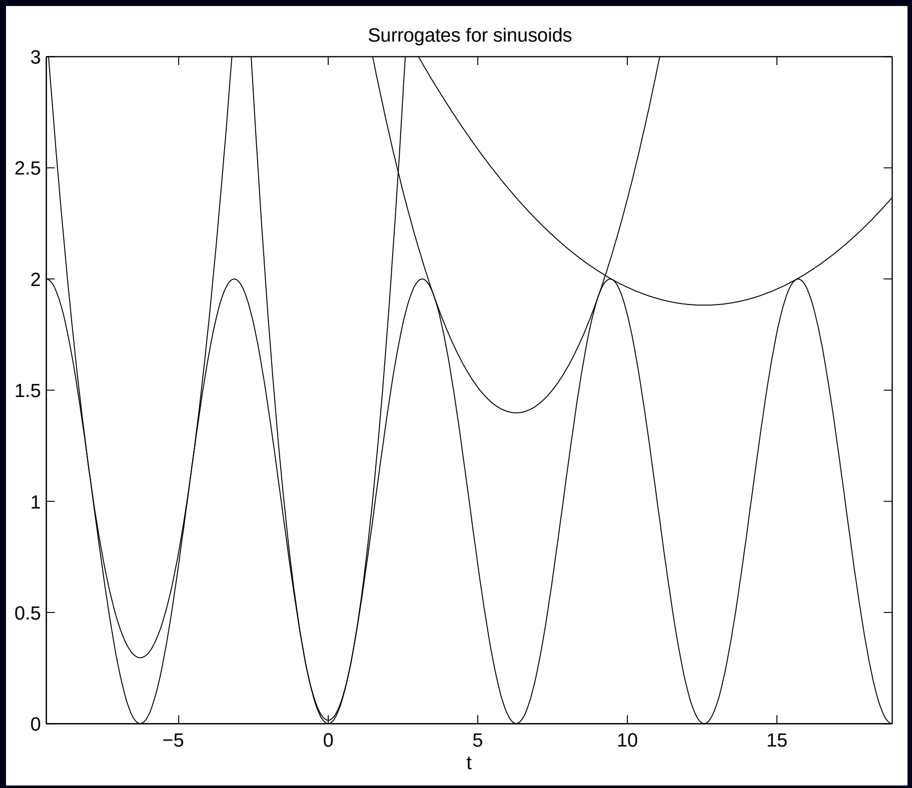
Intrinsically guaranteed to monotonically decrease the cost function.

# 1D Minimization: cos + quadratic

$$\ldots\; Q_j(x_j; \boldsymbol{x}^{(n)}) \equiv -\left|r_j^{(n)}\right| \cos\left(x_j - x_j^{(n)} - \angle r_j^{(n)}\right),$$

$$r_j^{(n)} = \left(f_j^{(n)}\right)^* [\boldsymbol{A}'(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}^{(n)})]_j + |m_j|^2 M \sum_{i=1}^{N} \left|B(\vec{k}(t_i))\right|^2$$




Simple 1D optimization transfer iterations...

# Final Algorithm for Phase Update

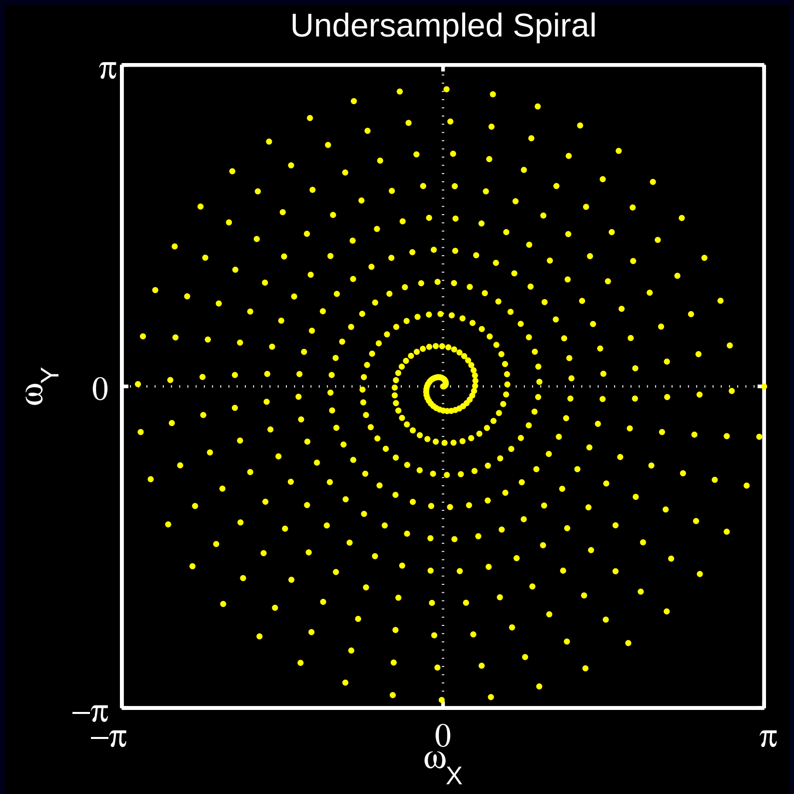
Diagonally preconditioned gradient descent:

