Image Registration using Constrained Optimization

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SIAM Imaging Science Conference
May 4, 2004

∗ This work was supported by NIH grant CA60711, PO1-CA59827 and RO1-CA81161.

Nonrigid Image Registration

- Estimating geometric transformation that aligns objects in two images

$$\hat{\theta} = \arg\max_{\theta \in \mathcal{K}} \Phi(A(T_\theta(\cdot)), B(\cdot)) - \beta \mathcal{R}(\theta),$$

where, $T_\theta : R^3 \longrightarrow R^3$ denotes a parametric nonrigid deformation model, $\Phi(A(\cdot), B(\cdot))$ is a similarity measure, $\mathcal{K}$ is a constraint set, β is a regularization parameter and $\mathcal{R}(\theta)$ is a penalty function.

Image Registration Problem

- Deformation model
- Similarity measure
- Penalty functions and/or constraint set
- Optimization methods (unconstrained or constrained)

Deformation Model using B-spline functions

- Deformation model using parameters $\theta = (\theta^x, \theta^y, \theta^z)$

$$T_\theta(x,y,z) = [x + f_{\theta^x}(x,y,z), y + g_{\theta^y}(x,y,z), z + h_{\theta^z}(x,y,z)], \quad (1)$$

$$f_{\theta^x}(x,y,z) = \sum_{ijk \in K_x} \theta^x_{ijk} \beta_3(\frac{x}{T_x} - i)\beta_3(\frac{y}{T_y} - j)\beta_3(\frac{z}{T_z} - k),$$

$$g_{\theta^y}(x,y,z) = \sum_{ijk \in K_y} \theta^y_{ijk} \beta_3(\frac{x}{T_x} - i)\beta_3(\frac{y}{T_y} - j)\beta_3(\frac{z}{T_z} - k),$$

$$h_{\theta^z}(x,y,z) = \sum_{ijk \in K_z} \theta^z_{ijk} \beta_3(\frac{x}{T_x} - i)\beta_3(\frac{y}{T_y} - j)\beta_3(\frac{z}{T_z} - k).$$

where, $\theta^x, \theta^y, \theta^z$ are the unknown coefficients, K_x, K_y, K_z are the sets of ”knot locations”, and T_x, T_y, T_z are expansion parameters.

Regularization and Invertibility

- The estimated deformation should be invertible.
- Jacobian determinants of the estimated deformation should be nonzero everywhere (inverse function theorem).
- Jacobian determinants should be positive by the continuity of the determinant (assuming there is a region without deformation).

Goals: Constrain θ to ensure positive Jacobian determinants

Existing Methods- Unconstrained Optimization with a Penalty Function

- Bending energy ['99 Rueckert *et al*.]
- Smoothness penalty ['03 Rohfling *et al*.]
- Exponential function of Jacobian determinant ['00 Kybic *et al*.]
  - Invertibility is not guaranteed.
  - Regularization parameter tuning is required.
  - Jacobian determinants between grid points can be negative even if those are positive at grid points.

