Part 1: From Physics to Statistics

or

"What quantity is reconstructed?"

(in emission tomography)

Outline

- Decay phenomena and fundamental assumptions
- Idealized detectors
- Random phenomena
- Poisson measurement statistics
- State emission tomography reconstruction problem

What Object is Reconstructed?

In *emission imaging*, our aim is to image the *radiotracer distribution.*

The what?

At time $t = 0$ we inject the patient with some *radiotracer*, containing a "large" number N of metastable atoms of some radionuclide.

Let $\vec{X}_k(t) \in \mathbb{R}^3$ denote the position of the kth *tracer atom* at time t.
These positions are influenced by blood flow, patient physiology, and other unpredictable phenomena such as Brownian motion.

The ultimate imaging device would provide an exact list of the spatial locations $\vec{X}_1(t), \ldots, \vec{X}_N(t)$ of all tracer atoms for the entire scan.

Would this be enough?

Atom Positions or Statistical Distribution?

Repeating a scan would yield different tracer atom sample paths $\left\{\vec{X}_k(t)\right\}_{k=1}^{N}$.

$\therefore$ statistical formulation

Assumption 1. The spatial locations of individual tracer atoms at any time $t \geq 0$ are *independent* random variables that are all *identically distributed* according to a common probability density function (pdf) $f_{\vec{X}(t)}(\vec{x})$.

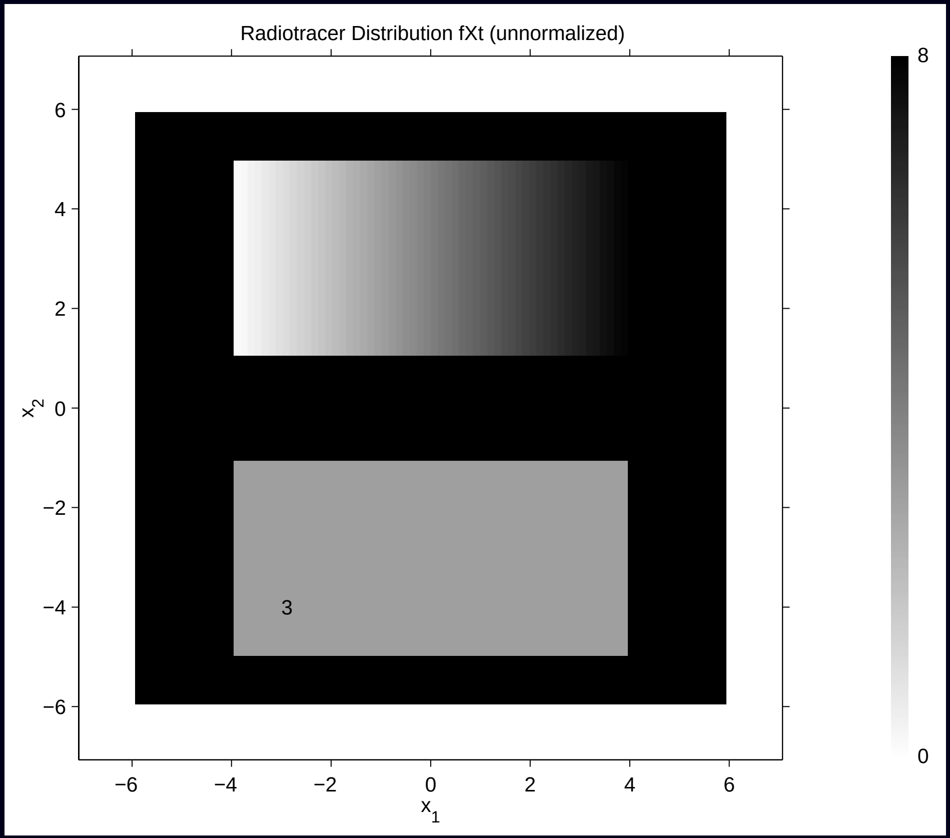
This pdf is determined by patient physiology and tracer properties.

