Reconstruction from Digital Holograms by Statistical Methods

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Outline

- Background on holography
- Image model
- Conventional numerical reconstruction
- Statistical holographic reconstruction
- Simulation results
- Conclusions and future work

Conventional Holography

- "Record both amplitude and phase of a wave field"
- Hologram = recorded interference pattern
  between an object beam and a reference beam

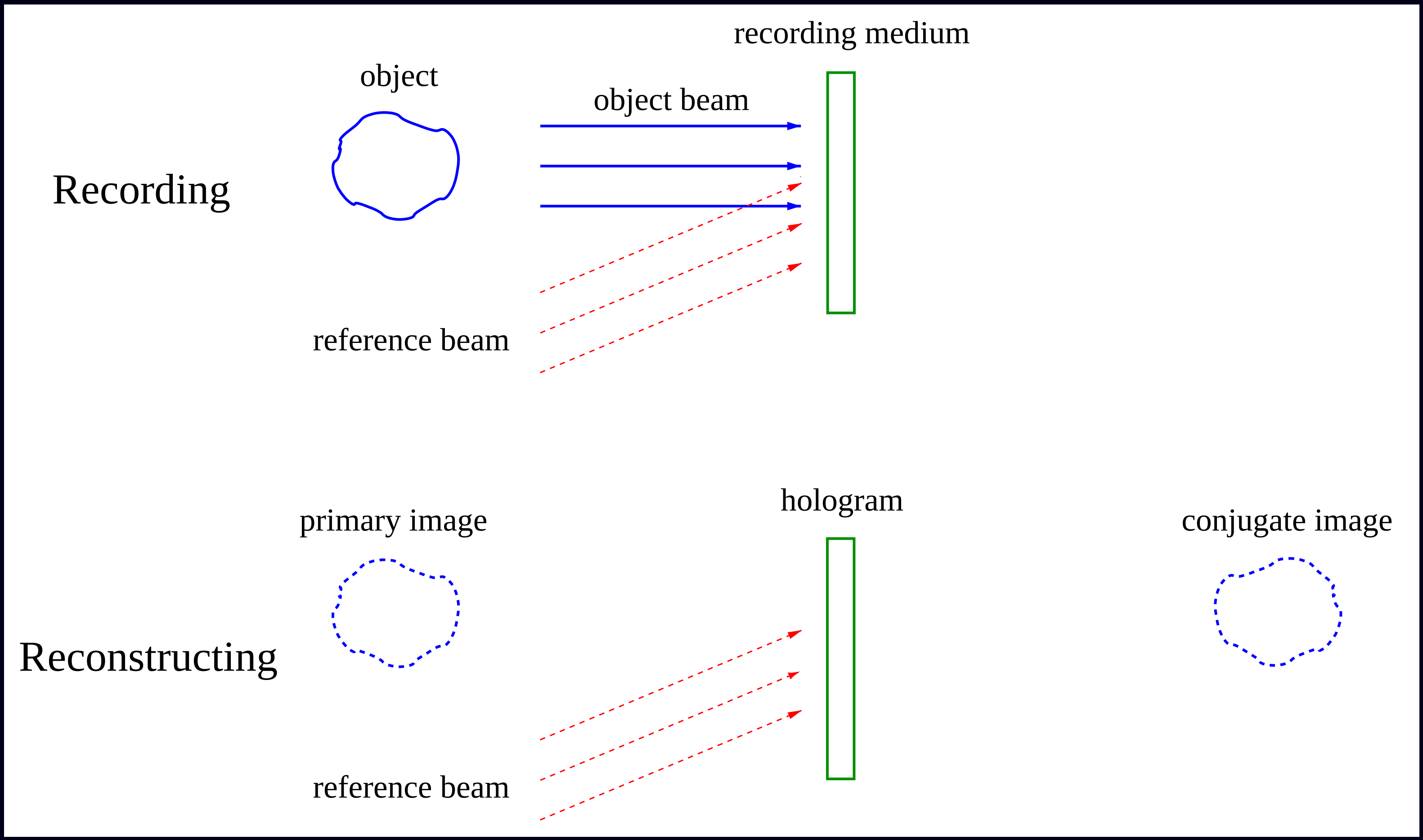
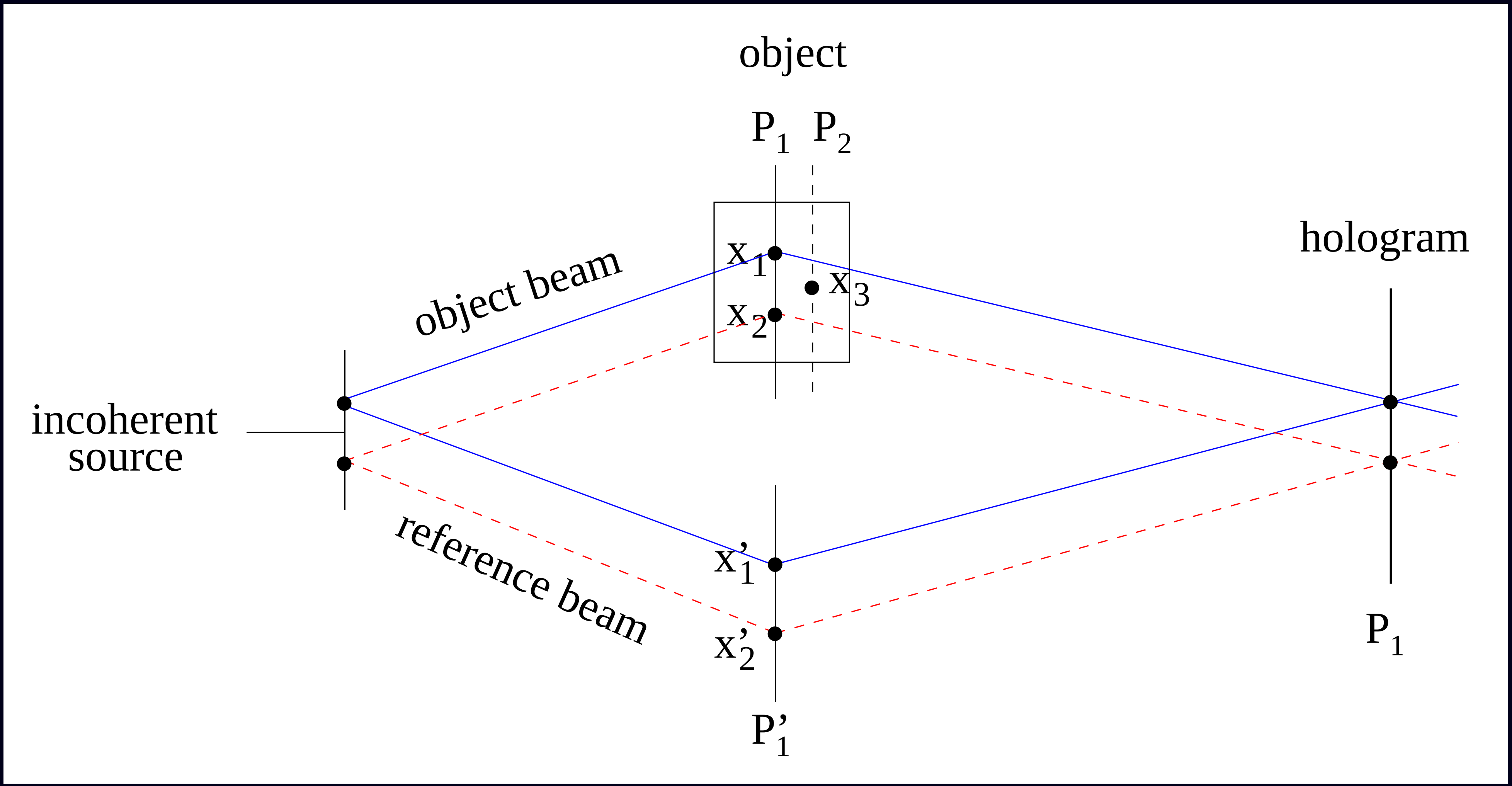