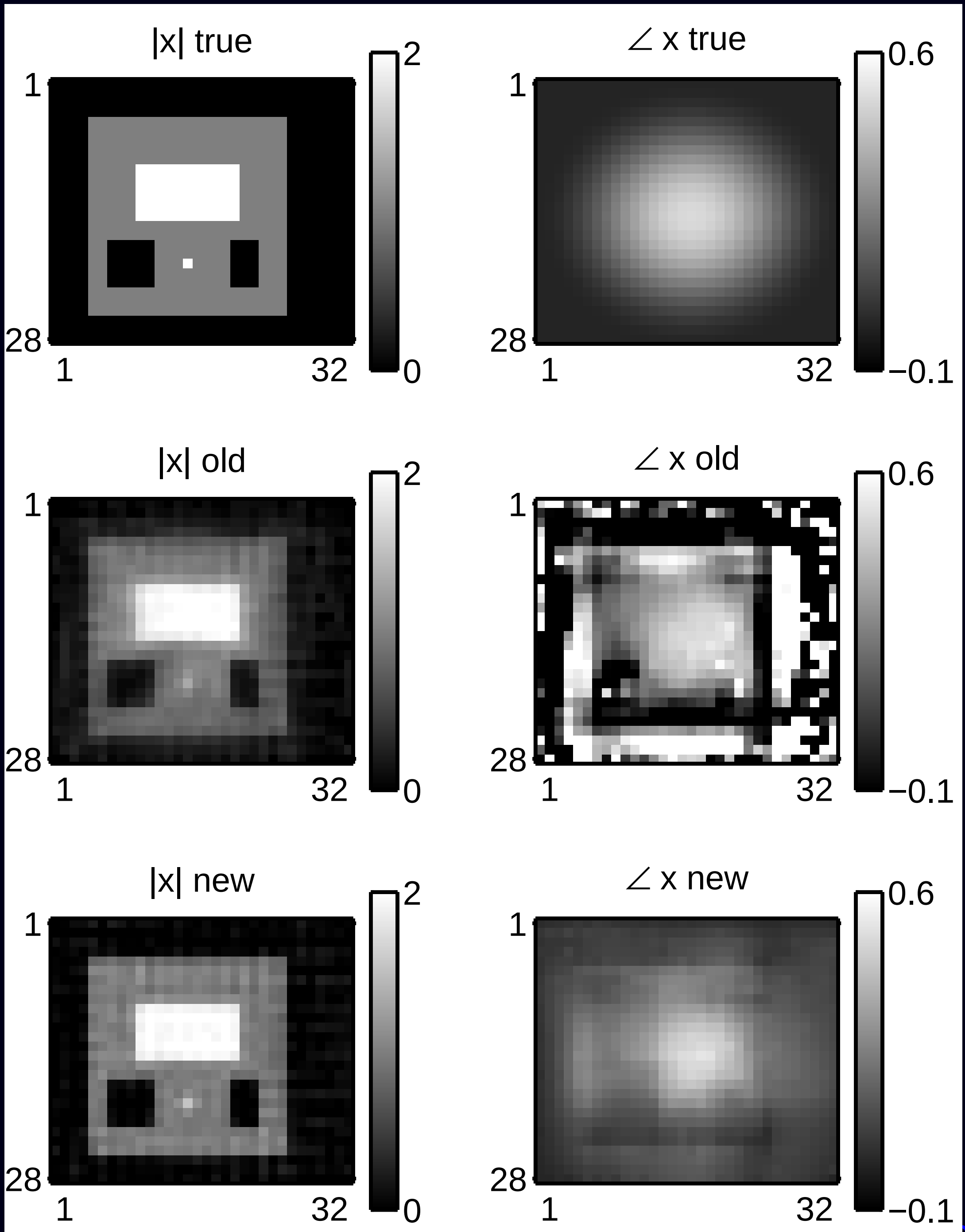
$$\boldsymbol{x}^{(n+1)} = \boldsymbol{x}^{(n)} - \boldsymbol{D}(\boldsymbol{x}^{(n)})\nabla\Phi(\boldsymbol{x}^{(n)})$$

where the diagonal matrix $\boldsymbol{D}$ has elements that ensure $\Phi$ decreases monotonically.

Alternate between magnitude and phase updates...

# Preliminary Simulation Example



# Summary

- Iterative reconstruction: much potential in MRI
- Computation: reduced by tools like NUFFT / temporal interpolation;
  combined with careful optimization algorithm design
  *cf.* Shepp and Vardi, 1982, PET
- Problems involving phase terms $e^{\iota x}$ suitable for optimization transfer.

# Future work

- Multiple receive coils (SENSE)
- Through-voxel field inhomogeneity gradients
- Motion (dynamic field maps...)
- Real data...