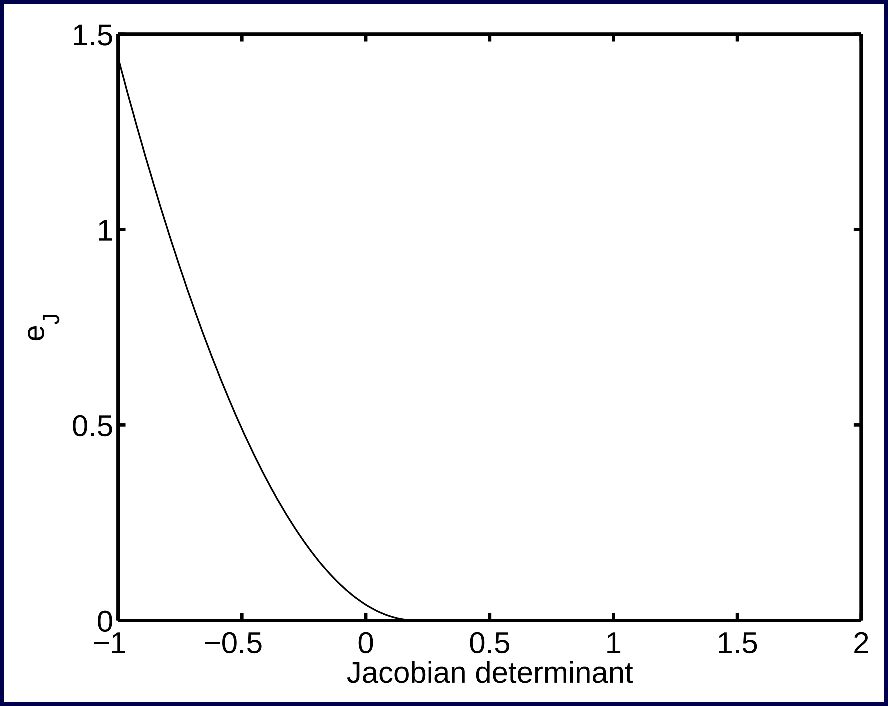
Penalty function

- Quadratically penalize Jacobian determinants smaller than threshold

$$E_J = \sum_{i,j,k} e_J(x_i, y_j, z_k), \tag{2}$$

$$e_J(x_i, y_j, z_k) = \begin{cases} 0 & \text{if } \det J(x,y,z) > J_t \\ (\det J(x_i, y_j, z_k) - J_t)^2 & \text{otherwise,} \end{cases}$$

where J_t is a threshold.



Existing Methods- Constrained Optimization subject to Constraints Ensuring Positive Jacobian Determinants

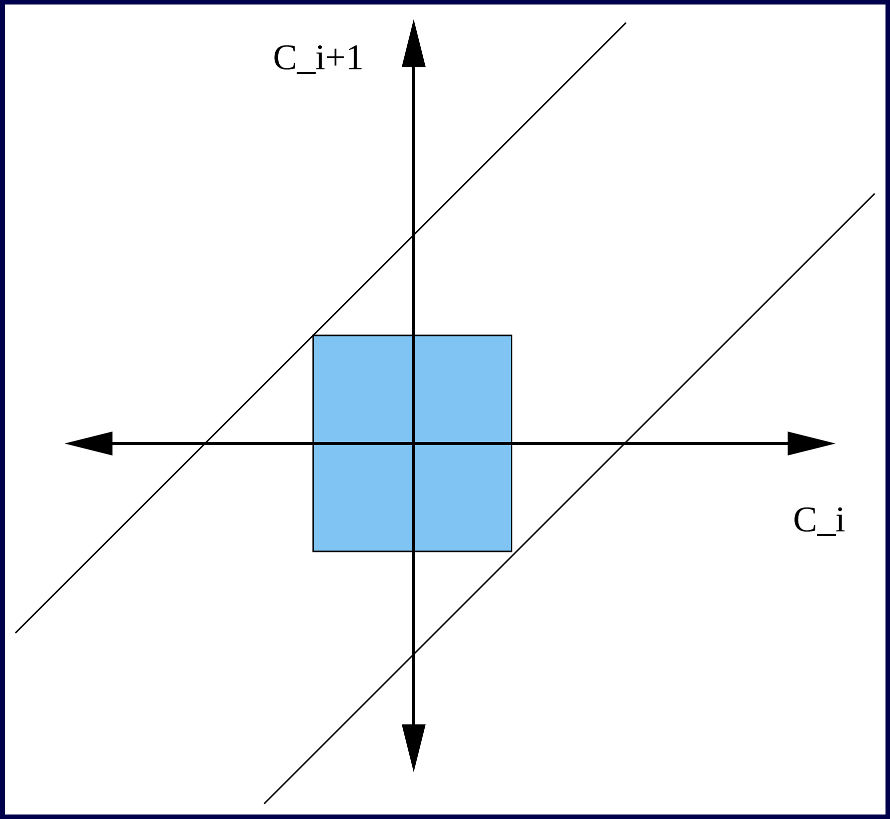
- Bounding gradients by 1/3 ensures positive Jacobian determinants
- Bound coefficients to bound gradients by 1/3 -['03 Rhode *et al.*]
  - Search space is too much restricted (large deformations with small gradients are precluded.)
  - Relationship between gradient bounds and Jacobian determinants bounds would be more desirable