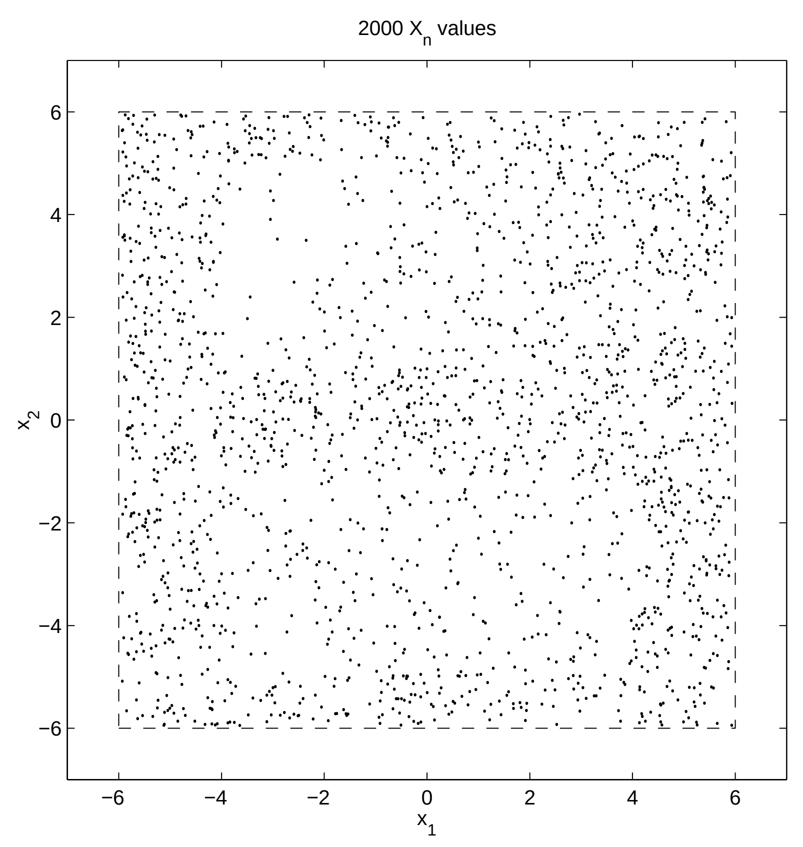
Larger values of $f_{\vec{X}(t)}(\vec{x})$ correspond to "hot spots" where the tracer atoms tend to be located at time t. Units: inverse volume, *e.g.*, atoms per cubic centimeter.

The *radiotracer distribution* $f_{\vec{X}(t)}(\vec{x})$ is the quantity of interest.

(Not $\left\{\vec{X}_k(t)\right\}_{k=1}^{N}$!)

Example: Perfect Detector



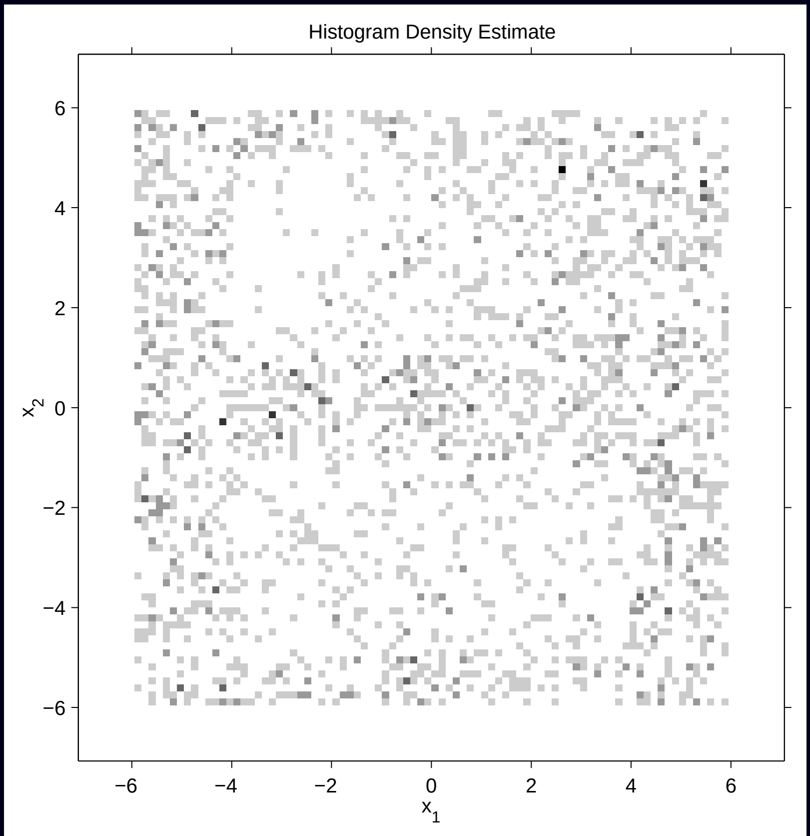


True radiotracer distribution $f_{\vec{X}(t)}(\vec{x})$ at some time t.

