Iterative Reconstruction in MRI Using Iterative Methods

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Apr. 14, 2004

Acknowledgements: Doug Noll, Brad Sutton

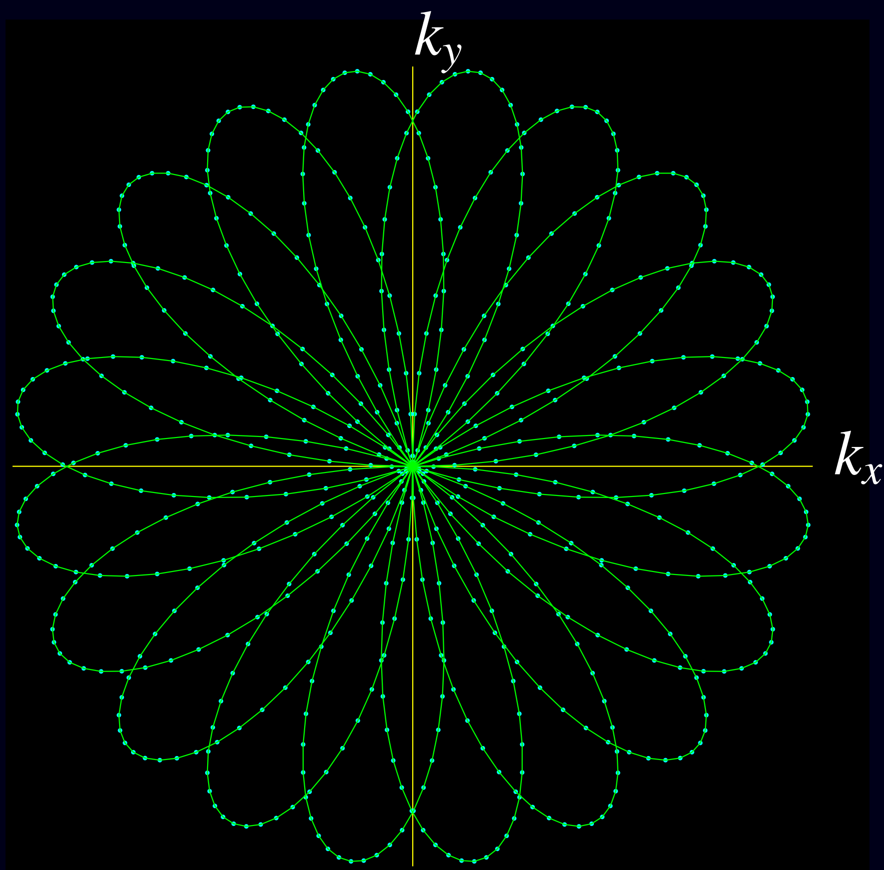
Outline

- MR image reconstruction
- Model-based reconstruction
- Iterations and Computation (NUFFT etc.)
- New regularization approach (ISBI '04)

MR Image Reconstruction

"k-space"

image



$\Longrightarrow$

Textbook MRI Measurement Model

Ignoring *lots* of things:

$$y_i = s(t_i) + \text{noise}_i, \qquad i = 1, \ldots, N$$

$$s(t) = \int f(\vec{r})\, e^{-\imath 2\pi \vec{k}(t) \cdot \vec{r}}\, d\vec{r},$$

where $\vec{k}(t)$ denotes the “k-space trajectory” of the MR pulse sequence.

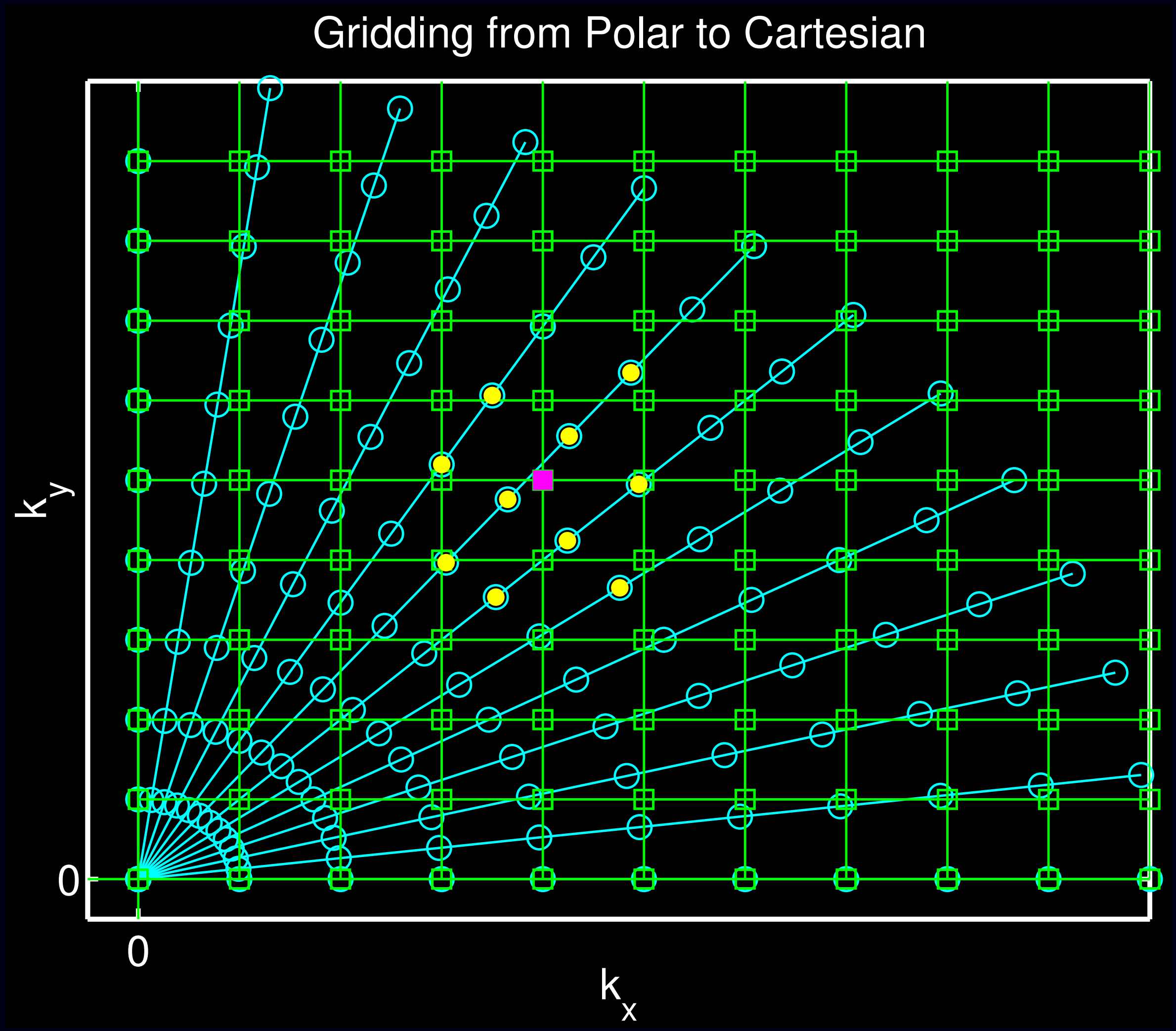
- MRI measurements are (roughly) samples of the Fourier transform of the object’s transverse magnetization $f(\vec{r})$.
- Reconstruction problem: recover $f(\vec{r})$ from measurements $\{y_i\}_{i=1}^{N}$.

Inherently under-determined (ill posed) problem

$\Longrightarrow$ no canonical solution.

Conventional MR Image Reconstruction

1. Interpolate measurements onto rectilinear grid ("gridding")
2. Apply inverse FFT to estimate samples of $f(\vec{r})$



Limitations of Gridding Reconstruction

1. Artifacts/inaccuracies due to interpolation
2. Contention about sample density “weighting”
3. Oversimplifications of Fourier transform signal model:
   - Magnetic field inhomogeneity
   - Magnetization decay (T_2)
   - Eddy currents
   - ...
4. Sensitivity encoding ?
5. ...

