Part 4. Performance Characteristics

- Spatial resolution properties
- Noise properties
- Detection properties

Spatial Resolution Properties

Choosing β can be painful, so ...

For true minimization methods:

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \Psi(\boldsymbol{x})$$

the *local impulse response* is approximately (Fessler and Rogers, T-MI, Sep. 1996):

$$l_j(\boldsymbol{x}) = \lim_{\delta \to 0} \frac{E[\hat{\boldsymbol{x}}|\boldsymbol{x}+\delta \boldsymbol{e}_j] - E[\hat{\boldsymbol{x}}|\boldsymbol{x}]}{\delta} \approx \left[-\nabla^{20}\Psi\right]^{-1} \nabla^{11}\Psi \frac{\partial}{\partial x_j} \bar{\boldsymbol{y}}(\boldsymbol{x}).$$

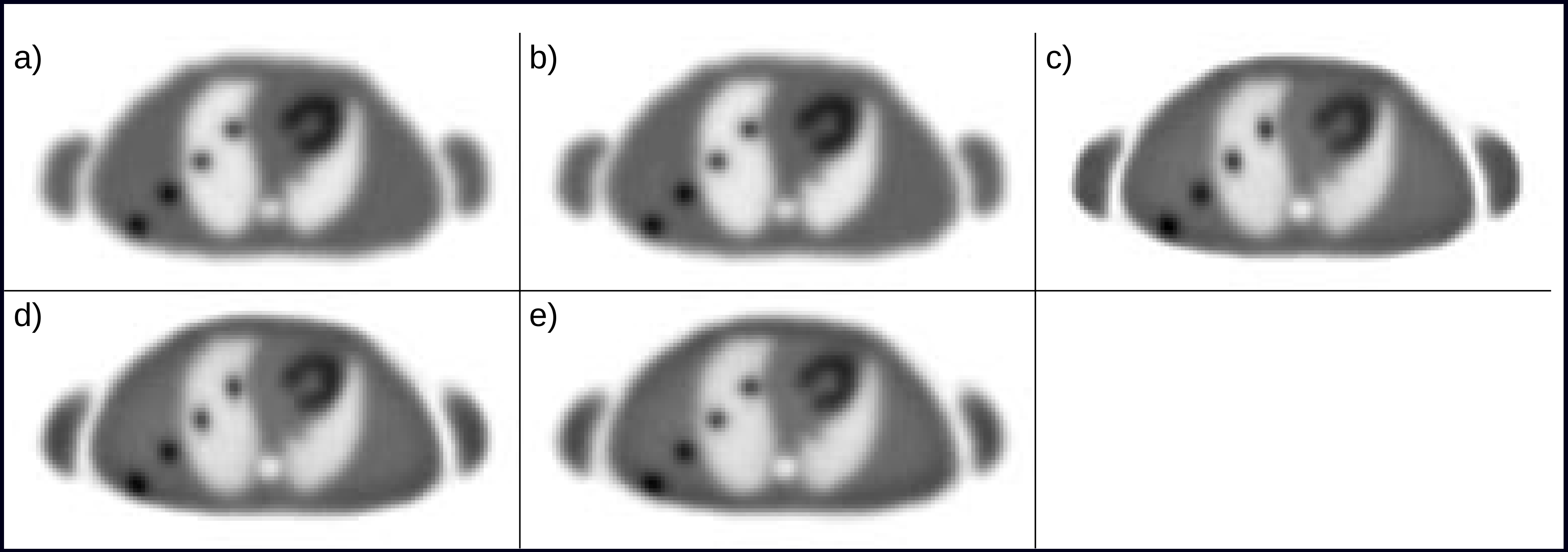
Depends only on chosen *cost function* and statistical model.
Independent of optimization algorithm.