A realization of $N = 2000$ i.i.d. atom positions (dots) recorded "exactly."

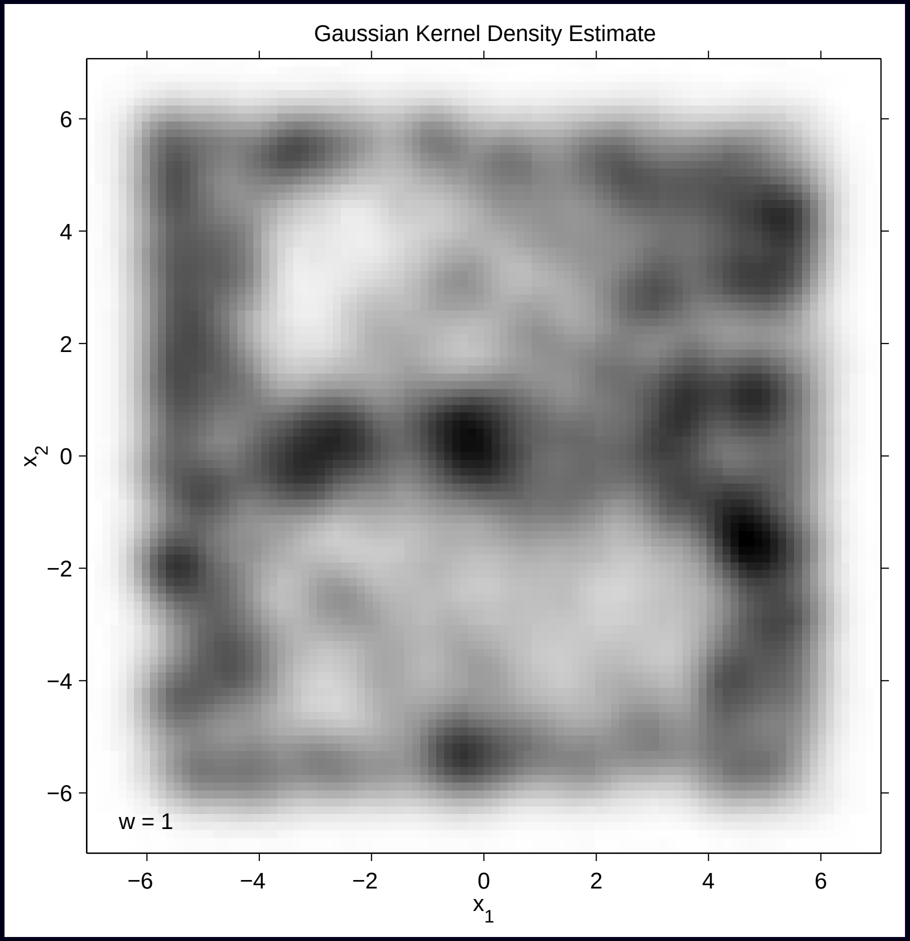
Little similarity!