Image Plane Holography

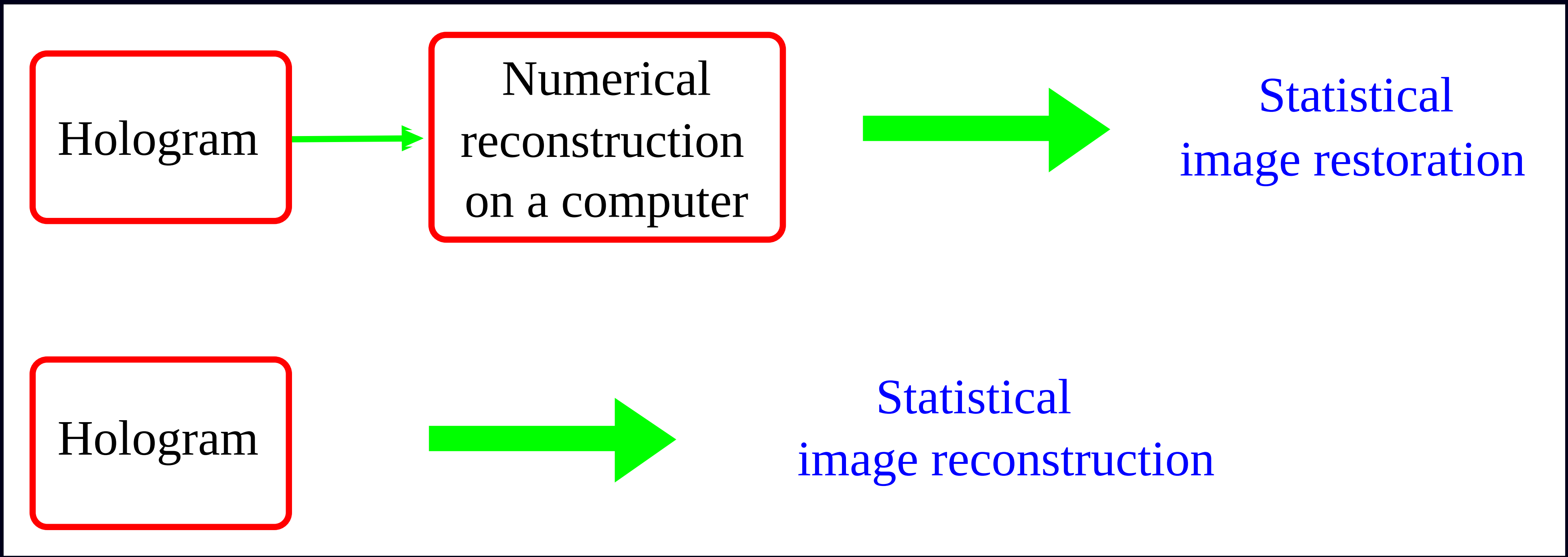


- Optical sectioning property as in confocal microscopy
- No *xy* scanning
- Larger source decreases coherence and improves spatial resolution

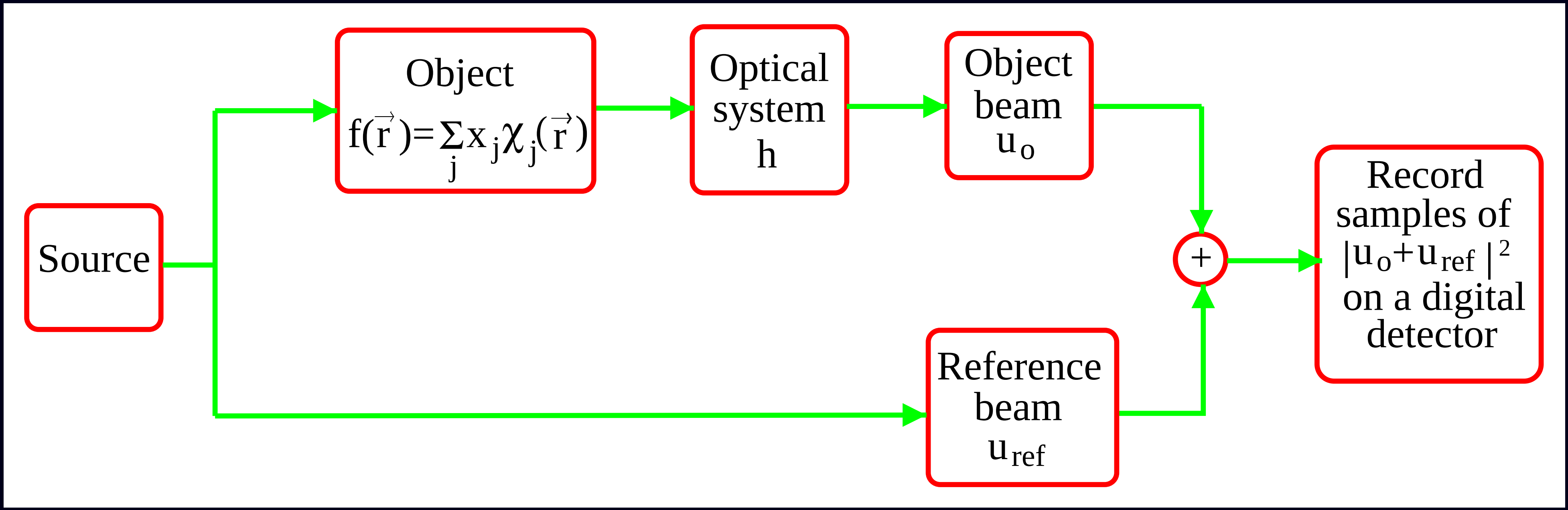
Challenges

- Finite source size limits spatial resolution
- Optical reconstruction inconvenient