Proposed Approach

- Relate Jacobian determinants (local volume change) bounds to displacement gradient bounds: Proposition 1
- Expand search space to include large deformation with small gradients by bounding differences between two neighboring coefficients: Proposition 2
- Constrained optimization subject to polyhedral constraints designed using Proposition 1 and 2

Search Space of 1D deformation

- Rhode’s constraint and proposed constraint



C_{i+1} and C_i are two neighboring coefficients.

Jacobian Determinants and Gradient Bounds

Proposition 1. Suppose that $\left|\frac{\partial f(x,y,z)}{\partial x}\right| \leq k_f$, $\left|\frac{\partial f(x,y,z)}{\partial y}\right| \leq k_f$, $\left|\frac{\partial f(x,y,z)}{\partial z}\right| \leq k_f$, $\left|\frac{\partial g(x,y,z)}{\partial x}\right| \leq k_g$, $\left|\frac{\partial g(x,y,z)}{\partial y}\right| \leq k_g$, $\left|\frac{\partial g(x,y,z)}{\partial z}\right| \leq k_g$ and $\left|\frac{\partial h(x,y,z)}{\partial x}\right| \leq k_h$, $\left|\frac{\partial h(x,y,z)}{\partial y}\right| \leq k_h$, $\left|\frac{\partial h(x,y,z)}{\partial z}\right| \leq k_h$, for $\forall x,y,z$. If $0 \leq k_f, k_g, k_h \leq \frac{1}{2}$, then $1-(k_f+k_g+k_h) \leq \det J(x,y,z) \leq (1+k_f)(1+k_g)(1+k_h)+(1+k_f)k_gk_h+(1+k_g)k_fk_h+(1+k_h)k_fk_g$.

- Derived using Kuhn-Tucker condition
- Rhode's result is a special case for minimum $\det J(x,y,z)$ when $k_f = k_g = k_h$.

Gradient Bounds and Constraints in Parameter Space

Proposition 2. If $\left|\theta^x_{i+1,j,k} - \theta^x_{i,j,k}\right| \leq b, \forall_{ijk} \in K_x$, then $\left|\frac{\partial f(x,y,z)}{\partial x}\right| \leq \frac{b}{T_x}$. Similarly, if $\left|\theta^x_{i,j+1,k} - \theta^x_{i,j,k}\right| \leq b, \forall_{ijk} \in K_x$, then $\left|\frac{\partial f(x,y,z)}{\partial y}\right| \leq \frac{b}{T_y}$ and $|\theta_{i,j,k+1} - \theta_{i,j,k}| \leq b, \forall_{ijk} \in K_z$ implies $\left|\frac{\partial f(x,y,z)}{\partial z}\right| \leq \frac{b}{T_z}$.

- Bounds on differences between two consecutive parameters (polyhedral convex set in parameter space).

Constrained Optimization

- Combining proposition 1 and 2 leads to polyhedral constraint set that bounds Jacobian determinants

$$\mathcal{H}_i = \{\theta \in X \,|\, \langle \theta, f_i \rangle \le c_i\}, \quad i = 1, \ldots, r \tag{3}$$

$$\mathcal{K} = \bigcap_{i=1}^{r} \mathcal{H}_i, \tag{4}$$

where, X is the parameter space, r is the number of constraints, f_i and c_i are appropriate vectors and scalars.

- Proposed image registration method

$$\hat{\theta} = \arg\max_{\theta \in \mathcal{K}} \Phi(A(T_\theta(\cdot)), B(\cdot)), \tag{5}$$

Gradient Projection Method

- Gradient projection method

$$\theta^{n+1} = P_{\mathcal{K}}(\theta^n - \alpha \nabla_\theta \Phi(A, B; \theta)), \tag{6}$$

where, $\mathcal{K}$ is the convex constraint set and $P_{\mathcal{K}}$ denotes the orthogonal projection onto the convex set $\mathcal{K}$.

