Analytical Approach to Regularization Design for Isotropic Spatial Resolution

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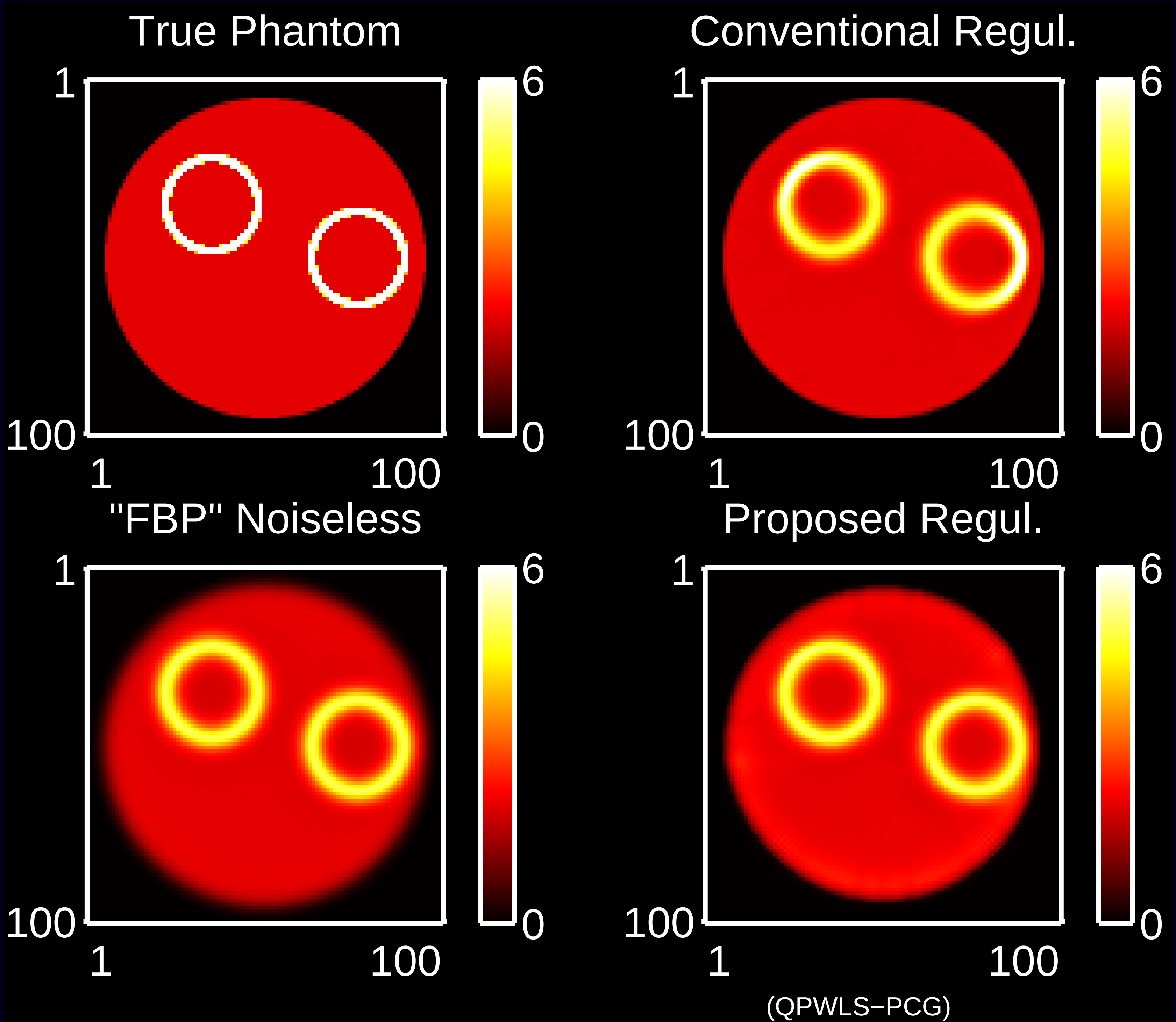
Motivation



True Phantom
Conventional Regul.
"FBP" Noiseless
Proposed Regul.
1
100
1
100
6
0
(QPWLS−PCG)

History

- 1994 MIC, Fessler and Rogers
  - Uniform quadratic penalties cause nonuniform image resolution
  - Simple “certainty-based” correction for shift-invariant systems
- 1998 ICIP, Stayman and Fessler
  - Improved regularization design for shift-invariant systems, compensating for anisotropy of local PSF
- 1999 Fully 3D
  - Qi and Leahy: design for uniform pixel contrast
  - Stayman and Fessler: design for 3D shift-invariant systems
- 2001 MIC, Stayman and Fessler
  - Improved (but complicated) design allowing negative weights
- 2002 MIC
  - Stayman and Fessler: faster method for space varying systems
  - Nuyts and Fessler: simplified design

All based on matrix analysis!