$\Rightarrow$ Use statistical image reconstruction



Measurement Model



- Object beam, superposition: $u_o(\vec{r}') = \int h(\vec{r}', \vec{r}) \, f(\vec{r}) \, d\vec{r}$
- Discretized object model $u_o(\vec{r}_i) = [\boldsymbol{A x}]_i, \quad a_{ij} = \int h(\vec{r}_i, \vec{r}) \, \chi_j(\vec{r}) \, d\vec{r}$
- Hologram intensity at recording plane

$$\begin{aligned} I(\vec{r}) &= |u_o(\vec{r}) + u_{ref}(\vec{r})|^2 \\ &= |u_o(\vec{r})|^2 + |u_{ref}(\vec{r})|^2 + u_o(\vec{r}) u_{ref}^*(\vec{r}) + u_o^*(\vec{r}) u_{ref}(\vec{r}) \end{aligned}$$

- Mean of *i*th measured sample

$$\begin{aligned} \mathrm{E}[Y_i] &= I(\vec{r}_i) + b_i \leftarrow \text{ background dark current} \\ &= |u_o(\vec{r}_i) + u_{ref}(\vec{r}_i)|^2 + b_i = |[\boldsymbol{A x}]_i + u_i|^2 + b_i, \qquad i = 1, \ldots, N \end{aligned}$$

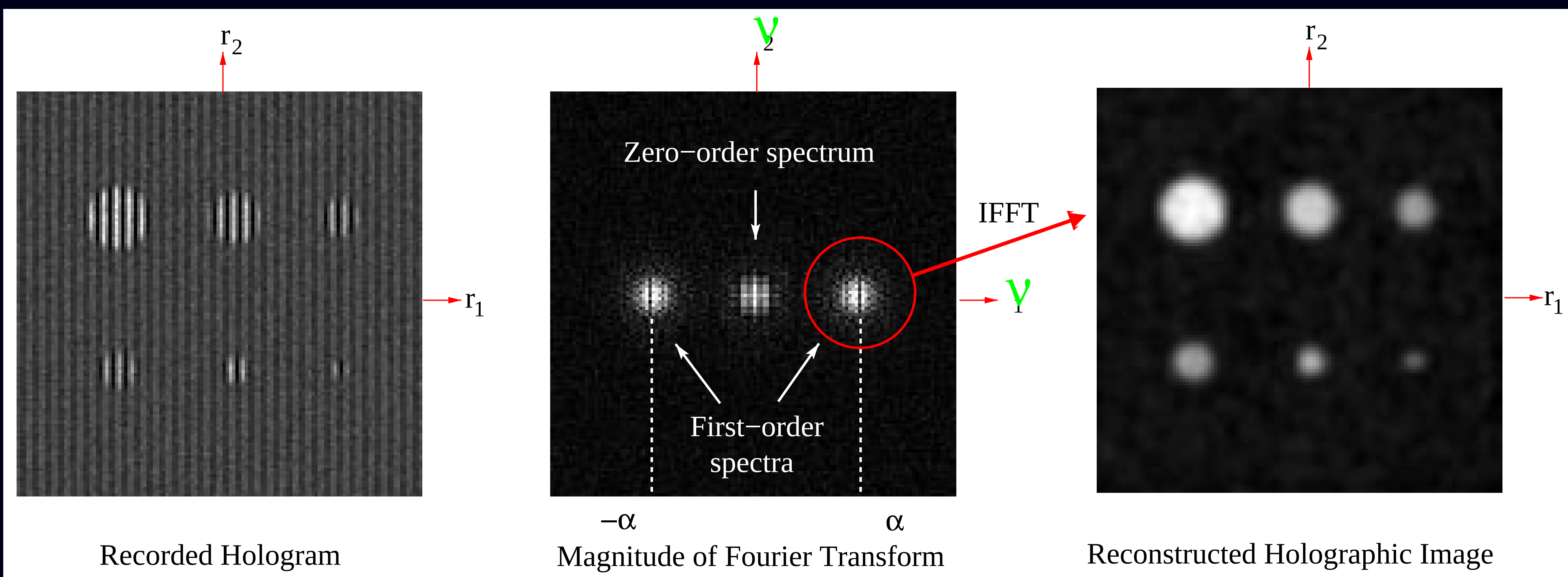
Conventional Reconstruction Method

- Plane-wave assumption: $u_{\mathrm{ref}}(\vec{r}) = U_{\mathrm{ref}}\, \mathrm{e}^{-\imath 2\pi \vec{r}\cdot\vec{\alpha}}$
- Hologram intensity

$$I(\vec{r}) = |u_{\mathrm{o}}(\vec{r})|^2 + U_{\mathrm{ref}}^2 + U_{\mathrm{ref}} u_{\mathrm{o}}(\vec{r})\, \mathrm{e}^{\imath 2\pi \vec{r}\cdot\vec{\alpha}} + U_{\mathrm{ref}} u_{\mathrm{o}}^*(\vec{r})\, \mathrm{e}^{-\imath 2\pi \vec{r}\cdot\vec{\alpha}}$$

- Fourier transform of hologram intensity

$$I(\vec{\nu}) = I_o(\vec{\nu}) + U_{\mathrm{ref}}^2 \delta(\vec{\nu}) + U_{\mathrm{ref}} U_o(\vec{\nu} - \vec{\alpha}) + U_{\mathrm{ref}} U_o^*(-\vec{\nu} - \vec{\alpha})$$



Statistical Model

- Poisson model of the hologram measurement data:

$$Y_i \sim \text{Poisson}\left\{ \left| [\boldsymbol{Ax}]_i + u_i \right|^2 + b_i \right\}, \qquad i = 1, \ldots, N$$