- Convergence is guaranteed, if α is chosen appropriately.
- In general, determining $P_{\mathcal{K}}$ is challenging.

Dykstra's Cyclic Projection Method

- Projection onto the intersection of convex sets can be computed by cyclic projections onto the convex sets
- Computing a projection onto a half space is easy.
- Dykstra's algorithm converges to $P_{\mathcal{K}}$ geometrically.

Inhale/exhale Lung CT Registration

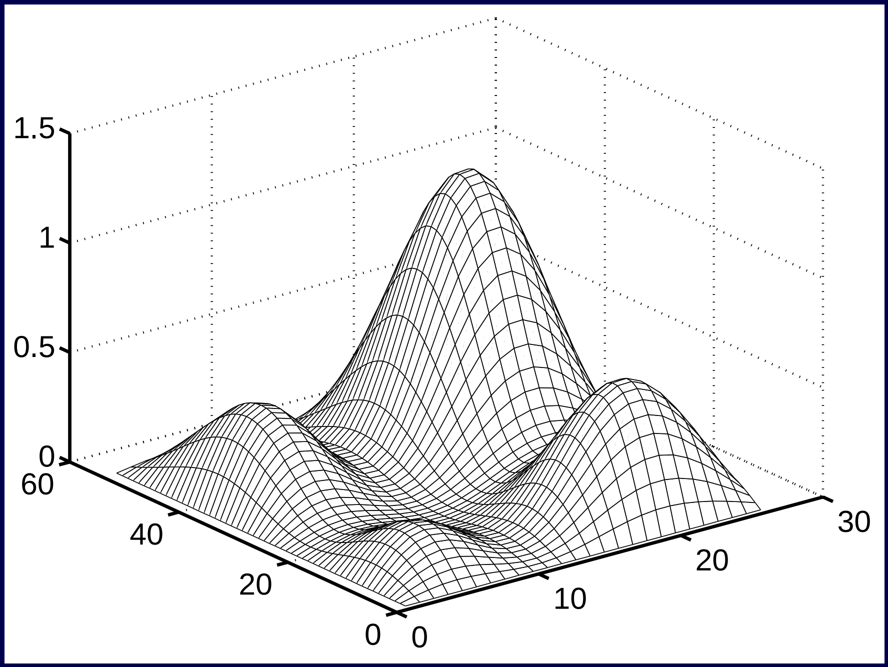
- Inhale/exhale CT images ($64 \times 36 \times 10$)
- Two synthetic deformations using sinusoidal basis function
- Constrained optimization method (gradient bound 1/3)
- Penalty based method using Jacobian determinants