Local Impulse Response

- Noisy measurement vector $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \text{noise}$
  - $\boldsymbol{y}$: measured projection data
  - $\boldsymbol{A}$: system matrix
  - $\boldsymbol{x}$: unknown image pixel values to reconstruct
- General image reconstruction method: $\hat{\boldsymbol{x}} = \hat{\boldsymbol{x}}(\boldsymbol{y})$
- Local impulse response for jth pixel:

$$\boldsymbol{l}^j = \lim_{\delta \to 0} \frac{\hat{\boldsymbol{x}}(\boldsymbol{y} + \delta \boldsymbol{A}\boldsymbol{e}^j) - \hat{\boldsymbol{x}}(\boldsymbol{y})}{\delta}$$

  $\boldsymbol{e}^j$ = point source in jth pixel
  "How does a small impulse in the jth pixel affect other pixels?"
- Useful for design of regularized reconstruction methods

Goal. Design the estimator $\hat{\boldsymbol{x}}$ to have good noise properties and spatial resolution properties that are isotropic and uniform, or ...

Penalized-Likelihood Reconstruction

Regularized estimator:

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} L(\boldsymbol{A}\boldsymbol{x}, \boldsymbol{y}) + R(\boldsymbol{x})$$

- $\boldsymbol{x}$: unknown image pixel values to reconstruct
- $\boldsymbol{y}$: measured projection data
- $\boldsymbol{A}$: system matrix
- L : negative log-likelihood (*e.g.*, Poisson statistical model)
- $R(\boldsymbol{x})$: quadratic regularizing roughness penalty $R(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}'\boldsymbol{R}\boldsymbol{x}$
  $\boldsymbol{R}$ is the Hessian of the penalty function $R(\boldsymbol{x})$

Local impulse response:

$$\boldsymbol{l}^j = [\boldsymbol{A}'\boldsymbol{W}\boldsymbol{A} + \boldsymbol{R}]^{-1}\boldsymbol{A}'\boldsymbol{W}\boldsymbol{A}\boldsymbol{e}^j$$

$\boldsymbol{W}$ depends on the log-likelihood and $\boldsymbol{y}$, *e.g.*, $\boldsymbol{W} = \mathrm{diag}\{1/y_i\}$.

This matrix form has been the foundation of most previous methods!

Local Discrete Fourier Approximations

Let $\boldsymbol{Q}$ denote the DFT matrix for image domain.

Local system frequency response:

$$\boldsymbol{A}'\boldsymbol{W}\boldsymbol{A}\boldsymbol{e}^j \approx \boldsymbol{Q}\,\mathrm{diag}\left\{\lambda_k^j\right\}\boldsymbol{Q}'\boldsymbol{e}^j, \qquad \boldsymbol{\lambda}^j = \mathrm{FFT}\left\{\boldsymbol{A}'\boldsymbol{W}\boldsymbol{A}\boldsymbol{e}^j\right\}$$

Local regularization frequency response:

$$\boldsymbol{R}\boldsymbol{e}^j \approx \boldsymbol{Q}\,\mathrm{diag}\left\{\omega_k^j\right\}\boldsymbol{Q}'\boldsymbol{e}^j, \qquad \boldsymbol{\omega}^j = \mathrm{FFT}\left\{\boldsymbol{R}\boldsymbol{e}^j\right\}$$

Local impulse response with local Fourier approximation:

$$\boldsymbol{l}^j = [\boldsymbol{A}'\boldsymbol{W}\boldsymbol{A}+\boldsymbol{R}]^{-1}\boldsymbol{A}'\boldsymbol{W}\boldsymbol{A}\boldsymbol{e}^j \approx \boldsymbol{Q}\,\mathrm{diag}\left\{\frac{\lambda_k}{\lambda_k+\omega_k}\right\}\boldsymbol{Q}'\boldsymbol{e}^j$$

Useful for design of the regularizer $\boldsymbol{R}$, but requires FFTs for every pixel.