Model-Based Image Reconstruction

More complete signal equation:

$$s(t) = \int f(\vec{r})\, s_{\text{coil}}(\vec{r})\, \mathrm{e}^{-\imath\omega(\vec{r})t}\, \mathrm{e}^{-R_2^*(\vec{r})t}\, \mathrm{e}^{-\imath 2\pi \vec{k}(t)\cdot\vec{r}}\, \mathrm{d}\vec{r}$$

$$y_i = s(t_i) + \text{noise}_i, \qquad i = 1, \ldots, N$$

- $s_{\text{coil}}(\vec{r})$ Receive-coil sensitivity pattern(s) (for SENSE)
- $\omega(\vec{r})$ Off-resonance frequency map
  (due to field inhomogeneity and susceptibility)
- $R_2^*(\vec{r})$ Relaxation map

Other factors (?)

- Eddy current effects; in $\vec{k}(t)$
- Concomitant gradient terms
- Chemical shift
- Motion

Goal?

(it depends)

Inhomogeneity-Corrected Reconstruction

$$s(t) = \int f(\vec{r})\, s_{\mathrm{coil}}(\vec{r})\, \mathrm{e}^{-\imath \omega(\vec{r}) t}\, \mathrm{e}^{-R_2^*(\vec{r}) t}\, \mathrm{e}^{-\imath 2\pi \vec{k}(t) \cdot \vec{r}}\, \mathrm{d}\vec{r}$$

Goal: reconstruct $f(\vec{r})$ given field map $\omega(\vec{r})$
(Assume all other terms are known or unimportant.)

(Sutton *et al.*, ISMRM 2001; T-MI 2003)

Sensitivity-Encoded (SENSE) Reconstruction

$$s(t) = \int f(\vec{r}) s_{\text{coil}}(\vec{r}) \, \mathrm{e}^{-\imath \omega(\vec{r}) t} \, \mathrm{e}^{-R_2^*(\vec{r}) t} \, \mathrm{e}^{-\imath 2\pi \vec{k}(t) \cdot \vec{r}} \, \mathrm{d}\vec{r}$$

Goal: reconstruct $f(\vec{r})$ given sensitivity maps $s_{\text{coil}}(\vec{r})$
(Assume all other terms are known or unimportant.)

Can combine SENSE with field inhomogeneity correction “easily”

(Sutton *et al.*, ISMRM 2001)

Joint Estimation of Image and Field-Map

$$s(t) = \int f(\vec{r})\, s_{\text{coil}}(\vec{r})\, \mathrm{e}^{-\imath \omega(\vec{r}) t}\, \mathrm{e}^{-R_2^*(\vec{r}) t}\, \mathrm{e}^{-\imath 2\pi \vec{k}(t) \cdot \vec{r}}\, \mathrm{d}\vec{r}$$

Goal: estimate *both* the image $f(\vec{r})$ and the field map $\omega(\vec{r})$
(Assume all other terms are known or unimportant.)

(Sutton *et al.*, ISMRM Workshop, 2001; ISBI 2002; ISMRM 2002; ISMRM 2003; MRM in review)

The Kitchen Sink

$$s(t) = \int f(\vec{r})\, s_{\text{coil}}(\vec{r})\, \mathrm{e}^{-\imath \omega(\vec{r}) t}\, \mathrm{e}^{-R_2^*(\vec{r}) t}\, \mathrm{e}^{-\imath 2\pi \vec{k}(t) \cdot \vec{r}}\, \mathrm{d}\vec{r}$$

Goal: estimate image $f(\vec{r})$, field map $\omega(\vec{r})$, *and* relaxation map $R_2^*(\vec{r})$

Requires “suitable” k-space trajectory.

(Sutton *et al.*, ISMRM 2002; Twieg, MRM, 2003)

Estimation of Dynamic Maps

$$s(t) = \int f(\vec{r})\, s_{\mathrm{coil}}(\vec{r})\, \mathrm{e}^{-\imath \omega(\vec{r}) t}\, \mathrm{e}^{-R_2^*(\vec{r}) t}\, \mathrm{e}^{-\imath 2\pi \vec{k}(t) \cdot \vec{r}}\, \mathrm{d}\vec{r}$$

Goal: estimate dynamic field map $\omega(\vec{r})$ and “BOLD effect” $R_2^*(\vec{r})$ given baseline image $f(\vec{r})$ in fMRI.

Motion...

Back to Basic Signal Model

$$s(t) = \int f(\vec{r})\, e^{-\imath 2\pi \vec{k}(t)\cdot\vec{r}}\, d\vec{r}$$

Goal: reconstruct $f(\vec{r})$ from $\boldsymbol{y} = (y_1, \ldots, y_N)$, where $y_i = s(t_i) + \varepsilon_i$.

Series expansion of unknown object:

$$f(\vec{r}) \approx \sum_{j=1}^{M} f_j b(\vec{r} - \vec{r}_j) \longleftarrow \text{usually 2D rect functions.}$$

$$y_i \approx \int \left[\sum_{j=1}^{M} f_j b(\vec{r} - \vec{r}_j)\right] e^{-\imath 2\pi \vec{k}(t_i)\cdot\vec{r}}\, d\vec{r} = \sum_{j=1}^{M} \left[\int b(\vec{r} - \vec{r}_j)\, e^{-\imath 2\pi \vec{k}(t_i)\cdot\vec{r}}\, d\vec{r}\right] f_j$$

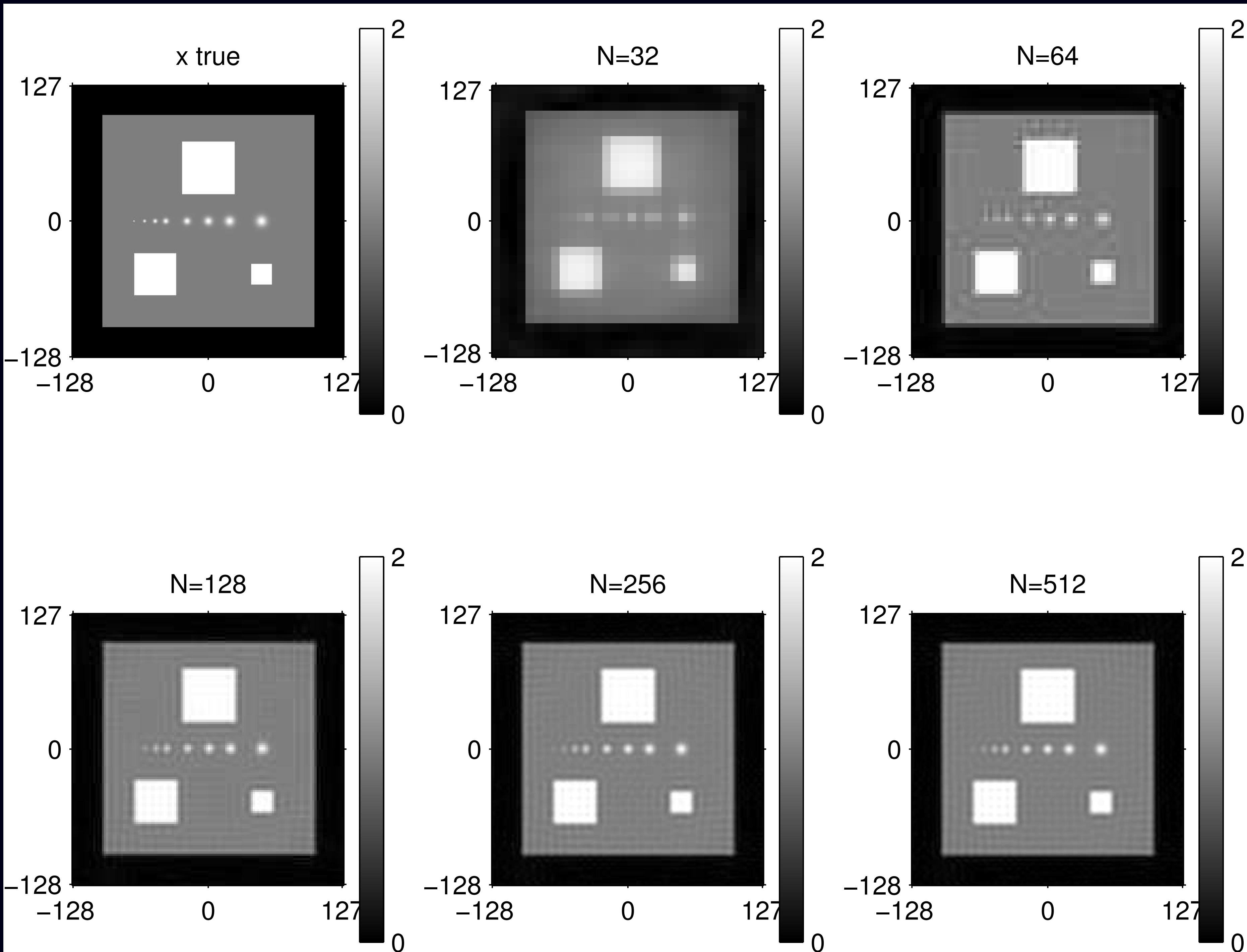
$$= \sum_{j=1}^{M} a_{ij} f_j, \qquad a_{ij} = B(\vec{k}(t_i))\, e^{-\imath 2\pi \vec{k}(t_i)\cdot\vec{r}_j}, \qquad b(\vec{r}) \overset{\text{FT}}{\Longleftrightarrow} B(\vec{k}).$$