Inhale CT image

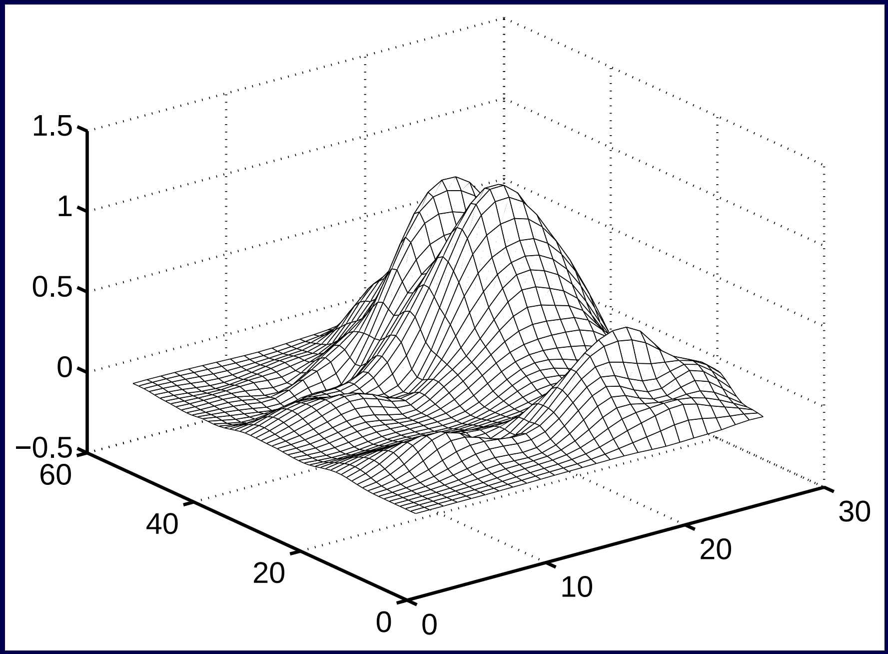
Exhale CT image

Simulation Results

- Number of B-splines: $30 \times 16 \times 8 \times 3$
- X-axis deformation evaluated at one slice