- Enables prediction of resolution properties (provided Ψ is minimized)
- Useful for designing regularization penalty functions with desired resolution properties

$$l_j(\boldsymbol{x}) \approx [\boldsymbol{A}'\boldsymbol{W}\boldsymbol{A} + \beta \boldsymbol{R}]^{-1} \boldsymbol{A}'\boldsymbol{W}\boldsymbol{A}\boldsymbol{x}^{\text{true}}.$$

- Helps choose β for desired spatial resolution

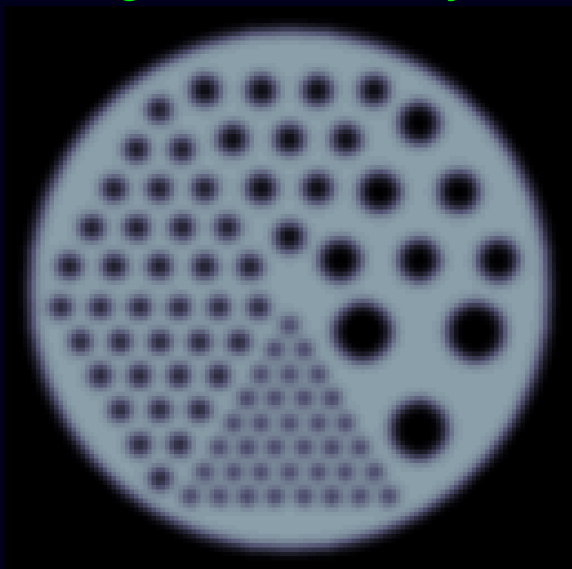
Modified Penalty Example, PET



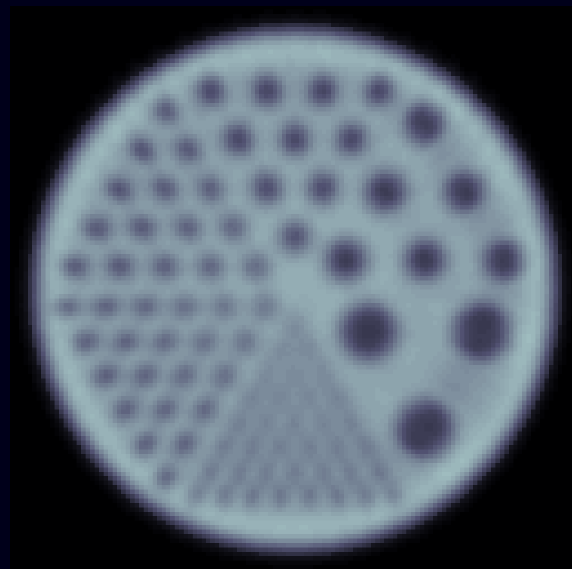
a) filtered backprojection
b) Penalized unweighted least-squares
c) PWLS with conventional regularization
d) PWLS with certainty-based penalty [25]
e) PWLS with modified penalty [143]