Discrete-discrete measurement model with system matrix $\boldsymbol{A} = \{a_{ij}\}$:

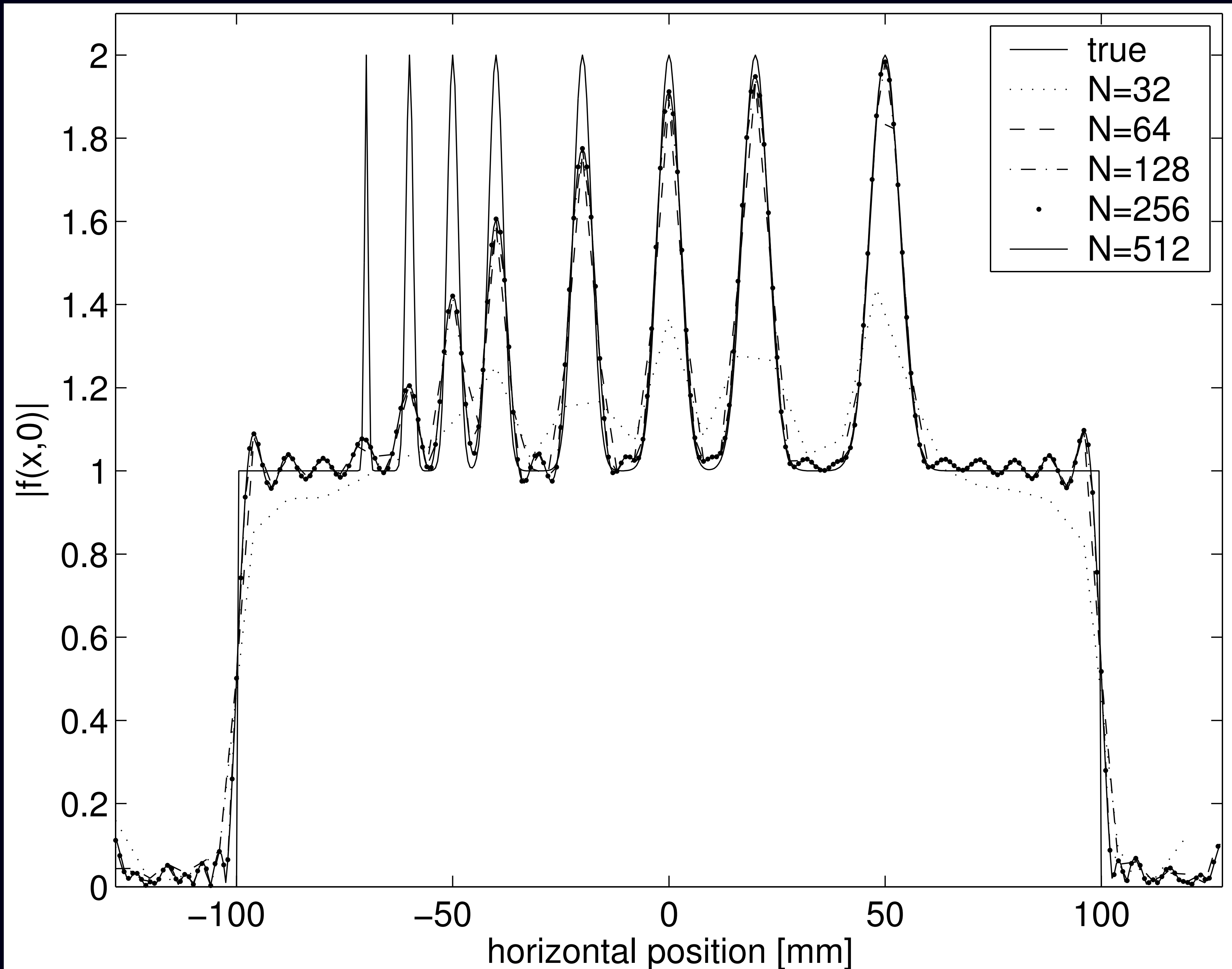
$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{f} + \boldsymbol{\varepsilon}.$$

Goal: estimate coefficients (pixel values) $\boldsymbol{f} = (f_1, \ldots, f_M)$ from $\boldsymbol{y}$.

Small Pixel Size Does Not Matter



Profiles



true
N=32
N=64
N=128
N=256
N=512
|f(x,0)|
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−100
−50
0
50
100
horizontal position [mm]

Regularized Least-Squares Estimation

$$\hat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f} \in \mathbb{C}^M} \Psi(\boldsymbol{f}), \qquad \Psi(\boldsymbol{f}) = \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{f}\|^2 + \alpha R(\boldsymbol{f})$$

- data fit term $\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{f}\|^2$
  corresponds to negative log-likelihood of Gaussian distribution
- regularizing roughness penalty term $R(\boldsymbol{f})$ controls noise

$$R(\boldsymbol{f}) \approx \int \|\nabla f\|^2 \, \mathrm{d}\vec{r}$$

- regularization parameter $\alpha > 0$
  controls tradeoff between spatial resolution and noise
  (Fessler & Rogers, IEEE T-IP, 1996)
- Equivalent to Bayesian MAP estimation with prior $\propto \mathrm{e}^{-\alpha R(\boldsymbol{f})}$

Quadratic regularization $R(\boldsymbol{f}) = \|\boldsymbol{C}\boldsymbol{f}\|^2$ leads to closed-form solution:

$$\hat{\boldsymbol{f}} = \left[\boldsymbol{A}'\boldsymbol{A} + \alpha \boldsymbol{C}'\boldsymbol{C}\right]^{-1} \boldsymbol{A}'\boldsymbol{y}$$

(a formula of limited practical use)

Iterative Minimization by Conjugate Gradients

Choose initial guess $\boldsymbol{f}^{(0)}$ (*e.g.*, fast conjugate phase / gridding).
Iteration (unregularized):

$$\boldsymbol{g}^{(n)} = \nabla\Psi\left(\boldsymbol{f}^{(n)}\right) = \boldsymbol{A}'(\boldsymbol{A}\boldsymbol{f}^{(n)} - \boldsymbol{y}) \quad \text{gradient}$$

$$\boldsymbol{p}^{(n)} = \boldsymbol{P}\boldsymbol{g}^{(n)} \quad \text{precondition}$$

$$\gamma_n = \begin{cases} 0, & n = 0 \\ \dfrac{\langle \boldsymbol{g}^{(n)}, \boldsymbol{p}^{(n)} \rangle}{\langle \boldsymbol{g}^{(n-1)}, \boldsymbol{p}^{(n-1)} \rangle}, & n > 0 \end{cases}$$

$$\boldsymbol{d}^{(n)} = -\boldsymbol{p}^{(n)} + \gamma_n \boldsymbol{d}^{(n-1)} \quad \text{search direction}$$

$$\boldsymbol{v}^{(n)} = \boldsymbol{A}\boldsymbol{d}^{(n)}$$

$$\alpha_n = \langle \boldsymbol{d}^{(n)}, -\boldsymbol{g}^{(n)} \rangle / \langle \boldsymbol{A}\boldsymbol{f}^{(n)}, \boldsymbol{A}\boldsymbol{f}^{(n)} \rangle \quad \text{step size}$$

$$\boldsymbol{f}^{(n+1)} = \boldsymbol{f}^{(n)} + \alpha_n \boldsymbol{d}^{(n)} \quad \text{update}$$

Bottlenecks: computing $\boldsymbol{A}\boldsymbol{f}$ and $\boldsymbol{A}'\boldsymbol{y}$.