$[\boldsymbol{Ax}]_i = \sum_{j=1}^{P} a_{ij} x_j$

- Known: imaging system $\boldsymbol{A}$, data $\{Y_i\}$,
  reference beam $\{u_i\}$, dark current offset $\{b_i\}$
- Cost function: $\Psi(\boldsymbol{x}) = L(\boldsymbol{x}) + V(\boldsymbol{x})$
- Negative likelihood function: $L(\boldsymbol{x}) = \sum_{i=1}^{N} h_i([\boldsymbol{Ax}]_i)$

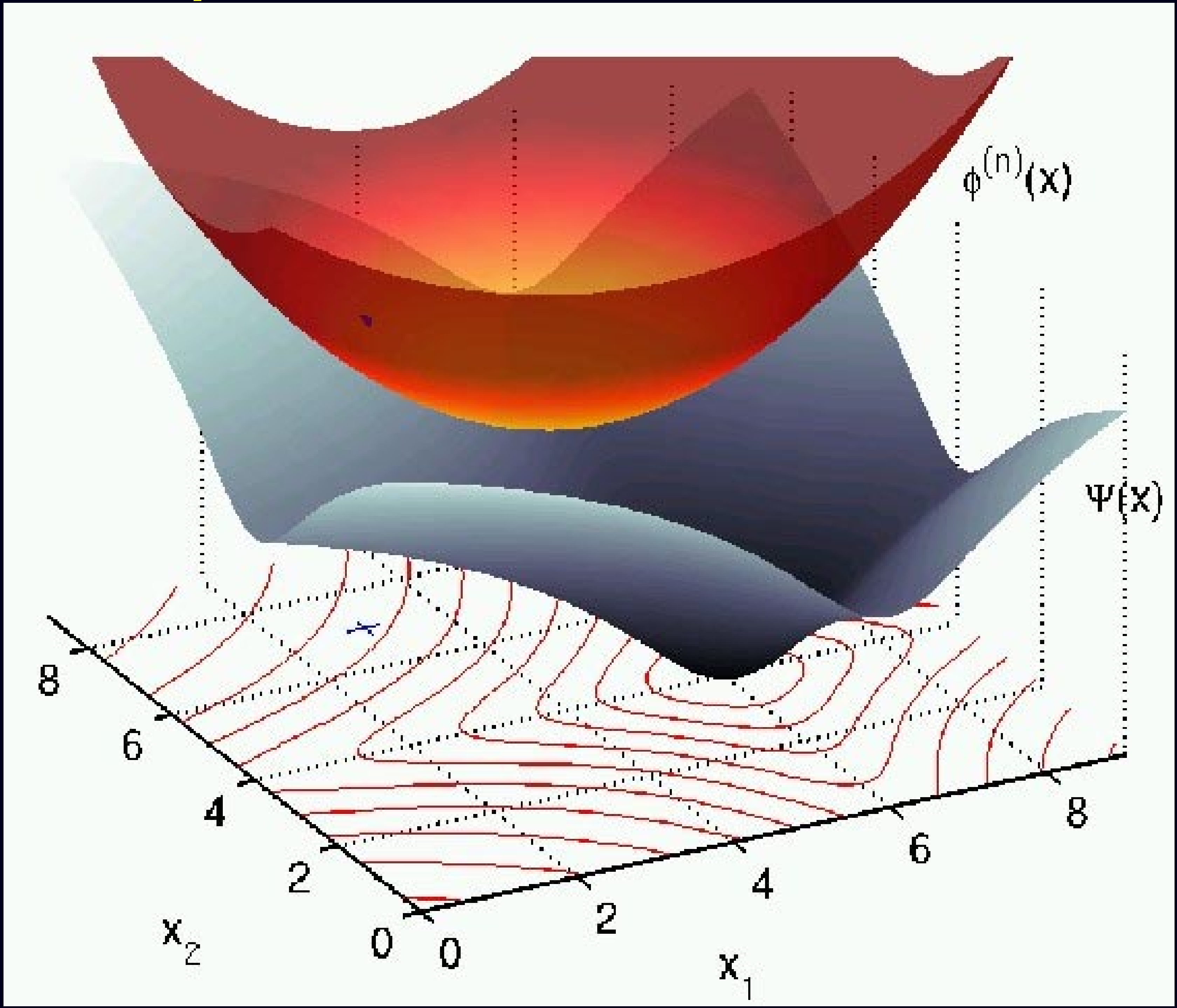
$$h_i(z) = (|z + u_i|^2 + b_i) - y_i \log(|z + u_i|^2 + b_i)$$

- Penalty: $V(\boldsymbol{x}) = \beta \sum_{i=1}^{r} \psi([\boldsymbol{Cx}]_i)$
- Penalized-likelihood (aka MAP) reconstruction:

$$\boldsymbol{x} \triangleq \arg\min_{\boldsymbol{x}} \Psi(\boldsymbol{x})$$

- No closed-form solution $\Rightarrow$ need an iterative algorithm

Optimization Transfer Illustration



Optimization Transfer Principle

- S-step: find a majorizing surrogate function $\phi^{(n)}$

$$1.\ \phi^{(n)}\left(\boldsymbol{x}^{(n)}\right) = \Psi\left(\boldsymbol{x}^{(n)}\right)$$

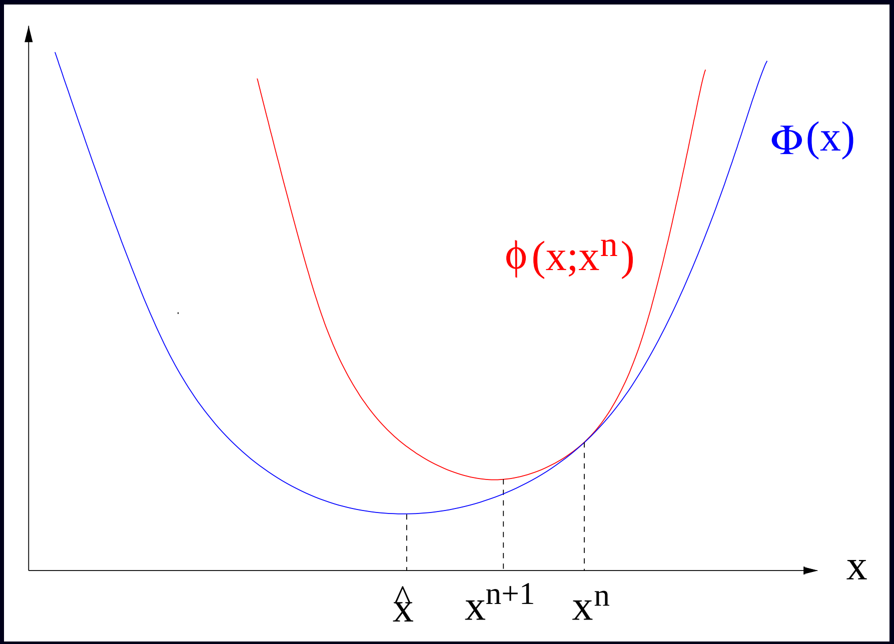
$$2.\ \phi^{(n)}(\boldsymbol{x}) \geq \Psi(\boldsymbol{x}), \qquad \forall \boldsymbol{x} \in \mathbb{C}^P$$