Modified Penalty Example, SPECT - Noiseless

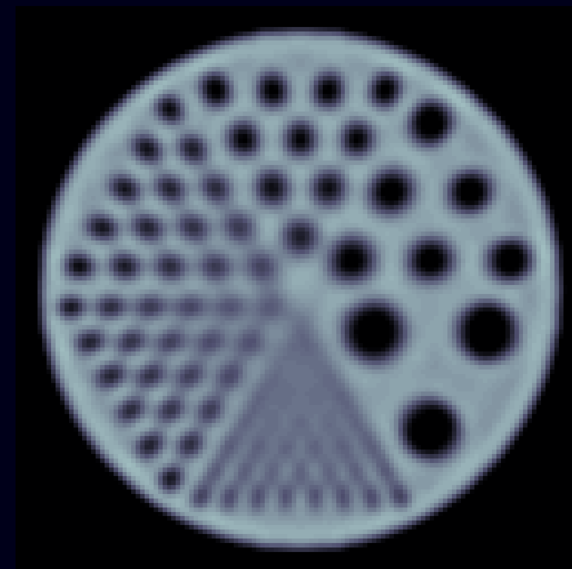
Target filtered object

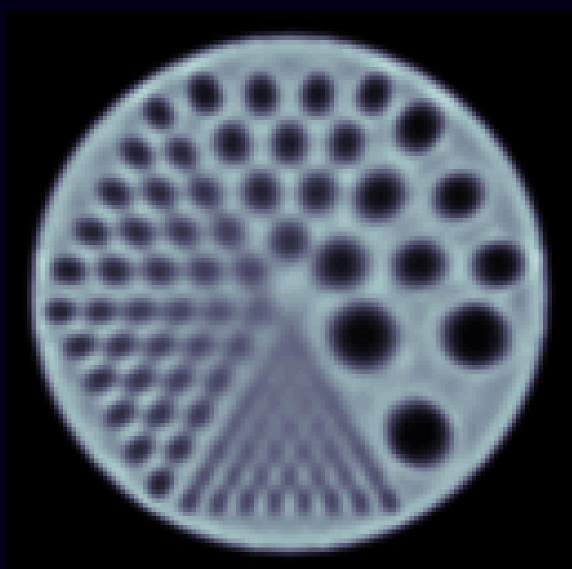


FBP

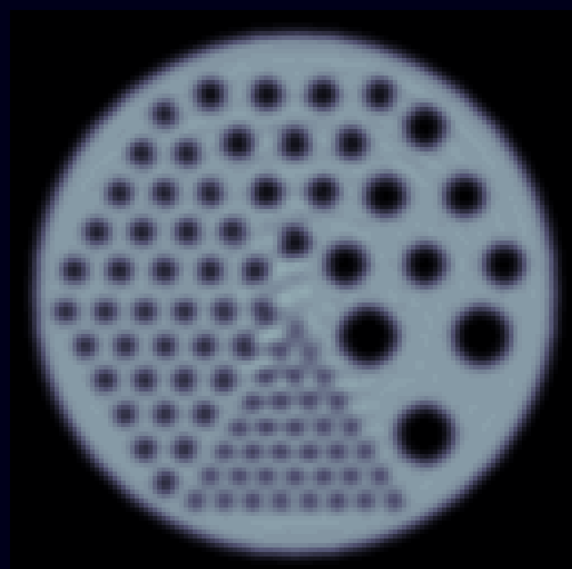


Conventional PWLS

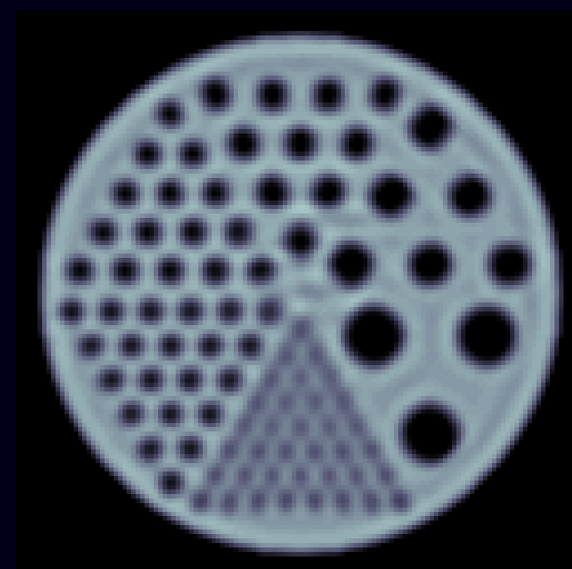




Truncated EM



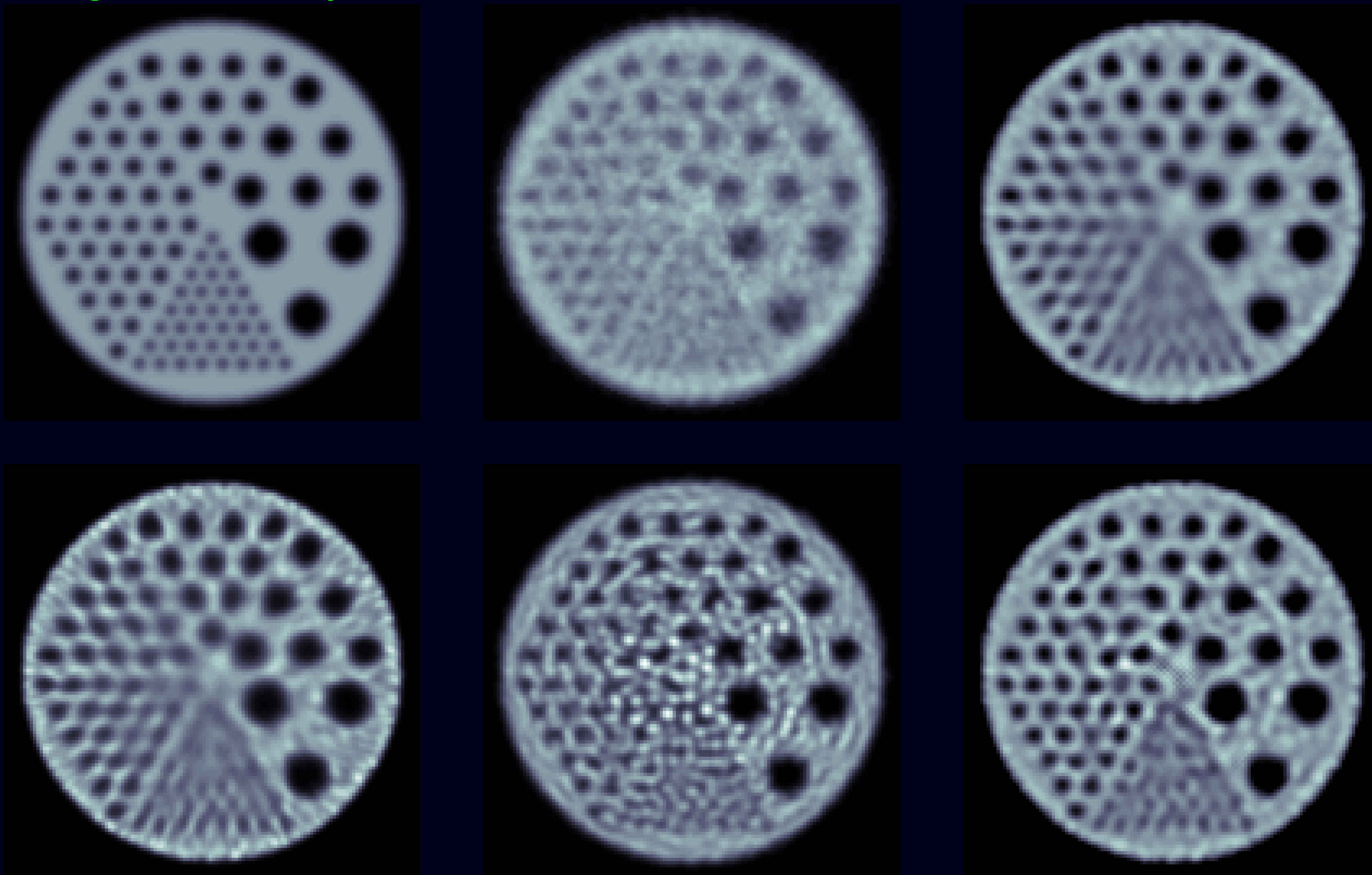
Post-filtered EM



Modified Regularization

Modified Penalty Example, SPECT - Noisy

Target filtered object | FBP | Conventional PWLS



Truncated EM | Post-filtered EM | Modified Regularization

Reconstruction Noise Properties

For unconstrained (converged) minimization methods, the estimator is *implicit*:

$$\hat{\boldsymbol{x}} = \hat{\boldsymbol{x}}(\boldsymbol{y}) = \arg\min_{\boldsymbol{x}} \Psi(\boldsymbol{x}, \boldsymbol{y}).$$

What is $\mathrm{Cov}\{\hat{\boldsymbol{x}}\}$? New simpler derivation.