- $\boldsymbol{A}$ is too large to store explicitly (not sparse)
- Even if $\boldsymbol{A}$ were stored, directly computing $\boldsymbol{A}\boldsymbol{f}$ is $O(NM)$, per iteration, whereas FFT is only $O(N \log N)$

Computing $\boldsymbol{Af}$ Rapidly

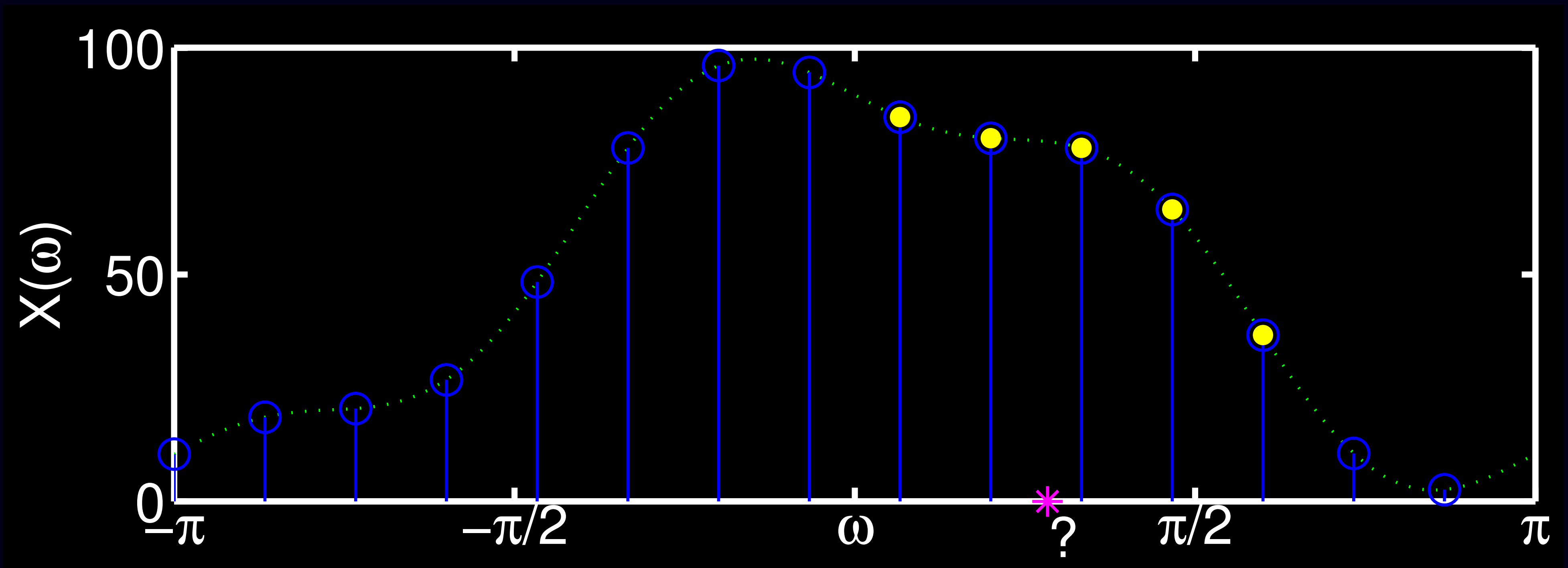
$$[\boldsymbol{A}\boldsymbol{f}]_i = \sum_{j=1}^{M} a_{ij} f_j = B(\vec{k}(t_i)) \sum_{j=1}^{M} \mathrm{e}^{-\imath 2\pi \vec{k}(t_i)\cdot\vec{r}_j} f_j, \qquad i = 1,\ldots,N$$

- Pixel locations $\{\vec{r}_j\}$ are uniformly spaced
- k-space locations $\left\{\vec{k}(t_i)\right\}$ are unequally spaced

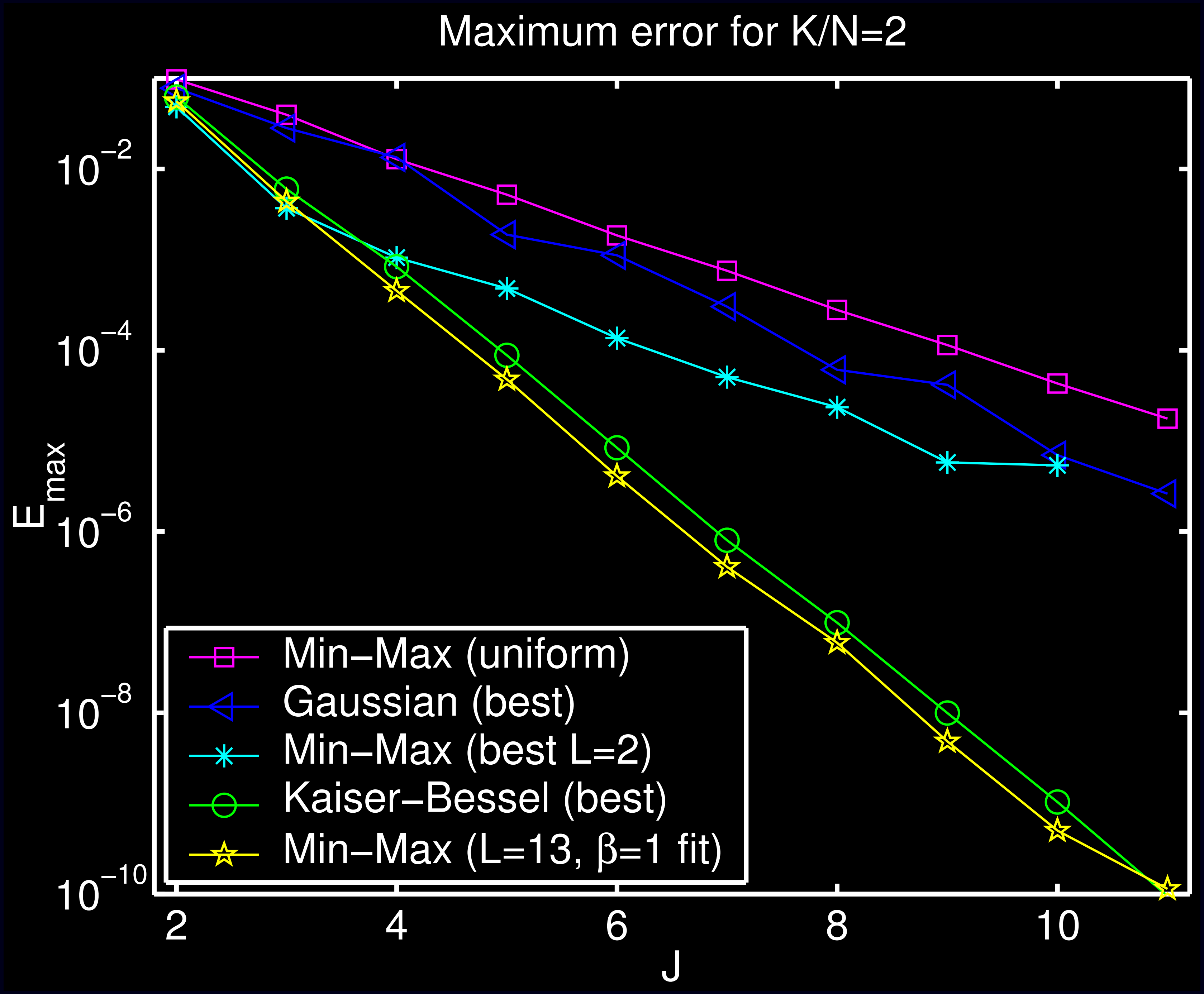
$\Longrightarrow$ needs nonuniform fast Fourier transform (NUFFT)

NUFFT (Type 2)

- Compute over-sampled FFT of equally-spaced signal samples
- Interpolate onto desired unequally-spaced frequency locations
- Dutt & Rokhlin, SIAM JSC, 1993, Gaussian bell interpolator
- Fessler & Sutton, IEEE T-SP, 2003, min-max interpolator
  and min-max optimized Kaiser-Bessel interpolator.
  NUFFT toolbox: `http://www.eecs.umich.edu/~fessler/code`



Worst-Case NUFFT Interpolation Error



Field inhomogeneity?