- M-step: minimize $\phi^{(n)}$ instead of Ψ

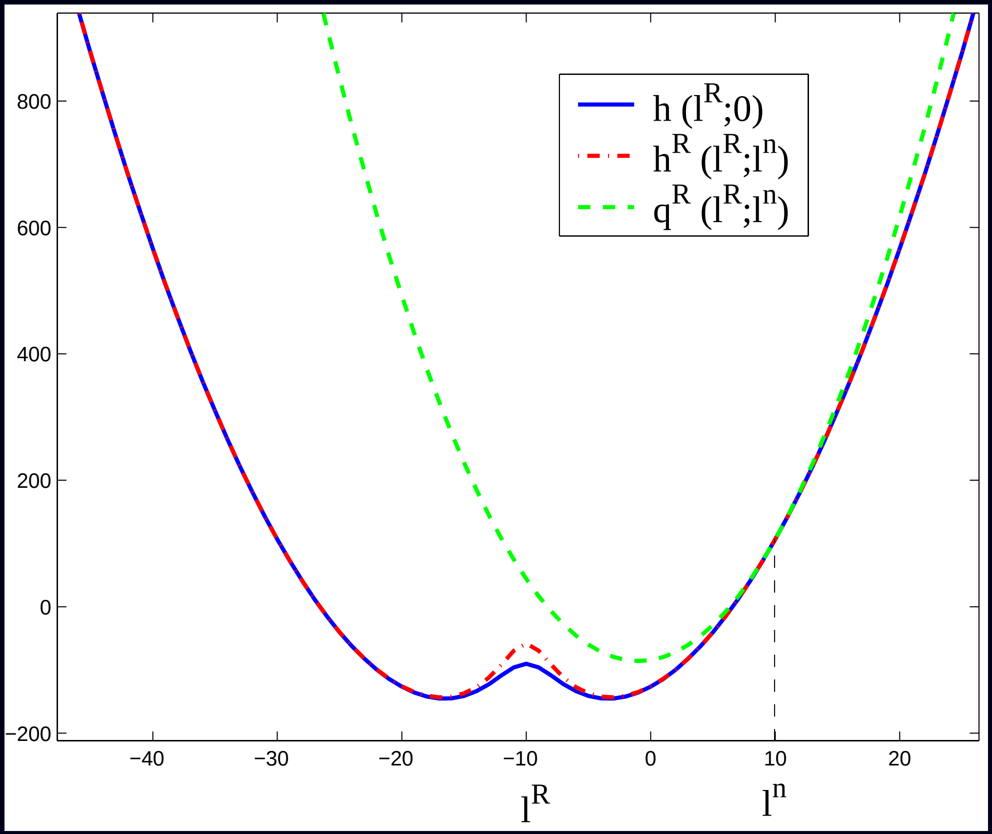
$$\boldsymbol{x}^{(n+1)} \triangleq \arg\min_{\boldsymbol{x}} \phi^{(n)}(\boldsymbol{x})$$

- Fact: algorithm monotonically decreases cost function (*cf.* Newton)

$$\Psi(\boldsymbol{x}^{(n+1)}) \leq \Psi(\boldsymbol{x}^{(n)})$$



Log-likelihood Surrogate Functions



$$L(\boldsymbol{x}) = \sum_{i=1}^{N} h_i([\boldsymbol{Ax}]_i) \qquad \text{(independence)}$$

$$h_i(z) = (|z+u_i|^2 + b_i) - y_i \log(|z+u_i|^2 + b_i)$$

$$\leq q_i(z; z_i^{(n)}) = h_i(z_i^{(n)}) + \nabla q_i(z_i^{(n)})(z - z_i^{(n)}) + \frac{1}{2}\breve{c}_i^{(n)} \left| z - z_i^{(n)} \right|^2$$

$z_i^{(n)} = [\boldsymbol{Ax}^{(n)}]_i$. Curvature: $\breve{c}_i^{(n)} = \breve{c}(y_i, u_i, b_i, [\boldsymbol{Ax}^{(n)}]_i)$ (JOSAA, in review)

Quadratic Surrogate Function

S-step:

$$L(\boldsymbol{x}) = \sum_{i=1}^{N} h_i([\boldsymbol{Ax}]_i) \le \sum_{i=1}^{N} q_i([\boldsymbol{Ax}]_i; [\boldsymbol{Ax}^{(n)}]_i) \triangleq Q(\boldsymbol{x}; \boldsymbol{x}^{(n)})$$

$$\begin{aligned} Q(\boldsymbol{x}; \boldsymbol{x}^{(n)}) &\triangleq \sum_{i=1}^{N} q_i([\boldsymbol{Ax}]_i; [\boldsymbol{Ax}^{(n)}]_i) \\ &= L(\boldsymbol{x}^{(n)}) + \nabla L(\boldsymbol{x}^{(n)})(\boldsymbol{x} - \boldsymbol{x}^{(n)}) + \frac{1}{2}(\boldsymbol{x} - \boldsymbol{x}^{(n)})' \boldsymbol{A}' \boldsymbol{D} \boldsymbol{A} (\boldsymbol{x} - \boldsymbol{x}^{(n)}) \end{aligned}$$

$$\boldsymbol{D} \triangleq \operatorname{diag}\left\{ \breve{c}_i^{(n)} \right\}$$

M-step:

$$\boldsymbol{x}^{(n+1)} = \arg\min_{\boldsymbol{x} \in \mathbb{C}^P} Q(\boldsymbol{x}; \boldsymbol{x}^{(n)}) + V(\boldsymbol{x})$$

Surrogate Q is quadratic, so minimize by PCG algorithm.