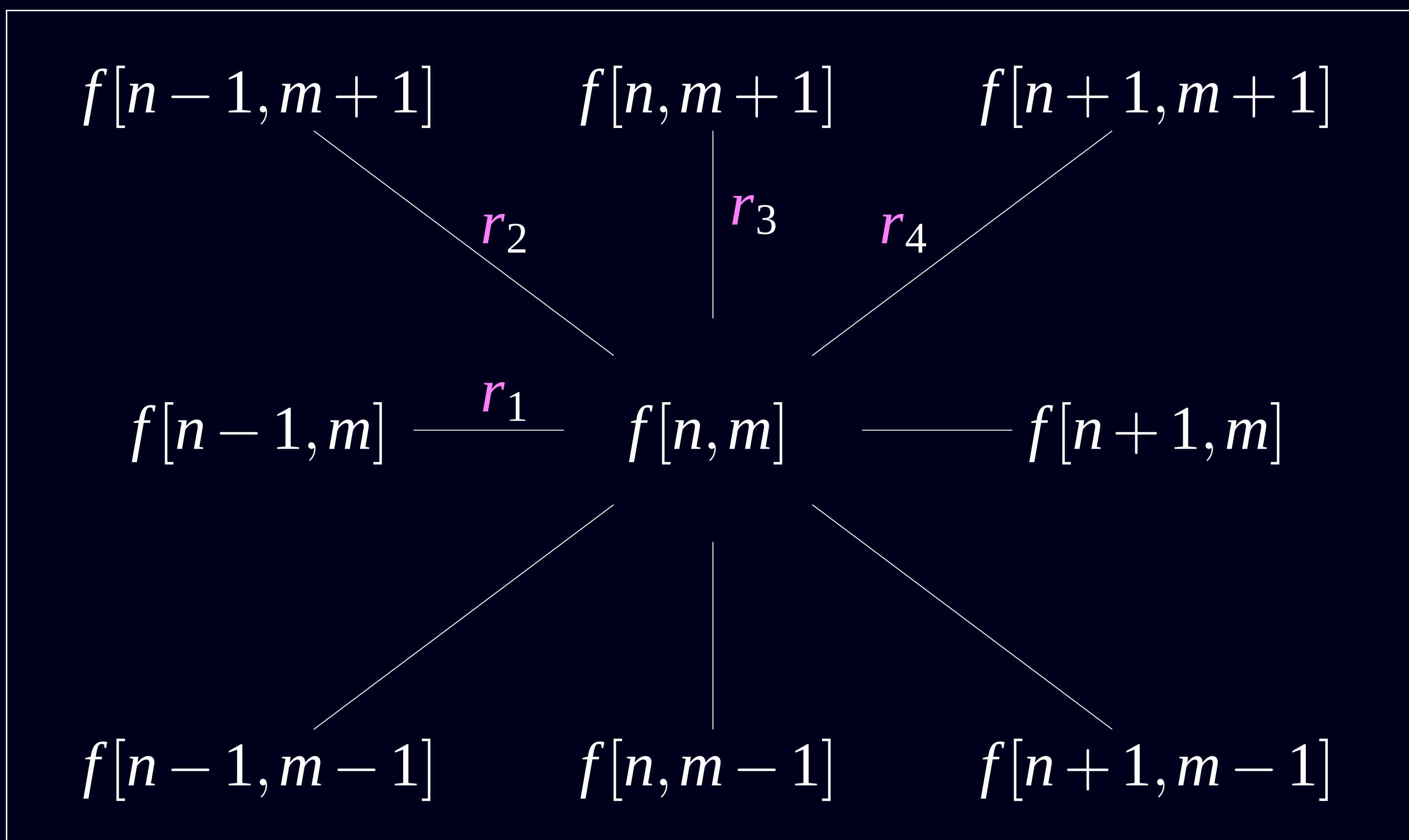
(And forward- / back-projections for each pixel for shift varying systems.)

Position-Dependent Regularization

$$R(f) = \sum_{n,m} \begin{array}{l} r_1^{(n,m)} \left| f[n,m] - f[n-1,m-0] \right|^2 + \\ r_2^{(n,m)} \left| f[n,m] - f[n-1,m+1] \right|^2 + \\ r_3^{(n,m)} \left| f[n,m] - f[n+0,m+1] \right|^2 + \\ r_4^{(n,m)} \left| f[n,m] - f[n+1,m-1] \right|^2 \end{array}$$

$\boldsymbol{r}^j = (r_1, \ldots, r_4)$: 4 penalty coefficients per pixel.

Conventional regularizer: $r_1 = r_3 = 1$, $r_2 = r_4 = 1/\sqrt{2}$.



Linearized Regularization Design

Goal: choose $\boldsymbol{R}$ (*i.e.*, $\{\boldsymbol{r}^j\}$) such that the resulting local impulse response $\boldsymbol{l}^j$ approximates some desired target PSF.