Combine NUFFT with min-max temporal interpolator
(Sutton *et al.*, IEEE T-MI, 2003)
(forward version of "time segmentation", Noll, T-MI, 1991)

Recall:

$$s(t) = \int f(\vec{r})\, e^{-\imath \omega(\vec{r}) t}\, e^{-\imath 2\pi \vec{k}(t)\cdot\vec{r}}\, d\vec{r}$$

Temporal interpolation approximation (aka "time segmentation"):

$$e^{-\imath \omega(\vec{r}) t} \approx \sum_{l=1}^{L} a_l(t)\, e^{-\imath \omega(\vec{r}) \tau_l}$$

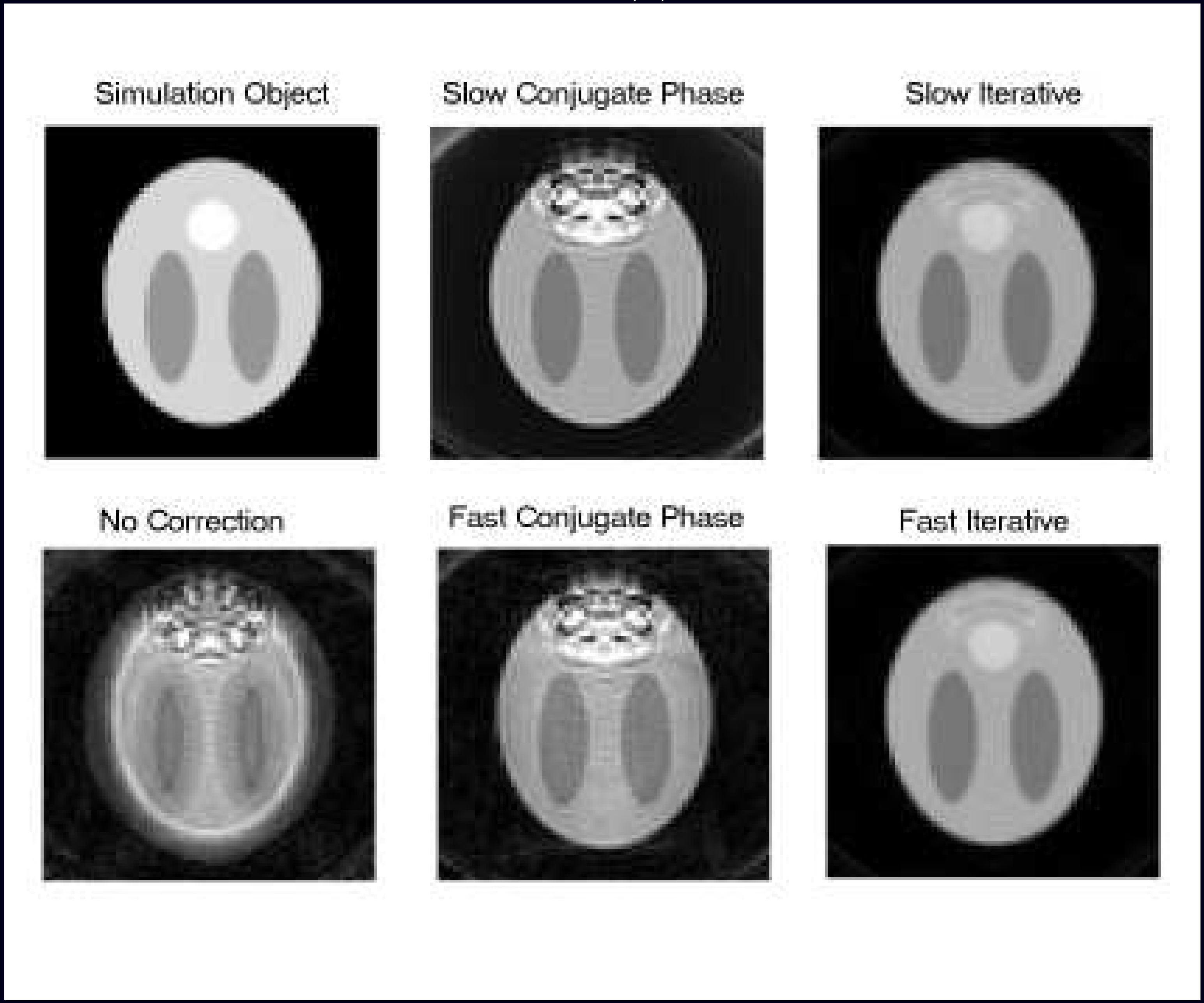
for min-max optimized temporal interpolation functions $\{a_l(\cdot)\}_{l=1}^{L}$.

$$s(t) \approx \sum_{l=1}^{L} a_l(t) \int \left[f(\vec{r})\, e^{-\imath \omega(\vec{r}) \tau_l} \right] e^{-\imath 2\pi \vec{k}(t)\cdot\vec{r}}\, d\vec{r}$$

Linear combination of L NUFFT calls.

Field Corrected Reconstruction Example

Simulation using known field map $\omega(\vec{r})$.

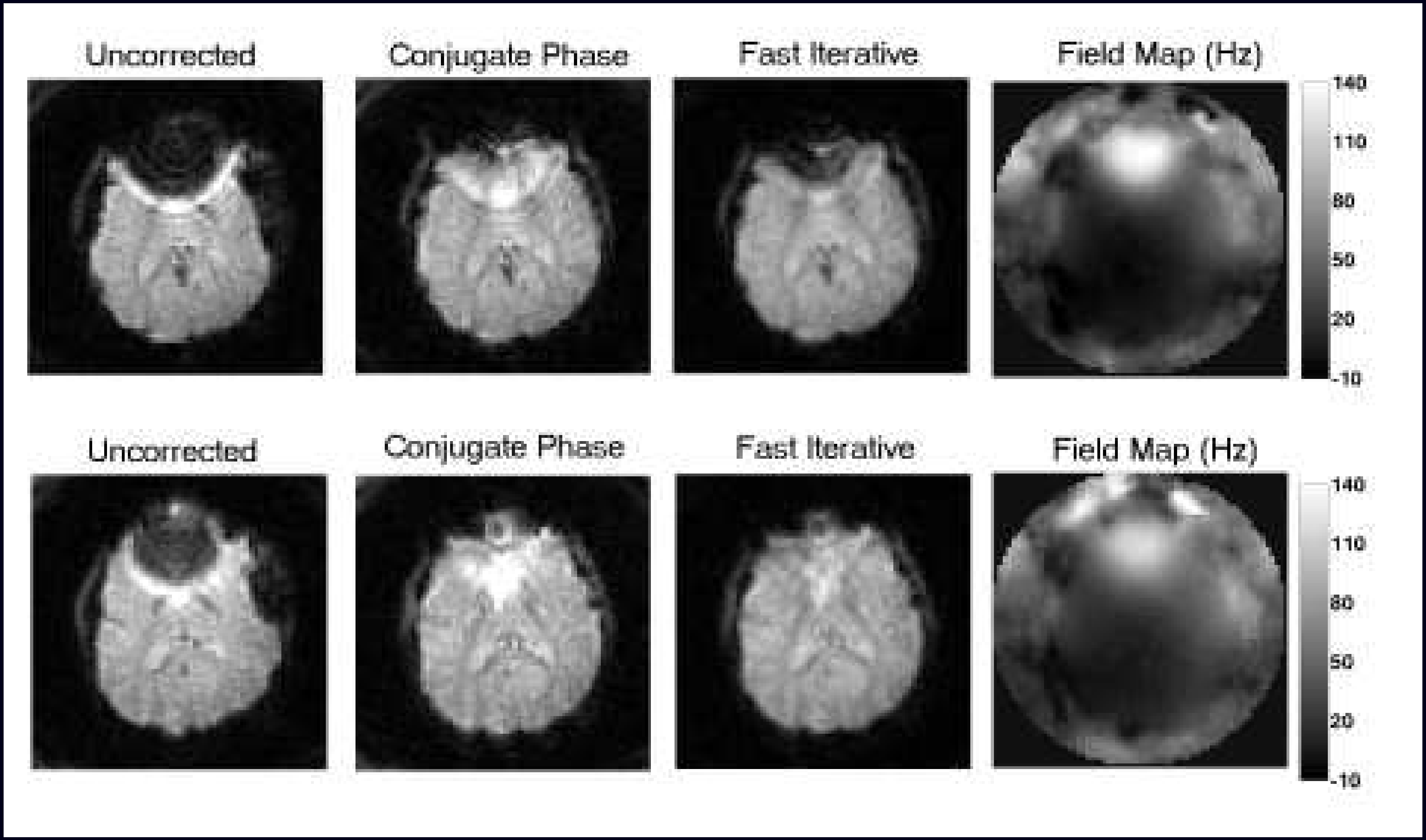


Simulation Quantitative Comparison

- Computation time?
- NRMSE between $\hat{f}$ and f^{true}?