Denote the column gradient by $g(\boldsymbol{x}, \boldsymbol{y}) \stackrel{\triangle}{=} \nabla_{\boldsymbol{x}} \Psi(\boldsymbol{x}, \boldsymbol{y})$.
Ignoring constraints, the gradient is zero at the minimizer: $g(\hat{\boldsymbol{x}}(\boldsymbol{y}), \boldsymbol{y}) = \boldsymbol{0}$.
First-order Taylor series expansion:

$$\begin{aligned} g(\hat{\boldsymbol{x}}, \boldsymbol{y}) &\approx g(\boldsymbol{x}^{\mathrm{true}}, \boldsymbol{y}) + \nabla_{\boldsymbol{x}} g(\boldsymbol{x}^{\mathrm{true}}, \boldsymbol{y})(\hat{\boldsymbol{x}} - \boldsymbol{x}^{\mathrm{true}}) \\ &= g(\boldsymbol{x}^{\mathrm{true}}, \boldsymbol{y}) + \nabla^2_{\boldsymbol{x}} \Psi(\boldsymbol{x}^{\mathrm{true}}, \boldsymbol{y})(\hat{\boldsymbol{x}} - \boldsymbol{x}^{\mathrm{true}}). \end{aligned}$$

Equating to zero:

$$\hat{\boldsymbol{x}} \approx \boldsymbol{x}^{\mathrm{true}} - \left[\nabla^2_{\boldsymbol{x}} \Psi(\boldsymbol{x}^{\mathrm{true}}, \boldsymbol{y})\right]^{-1} \nabla_{\boldsymbol{x}} \Psi(\boldsymbol{x}^{\mathrm{true}}, \boldsymbol{y}).$$

If the Hessian $\nabla^2 \Psi$ is weakly dependent on $\boldsymbol{y}$, then

$$\boxed{\mathrm{Cov}\{\hat{\boldsymbol{x}}\} \approx \left[\nabla^2_{\boldsymbol{x}} \Psi(\boldsymbol{x}^{\mathrm{true}}, \bar{\boldsymbol{y}})\right]^{-1} \mathrm{Cov}\left\{\nabla_{\boldsymbol{x}} \Psi(\boldsymbol{x}^{\mathrm{true}}, \boldsymbol{y})\right\} \left[\nabla^2_{\boldsymbol{x}} \Psi(\boldsymbol{x}^{\mathrm{true}}, \bar{\boldsymbol{y}})\right]^{-1}.}$$

If we further linearize w.r.t. the data: $g(\boldsymbol{x}, \boldsymbol{y}) \approx g(\boldsymbol{x}, \bar{\boldsymbol{y}}) + \nabla_{\boldsymbol{y}} g(\boldsymbol{x}, \bar{\boldsymbol{y}})(\boldsymbol{y} - \bar{\boldsymbol{y}})$, then

$$\mathrm{Cov}\{\hat{\boldsymbol{x}}\} \approx \left[\nabla^2_{\boldsymbol{x}} \Psi\right]^{-1} (\nabla_{\boldsymbol{x}} \nabla_{\boldsymbol{y}} \Psi) \; \mathrm{Cov}\{\boldsymbol{y}\} \; (\nabla_{\boldsymbol{x}} \nabla_{\boldsymbol{y}} \Psi)' \left[\nabla^2_{\boldsymbol{x}} \Psi\right]^{-1}.$$