Natural target PSF is from unweighted penalized least-squares:

$$\boldsymbol{l}^j = \underbrace{[\boldsymbol{A}'\boldsymbol{W}\boldsymbol{A} + \boldsymbol{R}]^{-1}\boldsymbol{A}'\boldsymbol{W}\boldsymbol{A}\boldsymbol{e}^j}_{\text{Local impulse resp.}} \approx \underbrace{[\boldsymbol{A}_0'\boldsymbol{A}_0 + \boldsymbol{R}_0]^{-1}\boldsymbol{A}_0'\boldsymbol{A}_0\boldsymbol{e}^j}_{\text{Target PSF}}.$$

Nonlinear in $\boldsymbol{R}$ $\Rightarrow$ complicated design.

Linearize by “cross multiplying:”

$$[\boldsymbol{A}_0'\boldsymbol{A}_0 + \boldsymbol{R}_0]\,\boldsymbol{A}'\boldsymbol{W}\boldsymbol{A}\boldsymbol{e}^j \approx [\boldsymbol{A}'\boldsymbol{W}\boldsymbol{A} + \boldsymbol{R}]\,\boldsymbol{A}_0'\boldsymbol{A}_0\boldsymbol{e}^j.$$

Simplify using “local shift invariance” approximations:

$$\boldsymbol{R}_0\boldsymbol{A}'\boldsymbol{W}\boldsymbol{A}\boldsymbol{e}^j \approx \boldsymbol{R}\boldsymbol{A}_0'\boldsymbol{A}_0\boldsymbol{e}^j.$$

“Linearized regularization design” (still with matrices):

$$\min_{\boldsymbol{R}\in\mathcal{R}} \left\| \boldsymbol{R}_0\boldsymbol{A}'\boldsymbol{W}\boldsymbol{A}\boldsymbol{e}^j - \boldsymbol{R}\boldsymbol{A}_0'\boldsymbol{A}_0\boldsymbol{e}^j \right\|.$$

Analytical Regularization Design

Matrix approach: $\min_{\boldsymbol{R} \in \mathcal{R}} \left\| \boldsymbol{R}_0 \boldsymbol{A}' \boldsymbol{W} \boldsymbol{A} \boldsymbol{e}^j - \boldsymbol{R} \boldsymbol{A}_0' \boldsymbol{A}_0 \boldsymbol{e}^j \right\|$

Key idea: replace 4 matrices with analytical Fourier approximations.

1. Nominal system transfer function

$$\boldsymbol{A}_0' \boldsymbol{A}_0 \equiv \frac{|B(\rho)|^2}{\rho}$$

- (ρ, φ) : polar coordinates in frequency space
- $B(\rho)$: "typical" detector frequency response

2. Weighted system transfer function

$$\boldsymbol{A}' \boldsymbol{W} \boldsymbol{A} \equiv \frac{w^j(\varphi) \left| B_\varphi^j(\rho) \right|^2}{\rho}$$

- $B_\varphi^j(\rho)$: detector response at projection angle φ for jth pixel
- $w^j(\varphi)$: angular weighting (certainty) for jth pixel (from $\boldsymbol{W}$)

Analytical Regularization Design

3. Isotropic 1st-order roughness: $R_0(f) = \int \|\nabla f\|^2$

$$\boldsymbol{R}_0 \equiv |2\pi\rho|^2$$

4. Local roughness penalty (simplified)

$$R(f) = \sum_{n,m} \sum_{l=1}^{L} r_l \frac{1}{2} \left| f[n,m] - f[n-n_l, m-m_l] \right|^2$$

Penalty coefficients $\boldsymbol{r}^j = (r_1, \ldots, r_L)$ to be designed (for each pixel).

After some Fourier analysis...:

$$\boldsymbol{R} \equiv (2\pi\rho)^2 \sum_{l=1}^{L} r_l \cos^2(\varphi - \varphi_l),$$

$$\varphi_l \triangleq \tan^{-1} \frac{m_l}{n_l}$$

$\varphi_l = (0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4})$ for $L = 4$

(Each penalty coefficient influences PSF shape along some direction.)

Analytical Regularization Design

Rewrite the “matrix” minimization using the 4 Fourier approximations.
Simplifying yields the following matrix-free design criterion:

$$\boldsymbol{r}^j = \arg\min_{\boldsymbol{r} \succeq \boldsymbol{0}} \int_0^\pi \left| w^j(\varphi) - \sum_{l=1}^L r_l \cos^2(\varphi - \varphi_l) \right|^2 d\varphi$$

$w^j(\varphi)$: angular “certainty” weighting for jth pixel, from data statistics.
$\cos^2(\varphi - \varphi_l)$: angular contribution for lth penalty direction.

No matrix inverses (*cf.* analytical $1/\rho$).