Reconstruction Method	Time (s)	NRMSE complex	NRMSE magnitude
No Correction	0.06	1.35	0.22
Full Conjugate Phase	4.07	0.31	0.19
Fast Conjugate Phase	0.33	0.32	0.19
Fast Iterative (10 iters)	2.20	0.04	0.04
Exact Iterative (10 iters)	128.16	0.04	0.04

Human Data: Field Correction



Regularization (ISBI ’04)

- Conventional regularization for MRI uses a roughness penalty for the *complex* voxel values:

$$R(\boldsymbol{f}) \approx \sum_{j=1}^{M} |f_j - f_{j-1}|^2 \qquad \text{(in 1D).}$$

- Regularizes the real and imaginary image components equally.
- In some MR studies, including BOLD fMRI:
  - magnitude of f_j carries the information of interest,
  - phase of f_j should be spatially smooth.
  - This *a priori* information is ignored by $R(\boldsymbol{f})$.
- Alternatives to $R(\boldsymbol{f})$:
  - Constrain $\boldsymbol{f}$ to be real?
    (Unrealistic: RF phase inhomogeneity, eddy currents, ...)
  - Determine phase of $\boldsymbol{f}$ “somehow,” then estimate its magnitude.
    - Non-iteratively (Noll, Nishimura, Macovski, IEEE T-MI, 1991)
    - Iteratively (Lee, Pauly, Nishimura, ISMRM, 2003)

Separate Magnitude/Phase Regularization

Decompose $\boldsymbol{f}$ into its "magnitude" $\boldsymbol{m}$ and phase $\boldsymbol{x}$:

$$f_j(\boldsymbol{m},\boldsymbol{x}) = m_j e^{\imath x_j}, \qquad m_j \in \mathbb{R}, \qquad x_j \in \mathbb{R}, \qquad j = 1,\ldots,M.$$

(Allow "magnitude" m_j to be negative.)

Proposed cost function with separate regularization of $\boldsymbol{m}$ and $\boldsymbol{x}$:

$$\Psi(\boldsymbol{m},\boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{f}(\boldsymbol{m},\boldsymbol{x})\|^2 + \gamma R_1(\boldsymbol{m}) + \beta R_2(\boldsymbol{x}).$$

Choose $\beta \gg \gamma$ to strongly smooth phase estimate.

Joint estimation of magnitude and phase via regularized LS:

$$(\hat{\boldsymbol{m}},\hat{\boldsymbol{x}}) = \underset{\boldsymbol{m}\in\mathbb{R}^M,\ \boldsymbol{x}\in\mathbb{R}^M}{\arg\min}\ \Psi(\boldsymbol{m},\boldsymbol{x})$$

Ψ is not convex $\Longrightarrow$ need good initial estimates $(\boldsymbol{m}^{(0)},\boldsymbol{x}^{(0)})$.

Alternating Minimization

Magnitude Update:

$$\boldsymbol{m}^{\text{new}} = \arg\min_{\boldsymbol{m}\in\mathbb{R}^M} \Psi\left(\boldsymbol{m}, \boldsymbol{x}^{\text{old}}\right)$$

Phase Update:

$$\boldsymbol{x}^{\text{new}} = \arg\min_{\boldsymbol{x}\in\mathbb{R}^M} \Psi(\boldsymbol{m}^{\text{new}}, \boldsymbol{x}),$$

Since $f_j = m_j \mathrm{e}^{\imath x_j}$ is linear in m_j, the magnitude update is easy.
Apply a few iterations of slightly modified CG algorithm
(constrain $\boldsymbol{m}$ to be real)

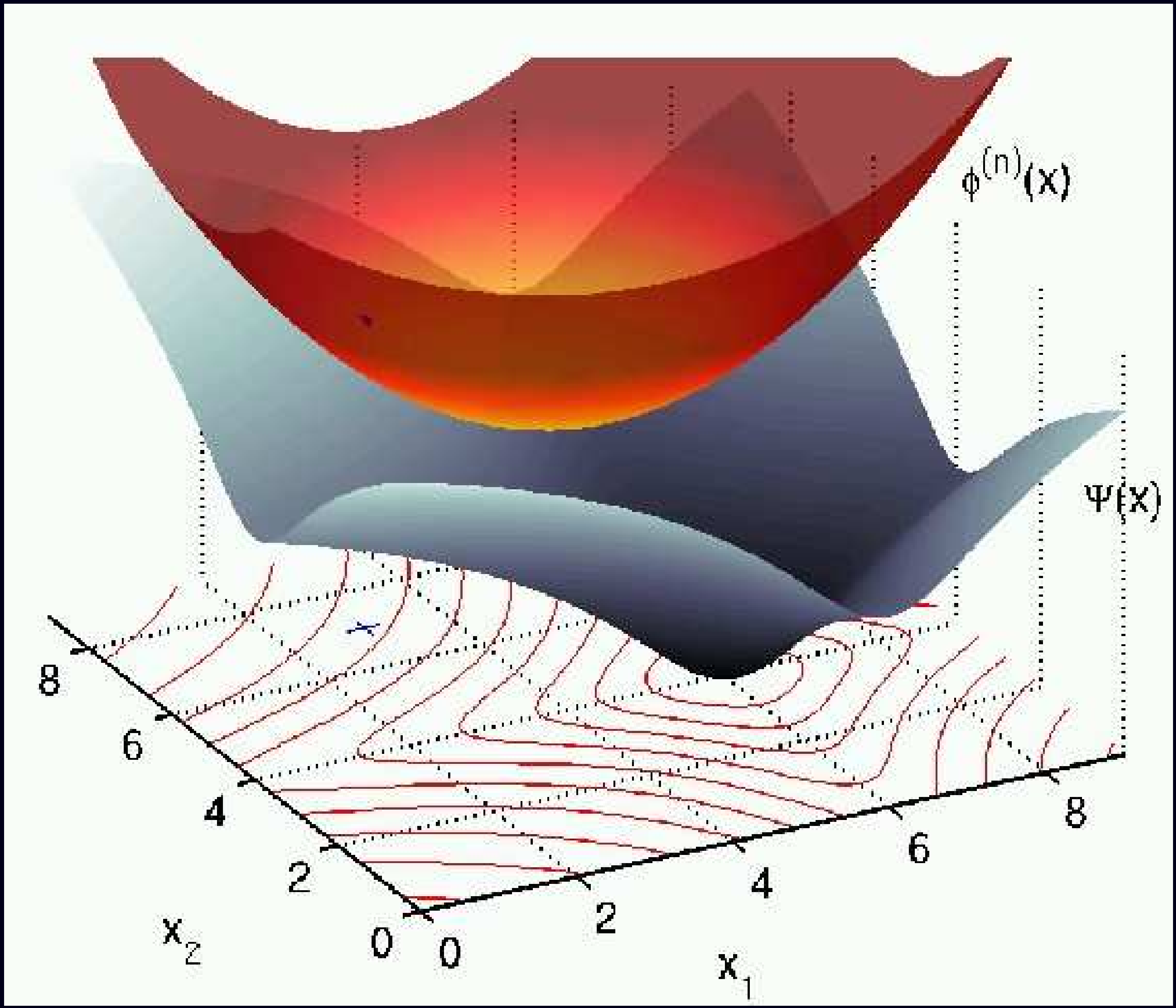
But $f_j = m_j \mathrm{e}^{\imath x_j}$ is highly nonlinear in $\boldsymbol{x}$. Complicates “argmin.”

Steepest descent?

$$\boldsymbol{x}^{(n+1)} = \boldsymbol{x}^{(n)} - \lambda \nabla_{\boldsymbol{x}} \Psi\left(\boldsymbol{m}^{\text{old}}, \boldsymbol{x}^{(n)}\right).$$

Choosing the stepsize λ is difficult.

Optimization Transfer



Surrogate Functions

To minimize a cost function $\Phi(\boldsymbol{x})$, choose surrogate functions $\phi^{(n)}(\boldsymbol{x})$ that satisfy the following *majorization* conditions:

$$\begin{aligned} \phi^{(n)}(\boldsymbol{x}^{(n)}) &= \Phi(\boldsymbol{x}^{(n)}) \\ \phi^{(n)}(\boldsymbol{x}) &\geq \Phi(\boldsymbol{x}), \qquad \forall \boldsymbol{x} \in \mathbb{R}^M. \end{aligned}$$

Iteratively minimize the surrogates as follows:

$$\boldsymbol{x}^{(n+1)} = \arg\min_{\boldsymbol{x}^{(n)} \in \mathbb{R}^M} \phi^{(n)}(\boldsymbol{x}) \,.$$

This will decrease Φ monotonically; $\Phi(\boldsymbol{x}^{(n+1)}) \leq \Phi(\boldsymbol{x}^{(n)})$.