Binning/Histogram Density Estimator

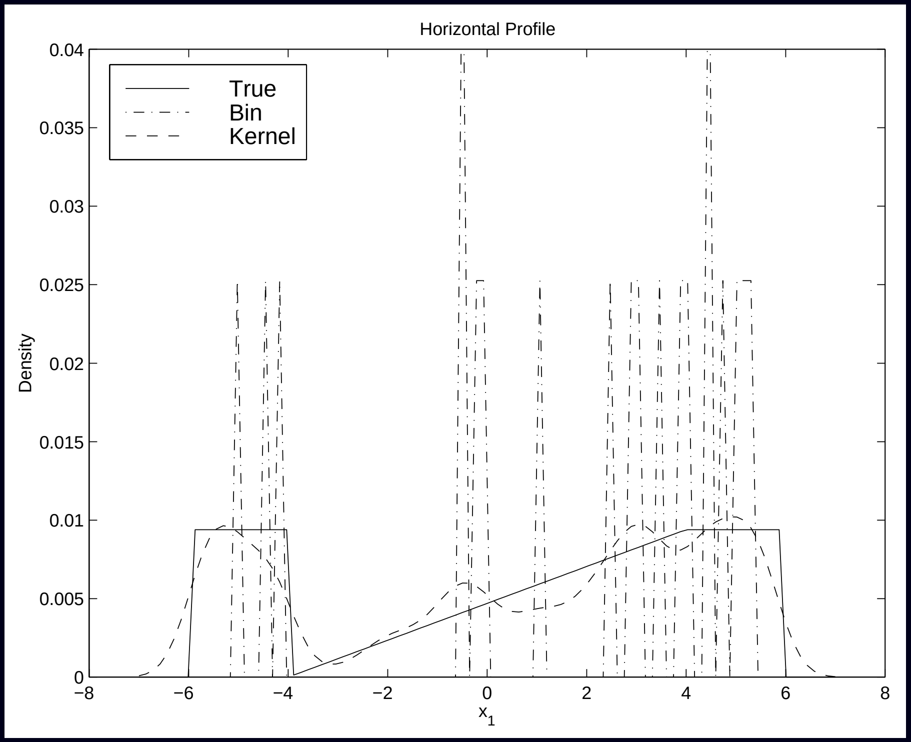


Estimate of $f_{\vec{X}(t)}(\vec{x})$ formed by histogram binning of $N = 2000$ points.
Ramp remains difficult to visualize.

Kernel Density Estimator



Gaussian kernel density estimator for $f_{\vec{X}(t)}(\vec{x})$ from $N = 2000$ points.



Horizontal profiles at $x_2 = 3$ through density estimates.

Poisson Spatial Point Process

Assumption 2. The number of injected tracer atoms N has a Poisson distribution with some mean

$$\mu_N \stackrel{\triangle}{=} E[N] = \sum_{n=0}^{\infty} n P[N = n].$$

Let $N(B)$ denote the number of tracer atoms that have spatial locations in any set $B \subset \mathbb{R}^3$ (VOI) at time t_0 after injection.

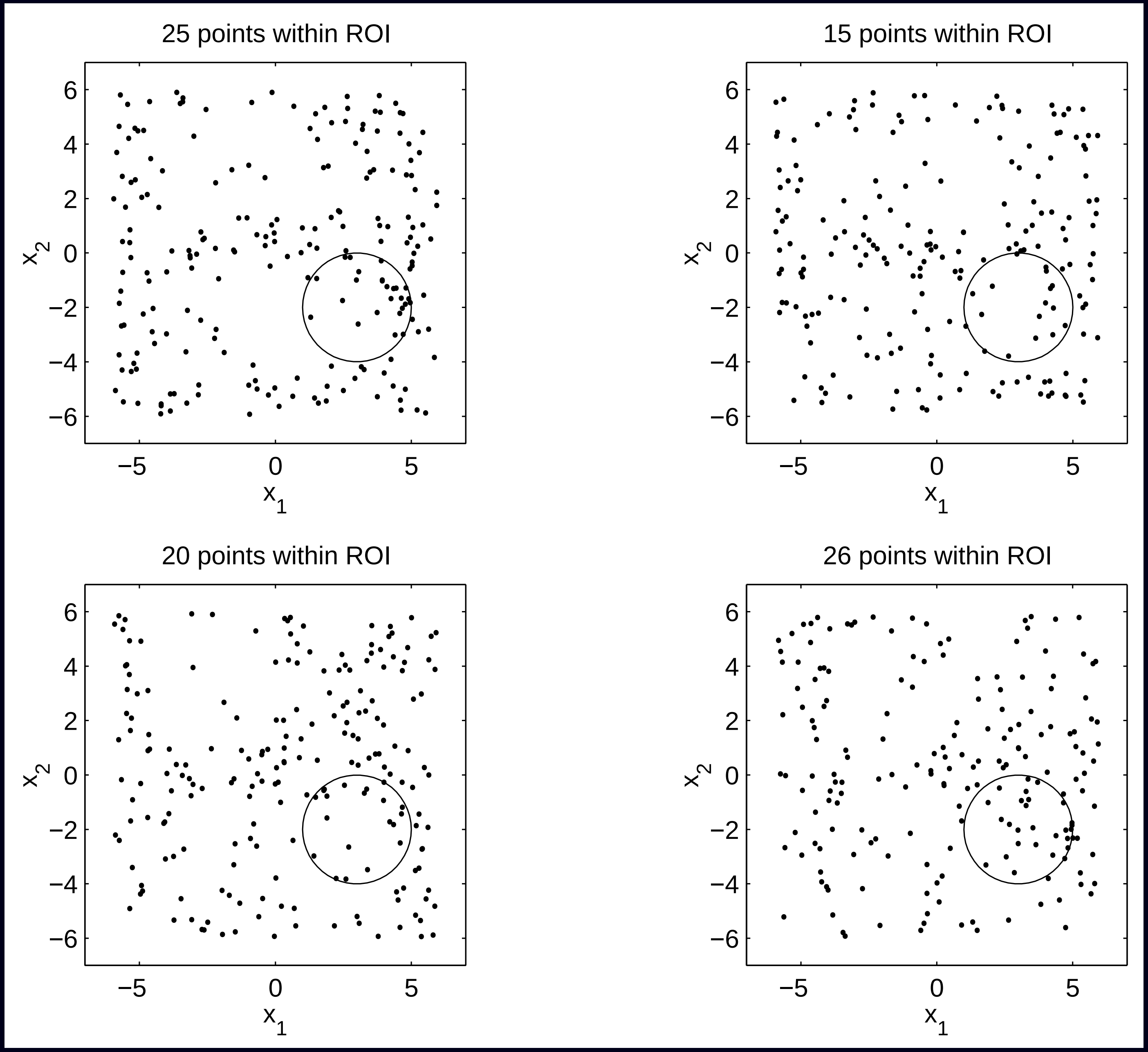
$N(\cdot)$ is called a *Poisson spatial point process*.