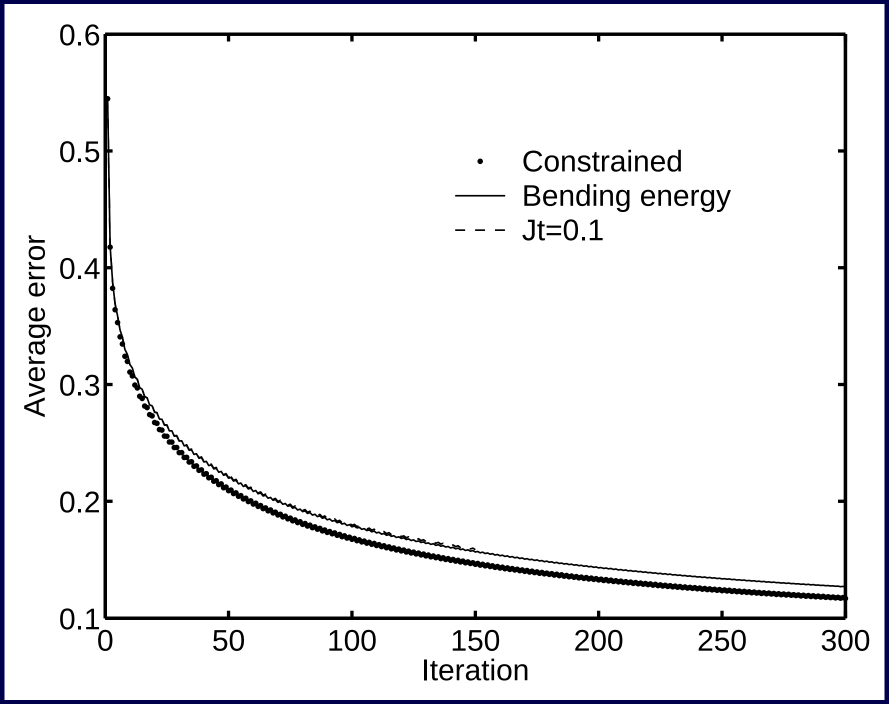
$f(x,y,z)$ evaluated at one slice



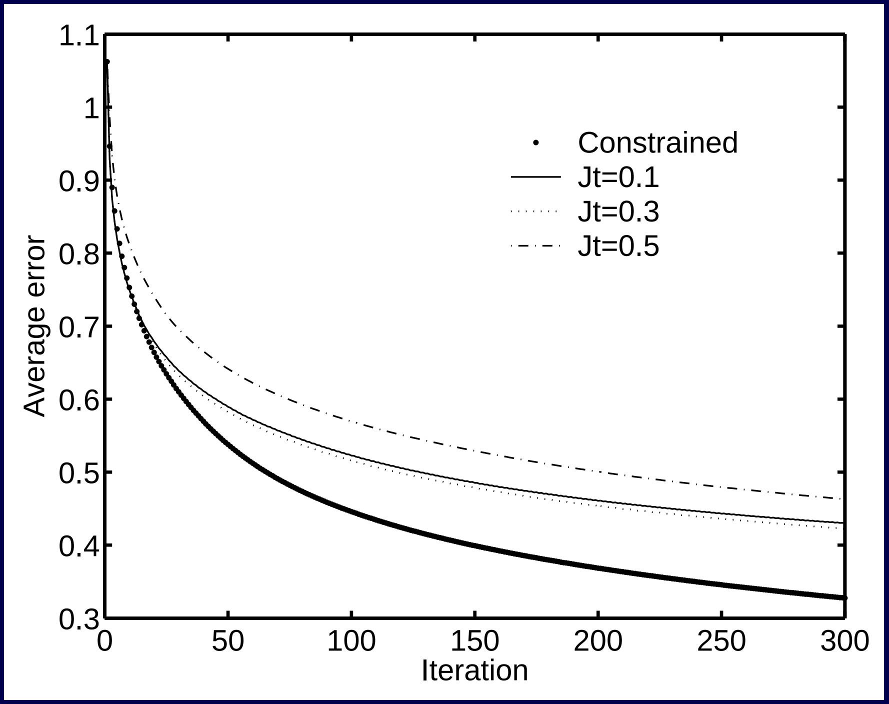
$\hat{f}(x,y,z)$ at the same slice

Simulation Results

- Two synthetic deformations: small and large gradient



Average error for low gradient deformation



Average error for high gradient deformation

Simulation Results

Characteristics of the estimated deformations

	Synthetic deformation 1	Proposed	E_J penalty	Synthetic deformation 2	Proposed	E_J penalty
$\min\|J\|$	0.807	0.648	0.231	0.201	0.433	0.029
$\max\|J\|$	1.324	1.398	1.939	1.457	1.912	3.887
$\max\left\|\frac{\partial f(x,y,z)}{\partial x}\right\|$	0.251	0.196	0.446	0.199	0.303	0.893
$\max\left\|\frac{\partial f(x,y,z)}{\partial y}\right\|$	0.229	0.204	0.264	0.551	0.328	1.547
$\max\left\|\frac{\partial f(x,y,z)}{\partial z}\right\|$	0.269	0.318	0.473	0.319	0.319	2.034
$\max\left\|\frac{\partial g(x,y,z)}{\partial x}\right\|$	0.192	0.174	0.321	0.445	0.331	3.002
$\max\left\|\frac{\partial g(x,y,z)}{\partial y}\right\|$	0.224	0.224	0.389	0.317	0.242	3.616
$\max\left\|\frac{\partial g(x,y,z)}{\partial z}\right\|$	0.329	0.323	0.586	0.609	0.333	2.442
$\max\left\|\frac{\partial h(x,y,z)}{\partial x}\right\|$	0.109	0.163	0.162	0.233	0.241	0.598
$\max\left\|\frac{\partial h(x,y,z)}{\partial y}\right\|$	0.201	0.191	0.212	0.204	0.276	0.889
$\max\left\|\frac{\partial h(x,y,z)}{\partial z}\right\|$	0.287	0.310	0.672	0.786	0.333	0.867

Experimental Results

- Inhale/Exhale CT registration for 8 patients
- Optimization parameter is tuned for PT01 (Registration after 150 iterations).

Lung CT registration results

(ρ is correlation coefficient between images)

	PT01	PT02	PT03	PT04	PT05	PT06	PT07	PT08
ρ before registration	0.701	0.678	0.852	0.722	0.888	0.755	0.956	0.930
ρ after registration	0.981	0.964	0.978	0.970	0.979	0.935	0.970	0.963
$\min\|J\|$	0.332	0.277	0.444	0.295	0.337	0.180	0.428	0.413
$\max\|J\|$	2.323	2.477	2.089	2.176	2.269	2.395	2.103	2.023

Summary

- Jacobian determinant penalty method yielded larger gradient deformation than truth.
- Different regularization parameters were required for different images.
- Proposed method performed well but required additional computation.

Future Work

- *A priori* information about gradient and Jacobian bound would be desirable.
- How to validate the estimated deformation in practice?
- How to remove manual tuning procedure for optimization?
- Comparison study with interior point methods for optimization