Quadratic surrogates for regularizing penalty function also available.
(*cf.* “half quadratic” methods)

Separable Quadratic Surrogate Algorithm

- Convexity of q_i allows separable surrogate (De Pierro, 1995):

$$q_i([\boldsymbol{Ax}]_i; z_i^{(n)}) = q_i\left(\sum_{j=1}^{P} \pi_{ij}\left[\frac{[a_{ij}(x_j - x_j^{(n)})]}{\pi_{ij}} + [\boldsymbol{Ax}^{(n)}]_i; z_i^{(n)}\right]\right)$$
$$\leq \sum_{j=1}^{P} \pi_{ij} q_i\left(\frac{[a_{ij}(x_j - x_j^{(n)})]}{\pi_{ij}} + [\boldsymbol{Ax}^{(n)}]_i; z_i^{(n)}\right)$$

$\pi_{ij} \geq 0$ and $\sum_{j=1}^{P} \pi_{ij} = 1$

- Separable quadratic surrogate function:

$$Q'(\boldsymbol{x}; \boldsymbol{x}^{(n)}) = \sum_{j=1}^{P} Q_j(x_j; \boldsymbol{x}^{(n)})$$

$$Q_j(x_j; \boldsymbol{x}^{(n)}) = \sum_{i=1}^{N} \pi_{ij} q_i\left(\frac{[a_{ij}(x_j - x_j^{(n)})]}{\pi_{ij}} + [\boldsymbol{Ax}^{(n)}]_i; z_i^{(n)}\right)$$

Parallelizable update: $x_j^{(n+1)} = \arg\min_{x_j \in \mathbb{C}} Q_j(x_j; \boldsymbol{x}^{(n)}) + V_j(x_j; \boldsymbol{x}^{(n)})$,

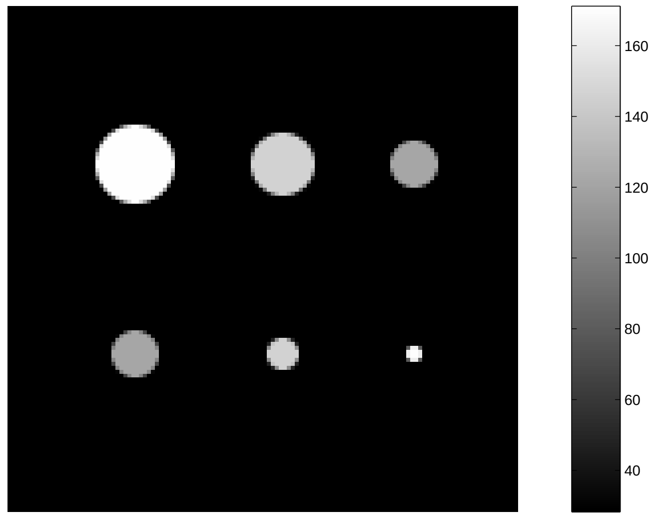
for $j = 1, \ldots, P$

How Many Holograms?

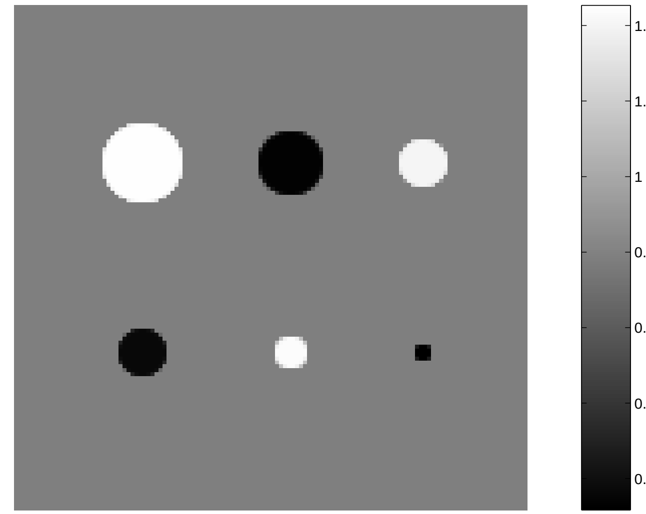
- Number of measurement elements N is fixed by the recorder (*e.g.*, CCD camera pixels)
- Number of reconstructed pixels P is chosen by algorithm designer.
- Natural choice is $P = N$, but this is under-determined!
  - ◦(Measured data values are real, reconstructed field is complex.)
  - ◦Need $P \leq N/2$ to avoid an under-determined problem.
  - ◦In conventional digital holography, FFT zero-padding is used.
- Possible strategies:
  - 1 recorded hologram; reconstruct half-size image: $P = N/2$
  - 1 recorded hologram; reconstruct full-size image: $P = N$ (Must rely on regularization.)
  - 2 recorded holograms with different reference beams; reconstruct full-size holographic image $P = N/2$