Covariance Continued

Covariance approximation:

$$\operatorname{Cov}\{\hat{\boldsymbol{x}}\} \approx \left[\nabla_{\boldsymbol{x}}^2 \Psi(\boldsymbol{x}^{\text{true}}, \bar{\boldsymbol{y}})\right]^{-1} \operatorname{Cov}\left\{\nabla_{\boldsymbol{x}} \Psi(\boldsymbol{x}^{\text{true}}, \boldsymbol{y})\right\} \left[\nabla_{\boldsymbol{x}}^2 \Psi(\boldsymbol{x}^{\text{true}}, \bar{\boldsymbol{y}})\right]^{-1}$$

Depends only on chosen cost function and statistical model.
Independent of optimization algorithm.

- Enables prediction of noise properties
- Can make variance images
- Useful for computing ROI variance (*e.g.*, for weighted kinetic fitting)
- Good variance prediction for quadratic regularization in nonzero regions
- Inaccurate for nonquadratic penalties, or in nearly-zero regions

Qi and Huesman's Detection Analysis

SNR of MAP reconstruction $>$ SNR of FBP reconstruction (T-MI, Aug. 2001)

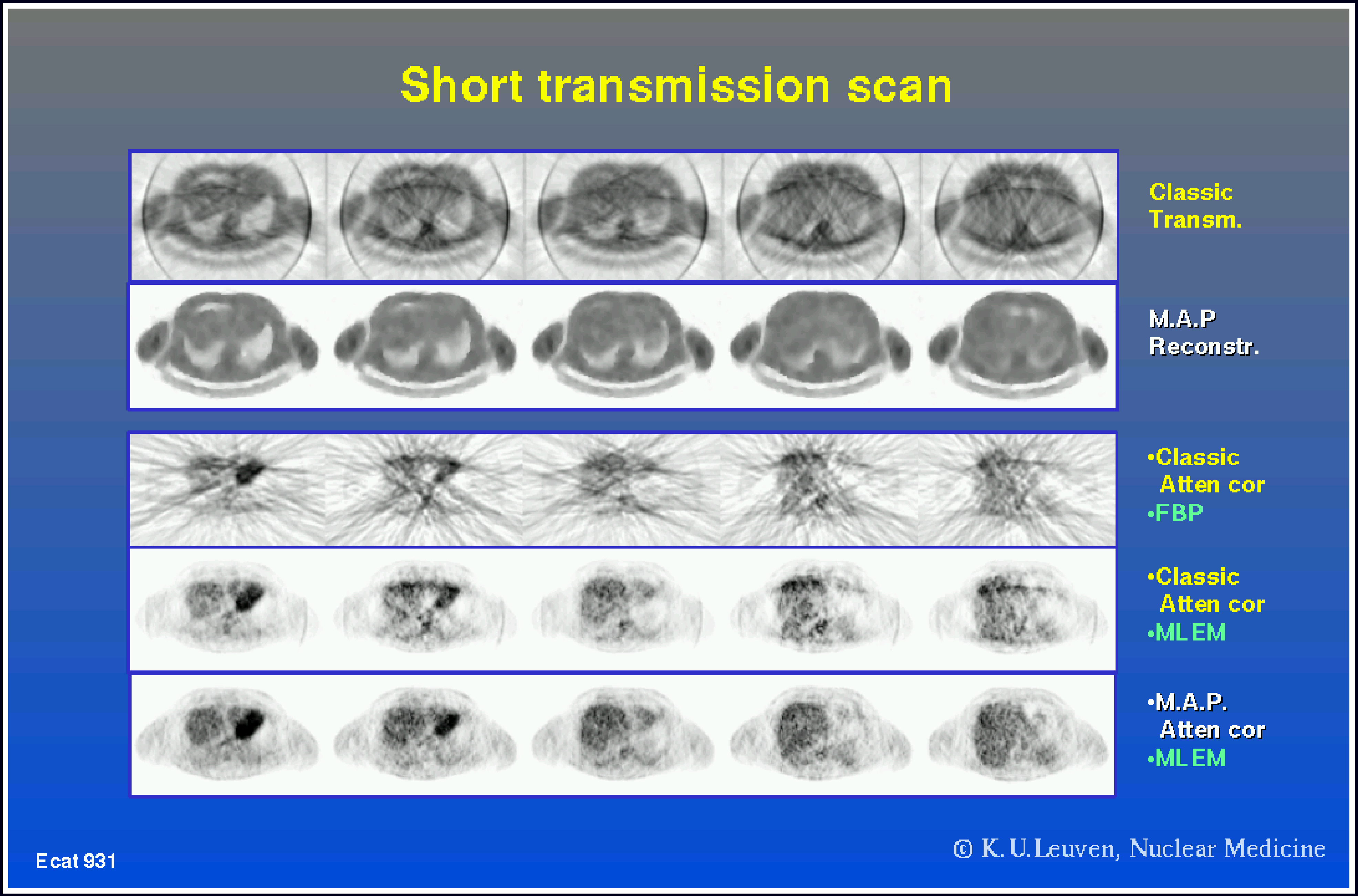
quadratic regularization
SKE/BKE task
prewhitened observer
non-prewhitened observer