Fact. For any set B, $N(B)$ is Poisson distributed with mean:

$$E[N(B)] = E[N] P[\vec{X} \in B] = \mu_N \int_B f_{\vec{X}(t_0)}(\vec{x}) \, d\vec{x}.$$

Poisson N injected atoms $+$ i.i.d. locations $\Rightarrow$ Poisson point process

Illustration of Point Process ($\mu_N = 200$)



Radionuclide Decay

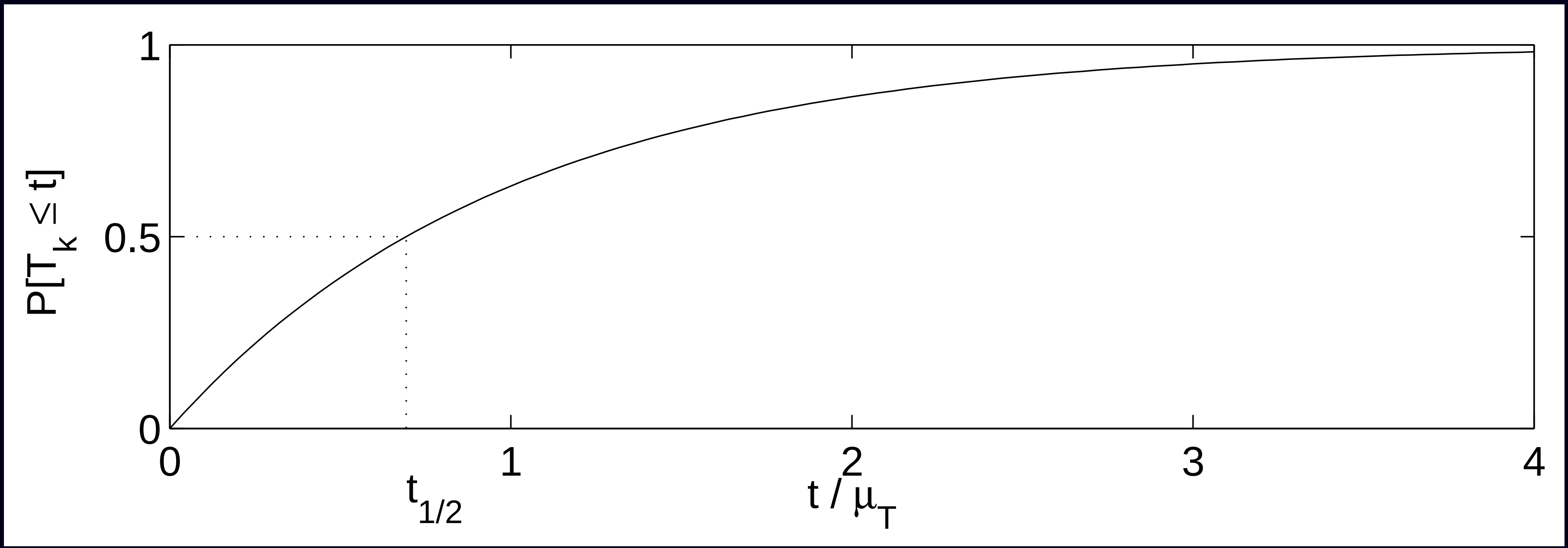
Preceding quantities are all unobservable.
We "observe" a tracer atom only when it decays and emits photon(s).