Simulation Data: Complex Object

Original Object

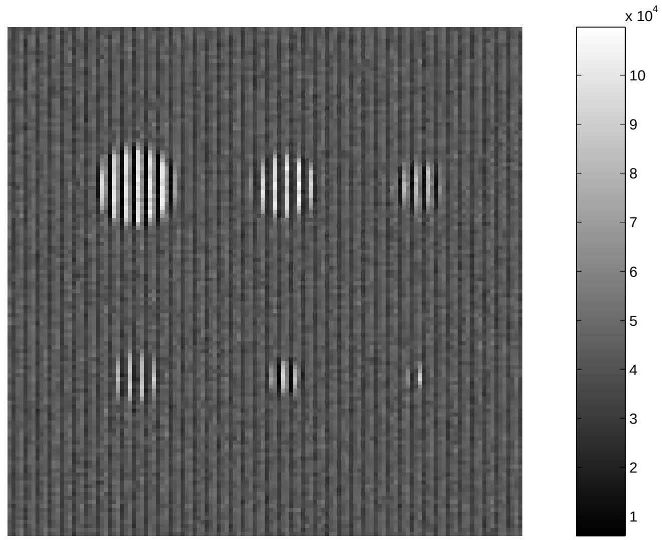


Magnitude

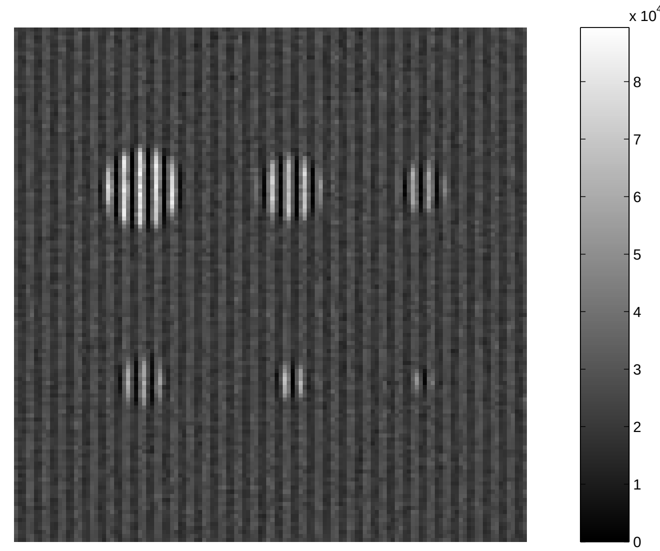


Phase

Holograms

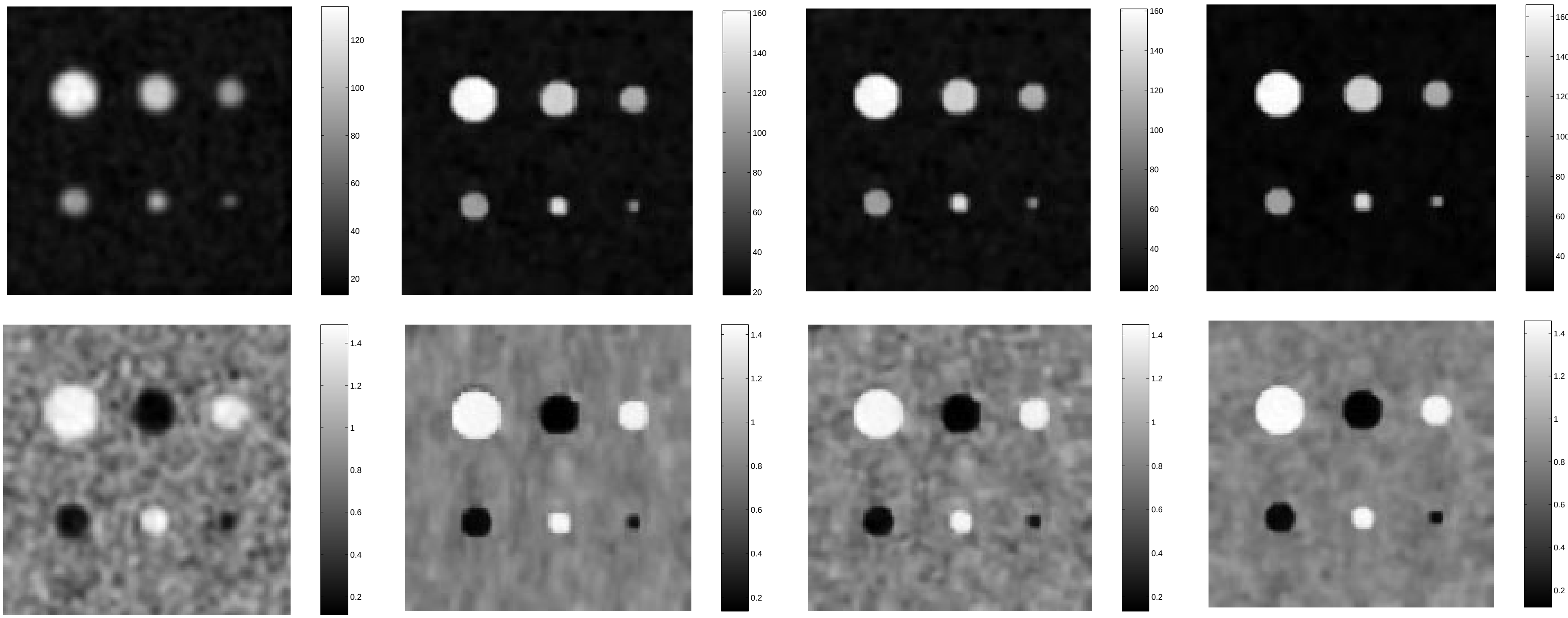


Data #1



Data #2

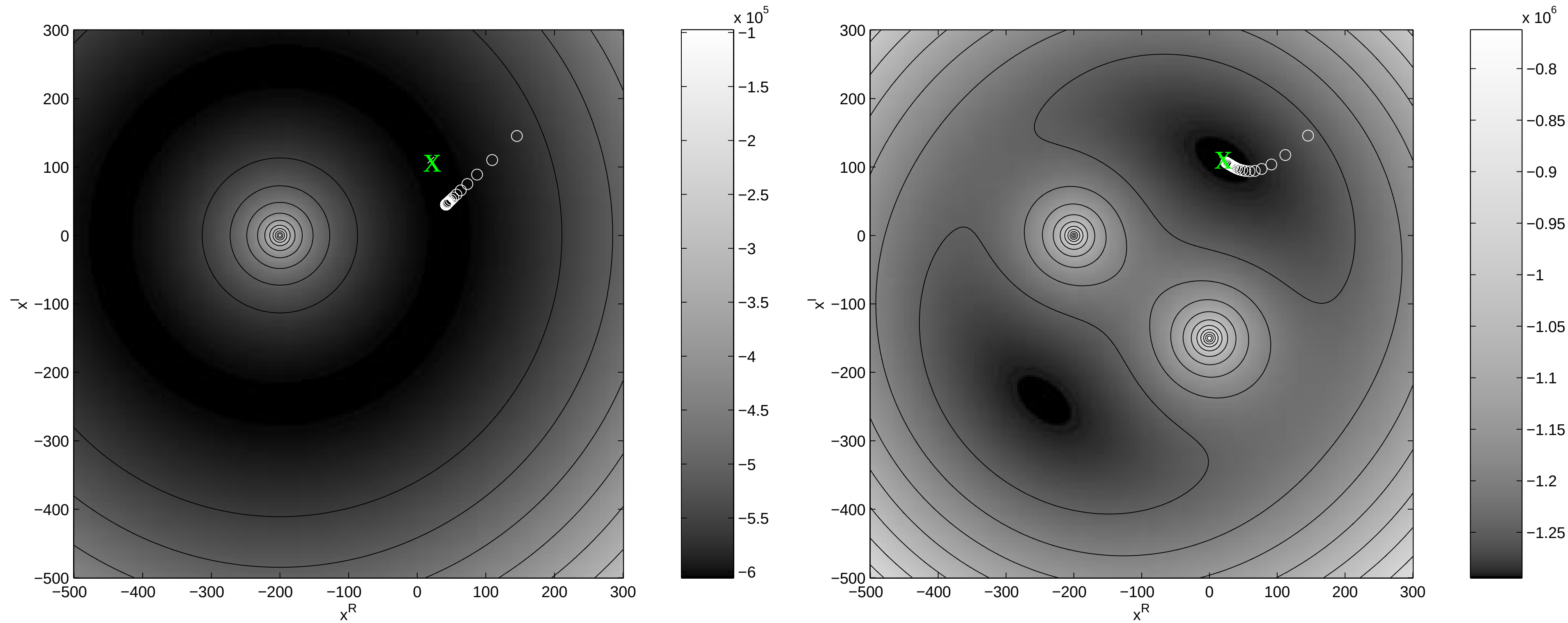
Reconstruction Results: Complex Object



Filtering Method (NMSE = 0.16) | One Data Set, P=N/2 (NMSE = 0.03) | One Data Set, P=N (NMSE = 0.03) | Two Data Sets, P=N/2 (NMSE = 0.02)