The art is in the design of surrogates.
Tradeoffs:
- ◦complexity
- ◦computation per iteration
- ◦convergence rate / number of iterations.

Surrogate Functions for MR Phase

$$\mathsf{L}(\boldsymbol{x}) \triangleq \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{f}(\boldsymbol{m},\boldsymbol{x})\|^2 = \sum_{i=1}^{N} h_i([\boldsymbol{A}\boldsymbol{f}(\boldsymbol{m},\boldsymbol{x})]_i),$$

where $h_i(t) \triangleq |y_i - t|^2$ is convex.

Extending De Pierro (IEEE T-MI, 1995), for $\pi_{ij} \geq 0$ and $\sum_{j=1}^{M} \pi_{ij} = 1$:

$$[\boldsymbol{A}\boldsymbol{f}(\boldsymbol{m},\boldsymbol{x})]_i = \sum_{j=1}^{M} b_{ij} \mathrm{e}^{\imath x_j} = \sum_{j=1}^{M} \pi_{ij} \left[\frac{b_{ij}}{\pi_{ij}} \left(\mathrm{e}^{\imath x_j} - \mathrm{e}^{\imath x_j^{(n)}}\right) + \bar{y}_i^{(n)}\right],$$

where $b_{ij} \triangleq a_{ij} m_j$, $\bar{y}_i^{(n)} \triangleq [\boldsymbol{A}\boldsymbol{f}(\boldsymbol{m},\boldsymbol{x}^{(n)})]_i$. Choose $\pi_{ij} \geq 0$ and $\sum_{j=1}^{M} \pi_{ij} = 1$.

Since h_i is convex:

$$\begin{aligned} h_i([\boldsymbol{A}\boldsymbol{f}(\boldsymbol{m},\boldsymbol{x})]_i) &= h_i\left(\sum_{j=1}^{M} \pi_{ij} \left[\frac{b_{ij}}{\pi_{ij}} \left(\mathrm{e}^{\imath x_j} - \mathrm{e}^{\imath x_j^{(n)}}\right) + \bar{y}_i^{(n)}\right]\right) \\ &\leq \sum_{j=1}^{M} \pi_{ij} h_i\left(\frac{b_{ij}}{\pi_{ij}} \left(\mathrm{e}^{\imath x_j} - \mathrm{e}^{\imath x_j^{(n)}}\right) + \bar{y}_i^{(n)}\right), \end{aligned}$$

with equality when $\boldsymbol{x} = \boldsymbol{x}^{(n)}$.

Separable Surrogate Function

$$\mathsf{L}(\boldsymbol{x}) = \sum_{i=1}^{N} h_i([\boldsymbol{A}\boldsymbol{f}(\boldsymbol{m},\boldsymbol{x})]_i) \leq \sum_{i=1}^{N}\sum_{j=1}^{M} \pi_{ij} h_i\left(\frac{b_{ij}}{\pi_{ij}}\left(\mathrm{e}^{\imath x_j} - \mathrm{e}^{\imath x_j^{(n)}}\right) + \bar{y}_i^{(n)}\right)$$
$$= \sum_{j=1}^{M} \underbrace{\sum_{i=1}^{N} \pi_{ij} h_i\left(\frac{b_{ij}}{\pi_{ij}}\left(\mathrm{e}^{\imath x_j} - \mathrm{e}^{\imath x_j^{(n)}}\right) + \bar{y}_i^{(n)}\right)}_{Q_j(x_j;\boldsymbol{x}^{(n)})}.$$

Construct similar surrogates $\{S_j\}$ for (convex) roughness penalty...

$$\text{Surrogate: } \phi^{(n)}(\boldsymbol{x}) = \sum_{j=1}^{M} Q_j(x_j;\boldsymbol{x}^{(n)}) + \beta S_j(x_j;\boldsymbol{x}^{(n)}).$$

Parallelizable (simultaneous) update, with 1D minimizations:

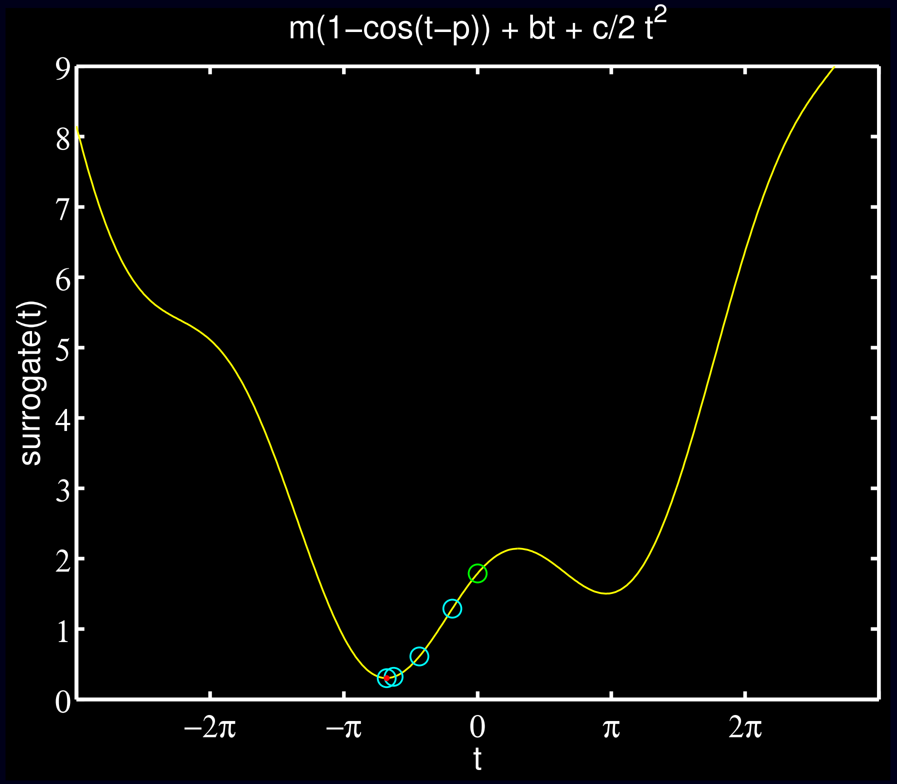
$$\boldsymbol{x}^{(n+1)} = \underset{\boldsymbol{x}^{(n)}\in\mathbb{R}^M}{\arg\min}\, \phi^{(n)}(\boldsymbol{x}) \implies x_j^{(n+1)} = \underset{x_j\in\mathbb{R}}{\arg\min}\, Q_j(x_j;\boldsymbol{x}^{(n)}) + \beta S_j(x_j;\boldsymbol{x}^{(n)}).$$

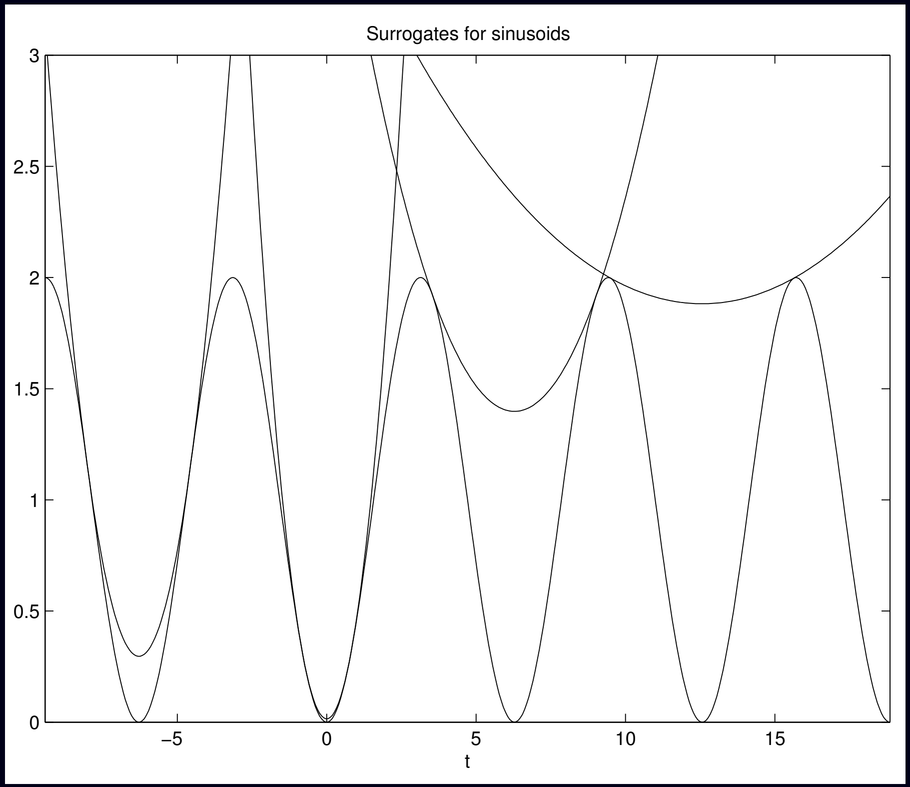
Intrinsically guaranteed to monotonically decrease the cost function.