The time that the kth tracer atom decays is a random variable T_k.

Assumption 3. The T_k's are statistically *independent* random variables, and are independent of the (random) spatial location.

Assumption 4. Each T_k has an exponential distribution with mean $\mu_T = t_{1/2} / \ln 2$.

Cumulative distribution function: $P[T_k \leq t] = 1 - \exp(-t / \mu_T)$



Statistics of an Ideal Decay Counter

Let $K(t, B)$ denote the number of tracer atoms that decay by time t,
and that were located in the VOI $B \subset \mathbb{R}^3$ at the time of decay.

Fact. $K(t, B)$ is a *Poisson counting process* with mean

$$E[K(t,B)] = \int_0^t \int_B \lambda(\vec{x}, \tau)\, d\vec{x}\, d\tau,$$

where the (nonuniform) *emission rate density* is given by

$$\lambda(\vec{x}, t) \triangleq \mu_N \frac{e^{-t/\mu_T}}{\mu_T} \cdot f_{\vec{X}(t)}(\vec{x}).$$

Ingredients: “dose,” “decay,” “distribution”

Units: “counts” per unit time per unit volume, *e.g.*, μCi/cc.

“Photon emission is a Poisson process”

What about the actual measurement statistics?

Idealized Detector Units

A nuclear imaging system consists of n_d conceptual *detector units*.

Assumption 5. Each decay of a tracer atom produces a recorded count in at most one detector unit.

Let $S_k \in \{0, 1, \ldots, n_d\}$ denote the index of the incremented detector unit for decay of kth tracer atom. ($S_k = 0$ if decay is undetected.)

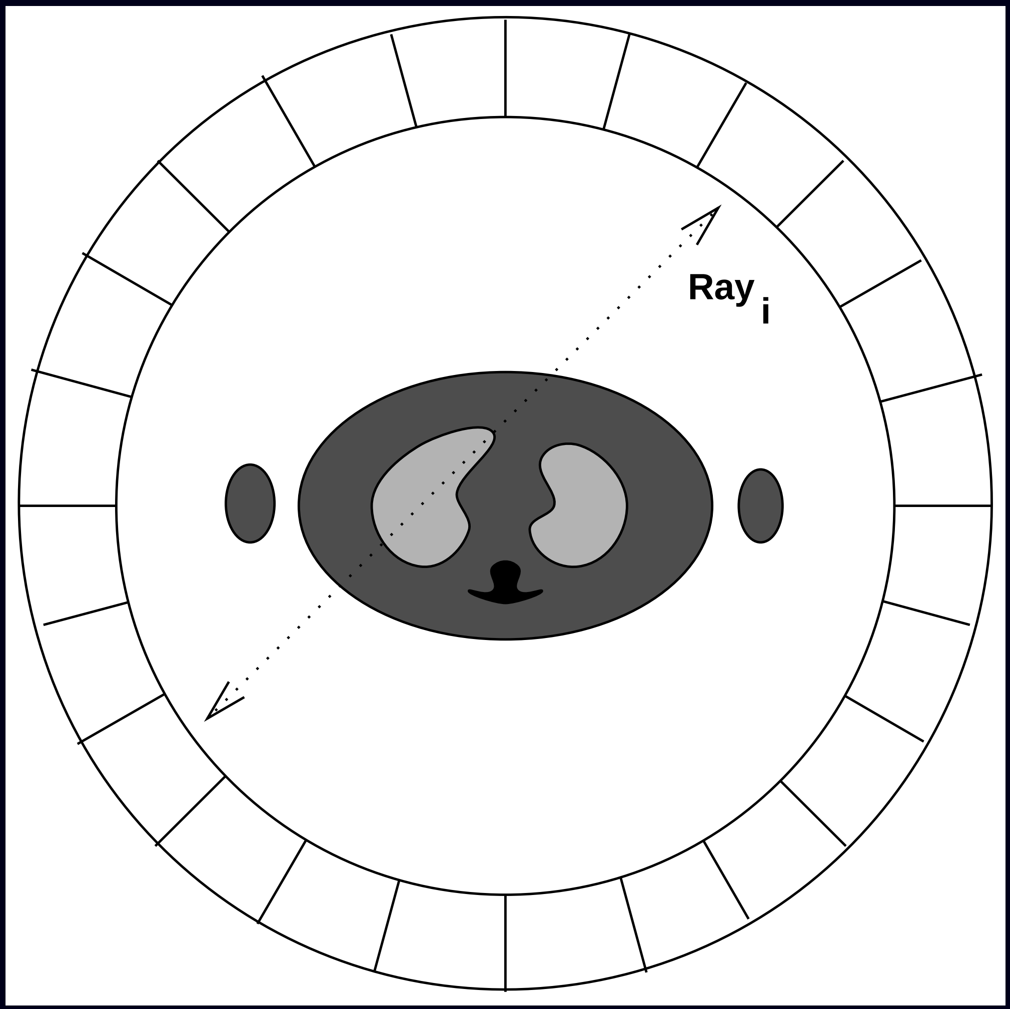
Assumption 6. The S_k's satisfy the following conditional independence:

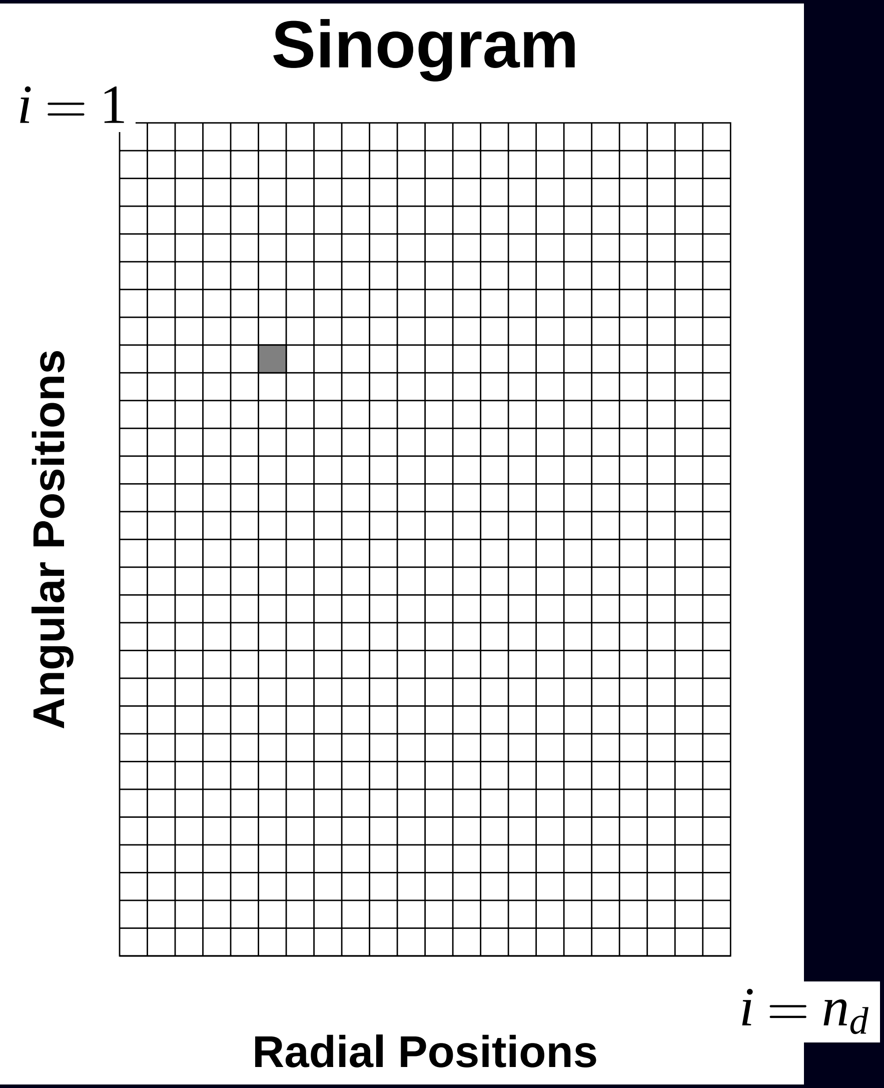
$$P\left(S_1, \ldots, S_N \,|\, N,\ T_1, \ldots, T_N,\ \vec{X}_1(\cdot), \ldots, \vec{X}_N(\cdot)\right) = \prod_{k=1}^{N} P\left(S_k | \vec{X}_k(T_k)\right).$$