- Nonquadratic, edge-preserving regularization
  $\psi(t) = |t/\delta| - \log(1+|t/\delta|)$, with δ chosen empirically
- 200 iterations of separable quadratic surrogate (SQS) algorithm

Contours of Marginal Objective Functions



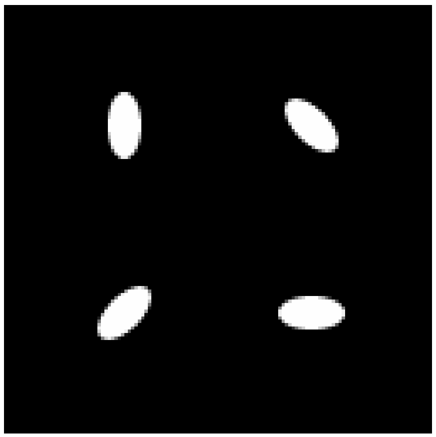
One Data Set

Two Data Sets

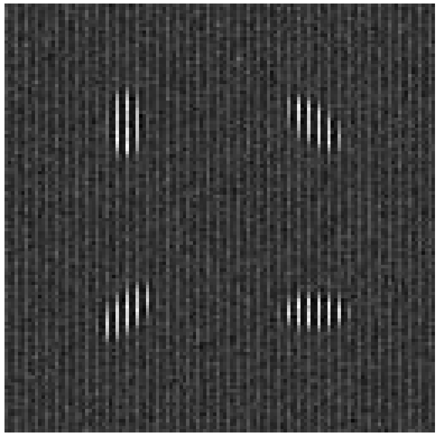
$h_i(z)$ for $z \in \mathbb{C}$, and $h_i([\boldsymbol{A}\boldsymbol{x}^{(n)}]_i)$ vs n

(Unregularized.)

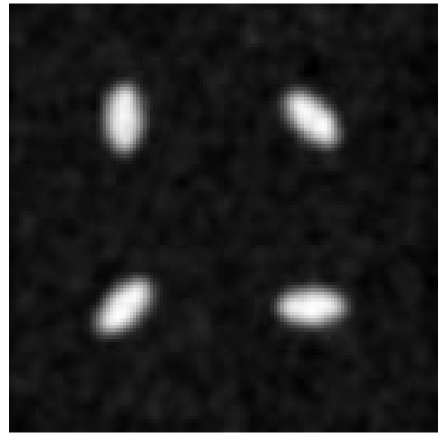
Real-Valued Object (Constrained)



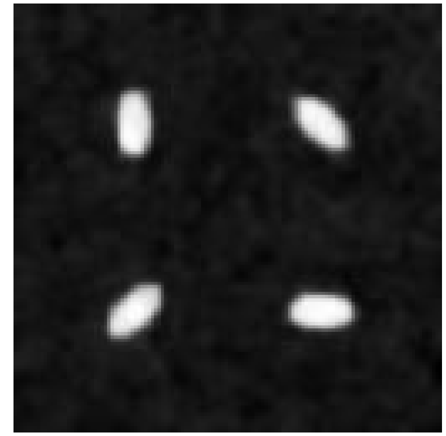
Original image



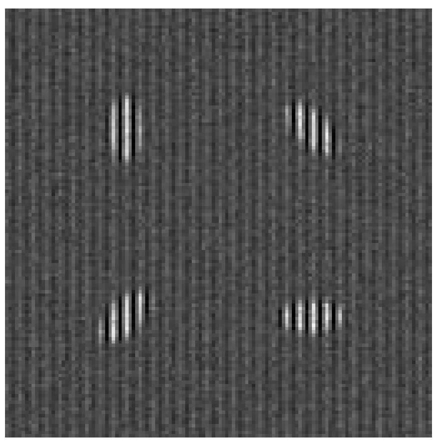
Hologram Data #1



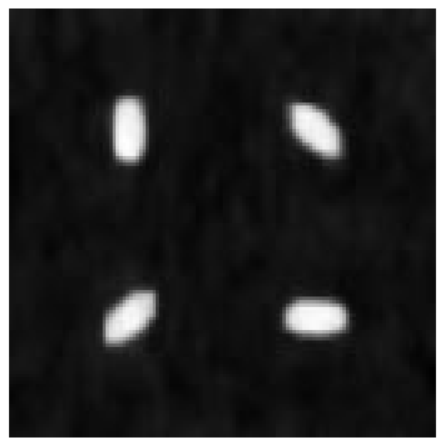
Filtering Method
(NMSE = 0.19)



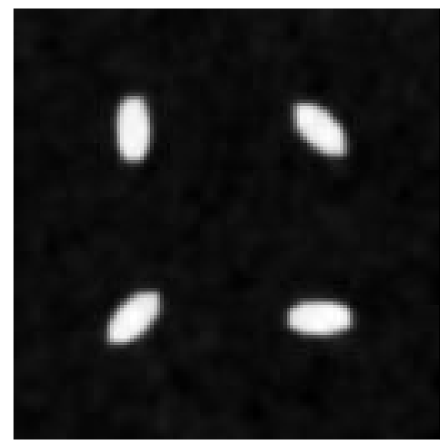
1 Data Set, P=N
(NMSE = 0.05)



Hologram Data #2



1 Data Set, P=N/2
(NMSE = 0.04)



2 Data Sets, P=N/2
(NMSE = 0.03)

Summary

- Statistical method for reconstructing a complex-valued object field from real-valued hologram intensity data
- Poisson statistical model.
  Extendable to others, *e.g.*, Poisson+Gaussian (Snyder *et al.*)
- Optimization transfer monotonically decreases the cost function
- Preliminary simulations suggest improved image quality compared with conventional digital holography reconstruction method.
- Does not assume the reference beam is a plane-wave!
- But requires reasonably accurate reference beam model...
- Can incorporate constraints such as real-valued object field
- Accommodates "hologram reference beam diversity," reducing the problem of multiple minima

Future Work?

The usual story

- Use a space-variant model
- Regularization parameter selection
- Faster converging algorithms
- Test with real data

Specific to this topic

- Different types of digital holography (*e.g.*, Fresnel, Fourier)
- Phase retrieval problems