Part 5. Miscellaneous Topics

(Pet peeves and more-or-less recent favorites)

- Short transmission scans
- 3D PET options
- OSEM of transmission data (ugh!)
- Precorrected PET data
- Transmission scan problems
- List-mode EM
- List of other topics I wish I had time to cover...

PET Attenuation Correction (J. Nuyts)



Iterative reconstruction for 3D PET

- Fully 3D iterative reconstruction
- Rebinning / 2.5D iterative reconstruction
- Rebinning / 2D iterative reconstruction
  - PWLS
  - OSEM with attenuation weighting
- 3D FBP
- Rebinning / FBP

OSEM of Transmission Data?

Bai and Kinahan *et al.* "'Post-injection single photon transmission tomography with ordered-subset algorithms for wholebody PET imaging"

- 3D penalty better than 2D penalty
- OSTR with 3D penalty better than FBP and OSEM
- standard deviation from a single realization to estimate noise can be misleading

Using OSEM for transmission data requires taking logarithm,
whereas OSTR does not.

Precorrected PET data

C. Michel examined shifted-Poisson model, “weighted OSEM” of various flavors.

concluded attenuation weighting matters especially

Transmission Scan Challenges

- Overlapping-beam transmission scans
- Polyenergetic X-ray CT scans
- Sourceless attenuation correction

All can be tackled with optimization transfer methods.

List-mode EM

$$
\begin{aligned}
x_j^{(n+1)} &= x_j^{(n)} \left[\sum_{i=1}^{n_d} a_{ij} \frac{y_i}{\bar{y}_i^{(n)}} \right] \Big/ \left(\sum_{i=1}^{n_d} a_{ij} \right) \\
&= \frac{x_j^{(n)}}{\sum_{i=1}^{n_d} a_{ij}} \sum_{i:y_i \neq 0} a_{ij} \frac{y_i}{\bar{y}_i^{(n)}}
\end{aligned}
$$

- Useful when $\sum_{i=1}^{n_d} y_i \leq \sum_{i=1}^{n_d} 1$
- Attenuation and scatter non-trivial
- Computing a_{ij} on-the-fly
- Computing sensitivity $\sum_{i=1}^{n_d} a_{ij}$ still painful
- List-mode ordered-subsets is naturally balanced

Misc

- 4D regularization (reconstruction of dynamic image sequences)
- "Sourceless" attenuation-map estimation
- Post-injection transmission/emission reconstruction
- μ-value priors for transmission reconstruction
- Local errors in $\hat{\mu}$ propagate into emission image (PET and SPECT)

Summary

- Predictability of resolution / noise and controlling spatial resolution argues for regularized *cost function*
- todo: Still work to be done...