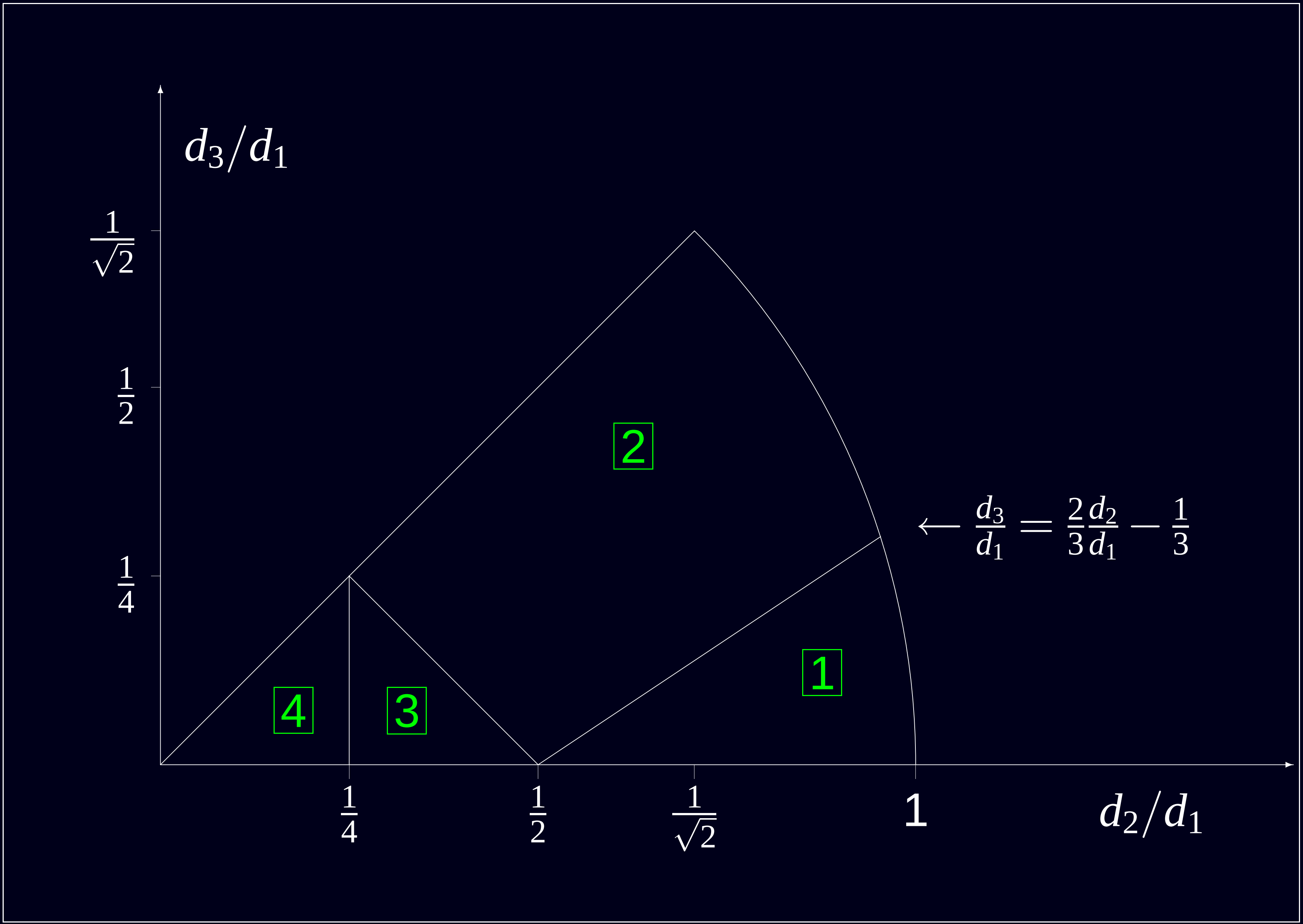
For 2nd-order neighborhood ($L = 4$), exact closed-form solution.
(No NNLS iterations needed.)

Solution requires just three sums (over projection angle) per pixel:

$$\begin{bmatrix} d_1^j \\ d_2^j \\ d_3^j \end{bmatrix} = \begin{bmatrix} \frac{1}{\pi}\int_0^\pi w^j(\varphi)\, d\varphi \\ \frac{1}{\pi}\int_0^\pi w^j(\varphi) \cos(2\varphi)\, d\varphi \\ \frac{1}{\pi}\int_0^\pi w^j(\varphi) \sin(2\varphi)\, d\varphi \end{bmatrix}$$

“average”
“0 and $\frac{\pi}{2}$”
“$\frac{\pi}{4}$ and $\frac{3\pi}{4}$”

Eight-fold symmetry



Analytical solution

Four penalty coefficients per pixel for 2nd-order neighborhood:

1

$$r_1 = \frac{4}{3}(d_1 + d_2), \qquad r_2 = r_3 = r_4 = 0$$

2

$$r_1 = \frac{8}{5}\left[\frac{1}{2}d_1 + \frac{3}{2}d_2 - d_3\right], \quad r_3 = \frac{12}{5}\left[d_3 - \left(\frac{2}{3}d_2 - \frac{1}{3}d_1\right)\right], \quad r_2 = r_4 = 0$$

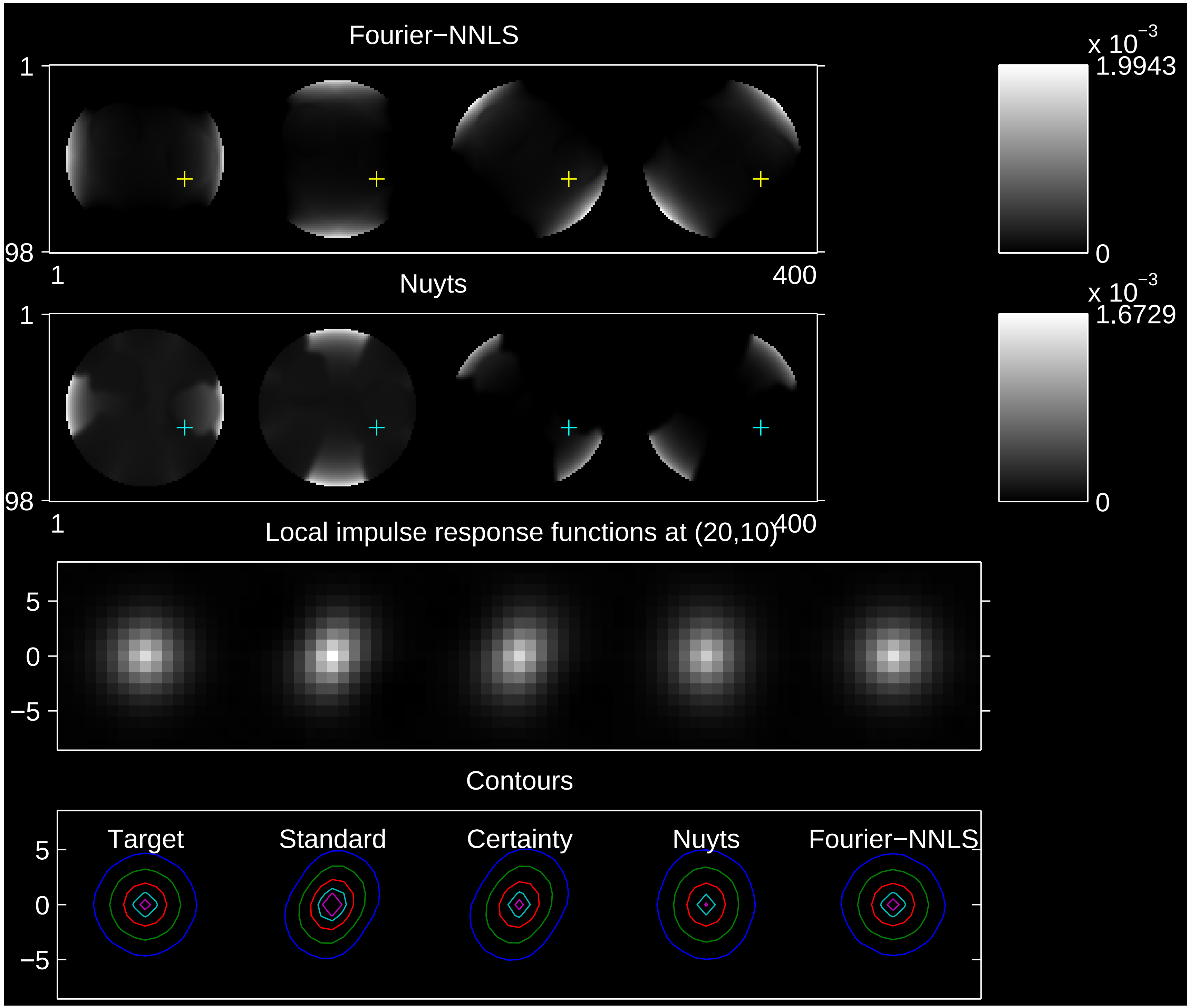
3

$$r_1 = 4d_2, \quad r_2 = 0, \quad r_3 = d_1 - 2d_2 + 2d_3, \quad r_4 = 2\left[\frac{1}{2}d_1 - (d_2 + d_3)\right]$$