1D Minimization: cos + quadratic

$$\ldots\; Q_j(x_j;\boldsymbol{x}^{(n)}) \equiv -\left|r_j^{(n)}\right| \cos\left(x_j - x_j^{(n)} - \angle r_j^{(n)}\right),$$

$$r_j^{(n)} = \left(f_j^{(n)}\right)^* [\boldsymbol{A}'(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}^{(n)})]_j + |m_j|^2 M \sum_{i=1}^{N} \left|B(\vec{k}(t_i))\right|^2$$





Simple 1D optimization transfer iterations...

Final Algorithm for Phase Update

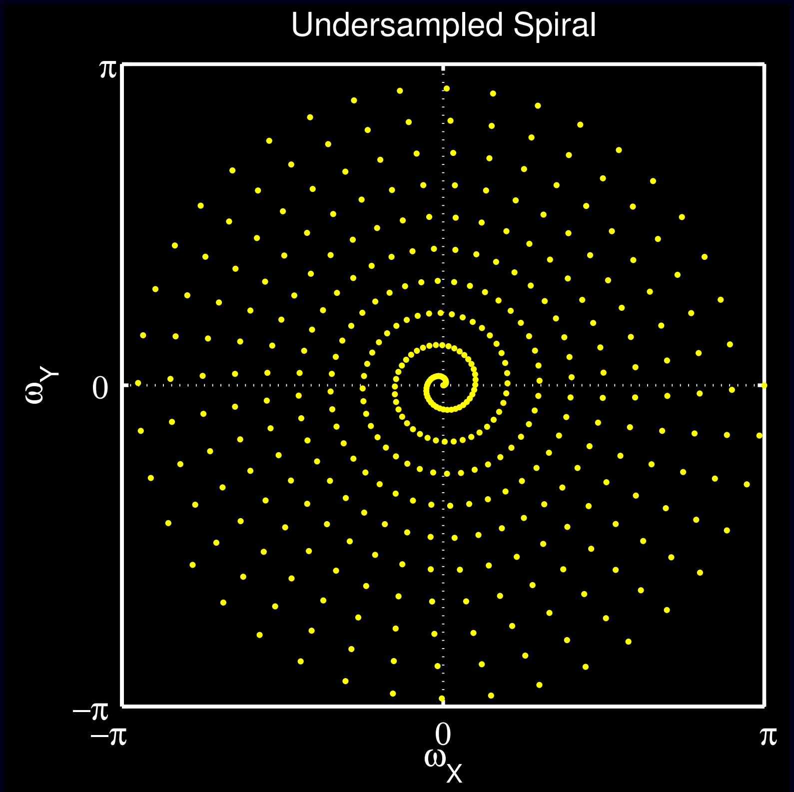
Diagonally preconditioned gradient descent:

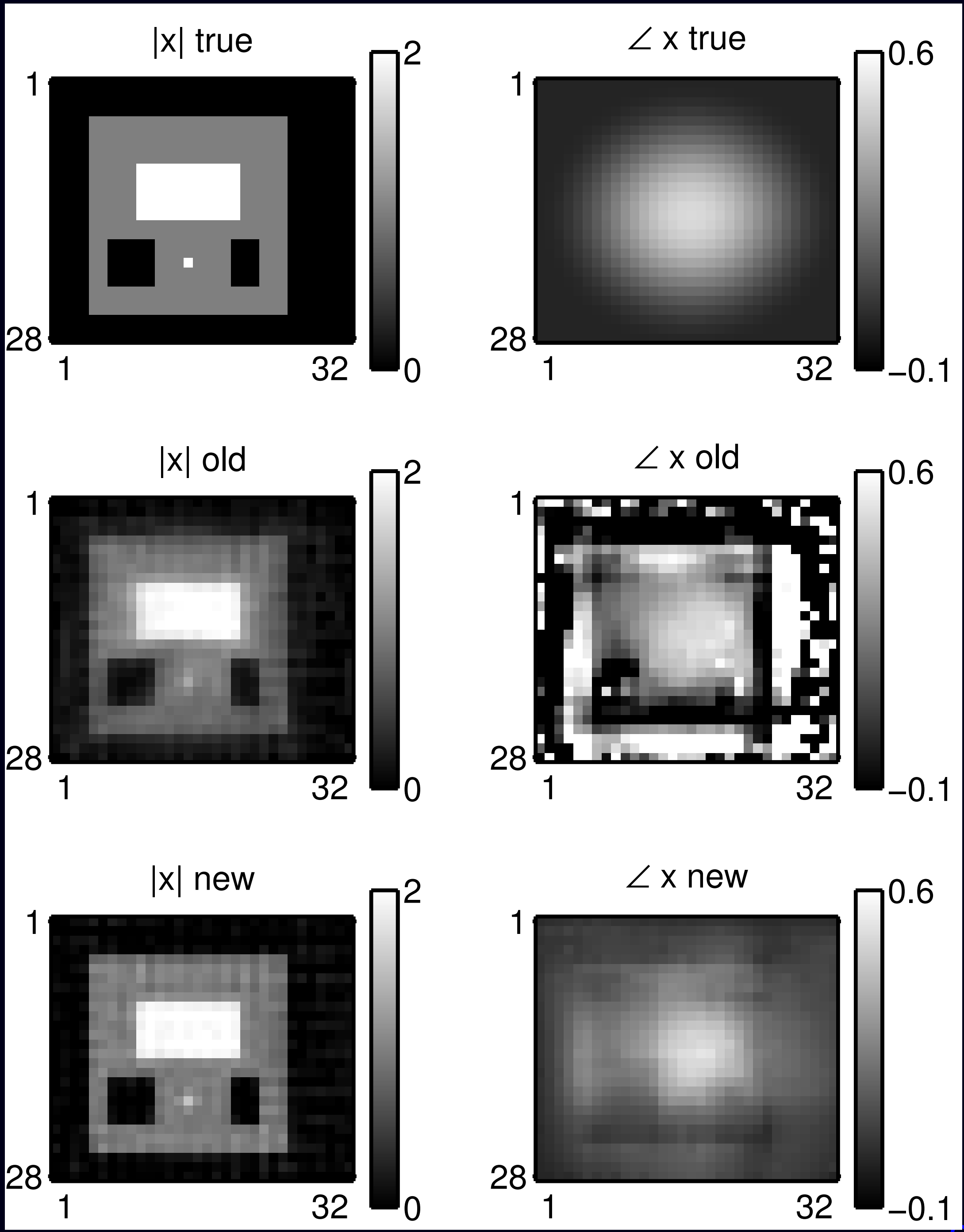
$$\boldsymbol{x}^{(n+1)} = \boldsymbol{x}^{(n)} - \boldsymbol{D}(\boldsymbol{x}^{(n)}) \nabla \Phi(\boldsymbol{x}^{(n)})$$

where the diagonal matrix $\boldsymbol{D}$ has elements that ensure Φ decreases monotonically.

Alternate between magnitude and phase updates...

Preliminary Simulation Example





Summary

- Iterative reconstruction: much potential in MRI
- Computation: reduced by tools like NUFFT / temporal interpolation; combined with careful optimization algorithm design
  cf. Shepp and Vardi, 1982, PET
- Problems involving phase terms $e^{\imath x}$ suitable for optimization transfer.

Future work

- Multiple receive coils (SENSE)
- Through-voxel field inhomogeneity gradients
- Motion (dynamic field maps...)
- Real data...