4

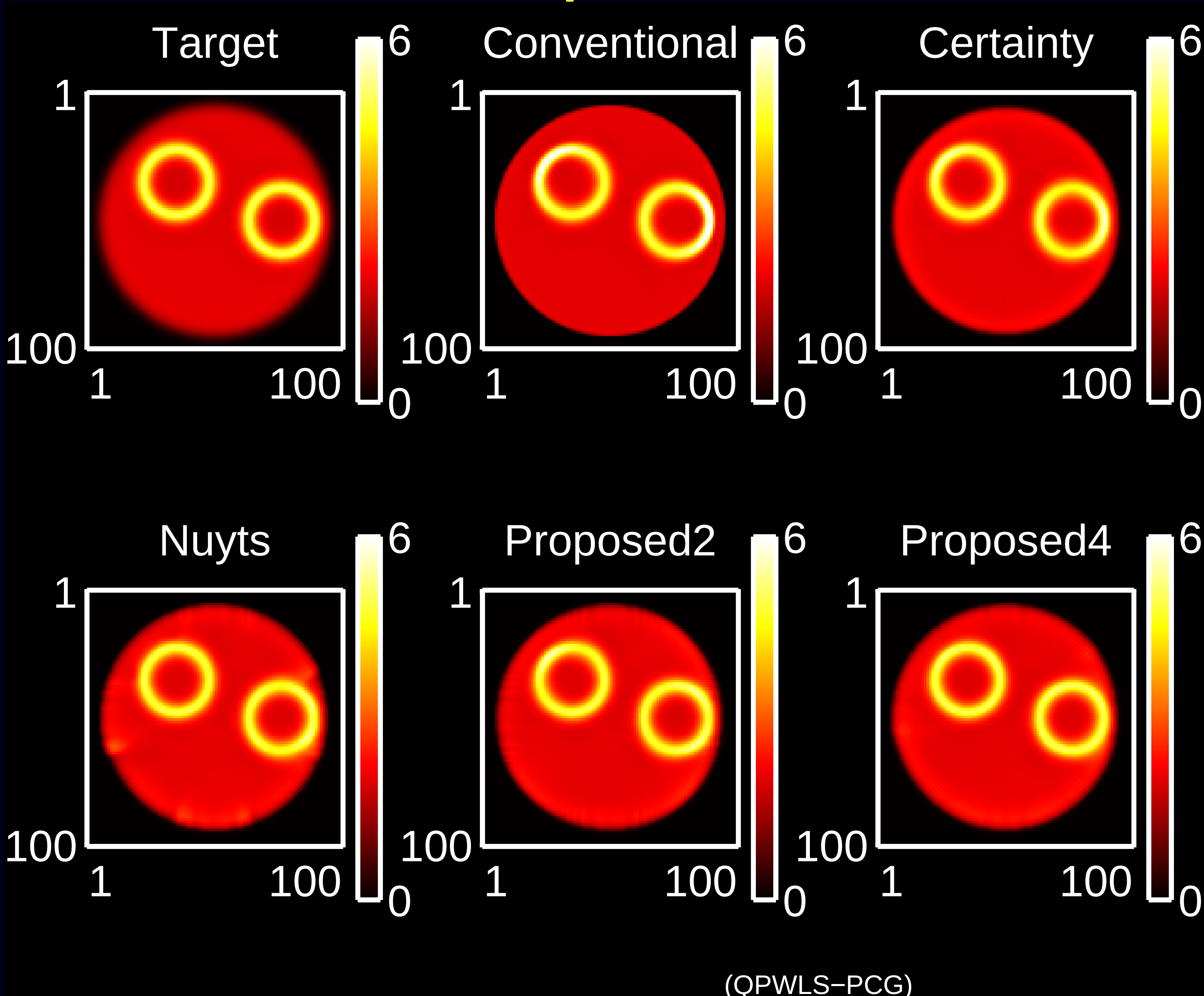
$$r_1 = 2\left(\tfrac{1}{4}d_1 + d_2\right), \quad r_2 = 2\left(\tfrac{1}{4}d_1 - d_2\right)$$
$$r_3 = 2\left(\tfrac{1}{4}d_1 + d_3\right), \quad r_4 = 2\left(\tfrac{1}{4}d_1 - d_3\right)$$

Example



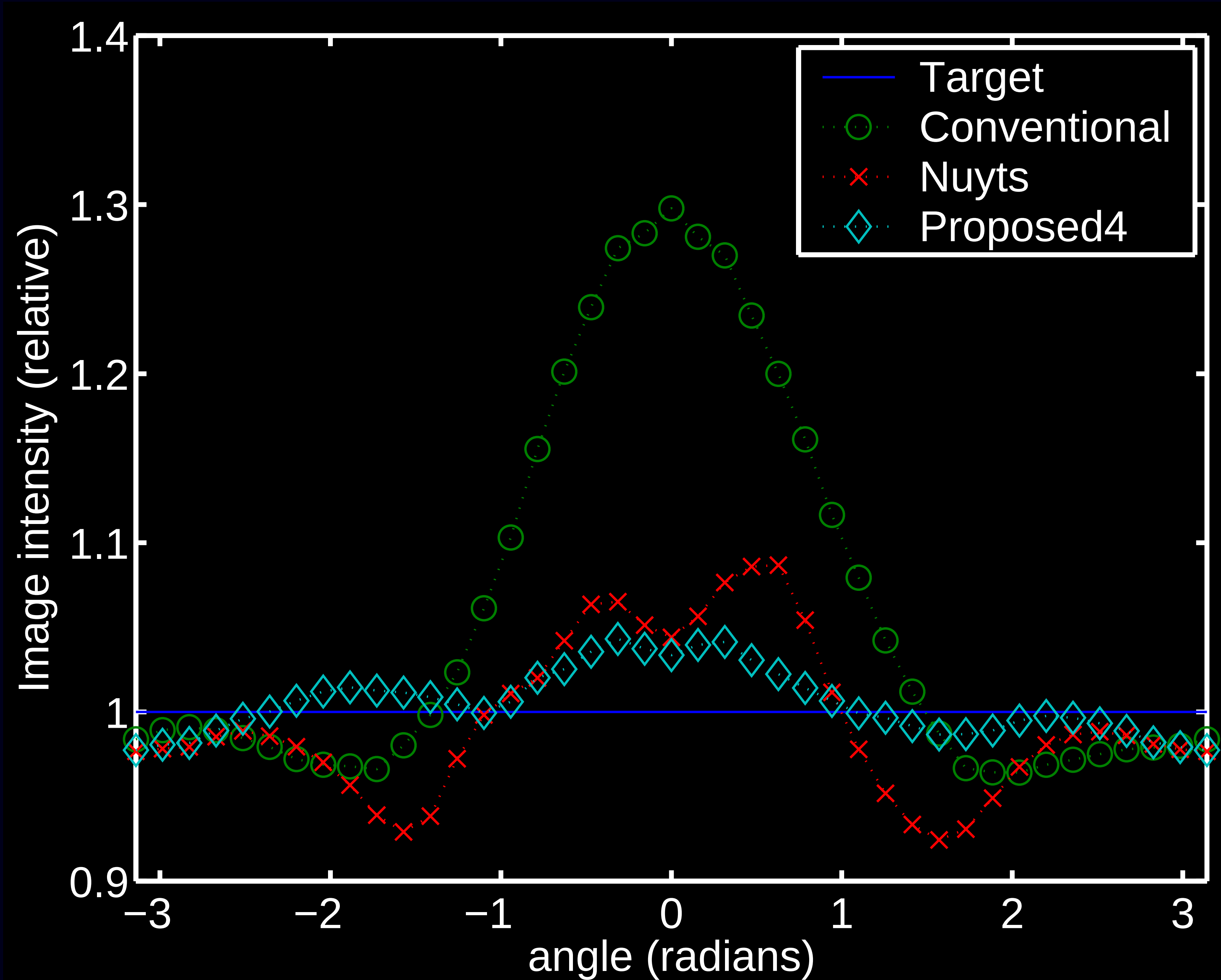
Fourier−NNLS
1
98
1
400
x 10^{-3}
1.9943
0
Nuyts
1
98
1
400
x 10^{-3}
1.6729
0
Local impulse response functions at (20,10)
5
0
−5
Contours
5
0
−5
Target
Standard
Certainty
Nuyts
Fourier−NNLS

Comparison



Target
Conventional
Certainty
Nuyts
Proposed2
Proposed4
1
100
1
100
6
0
(QPWLS−PCG)

Ring Profiles



Summary

- Simple, fast, effective regularization design for uniform, isotropic spatial resolution
- Analogy to FBP: solve first, discretize second. (*cf.* Fourier $(1/\rho)^{-1} = \rho$ versus matrix $[\boldsymbol{A}_0' \boldsymbol{A}_0]^{-1}$)
- Recommendation: combine modest regularization with post-filtering
- Extends to 3D and to shift-variant systems. Requires somewhat more computation for designing the regularizer, but is still more practical than alternatives.
- Analytical approximations also applicable to variance/autocorrelation predictions.
- Non-quadratic edge-preserving regularizers for transmission case?
- Matlab tomography toolbox:

  `http://www.eecs.umich.edu/~fessler`

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Anti-Acknowledgement

No portion of this research was performed using software from the Microsoft monopoly.