The recorded bin for the kth tracer atom's decay depends only on its position when it decays, and is independent of all other tracer atoms.

(No event pileup; no deadtime losses.)

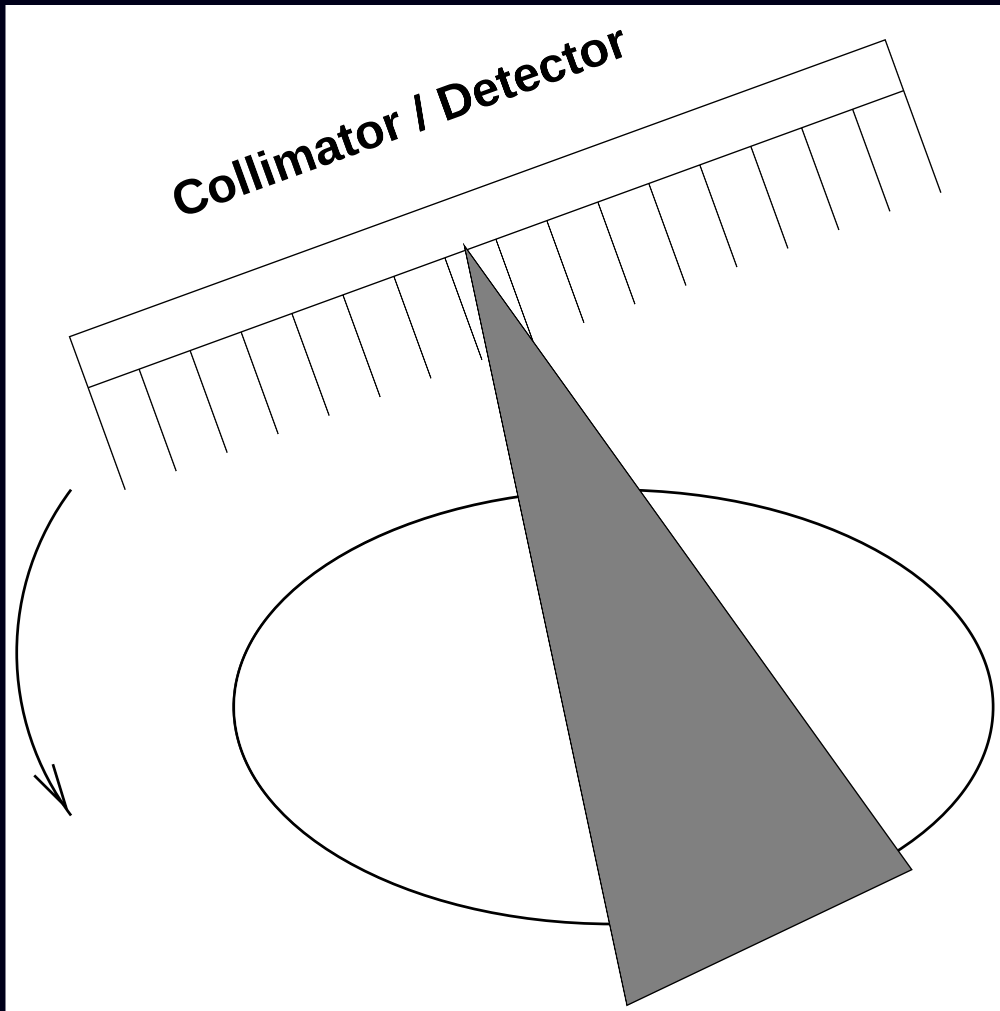
PET Example

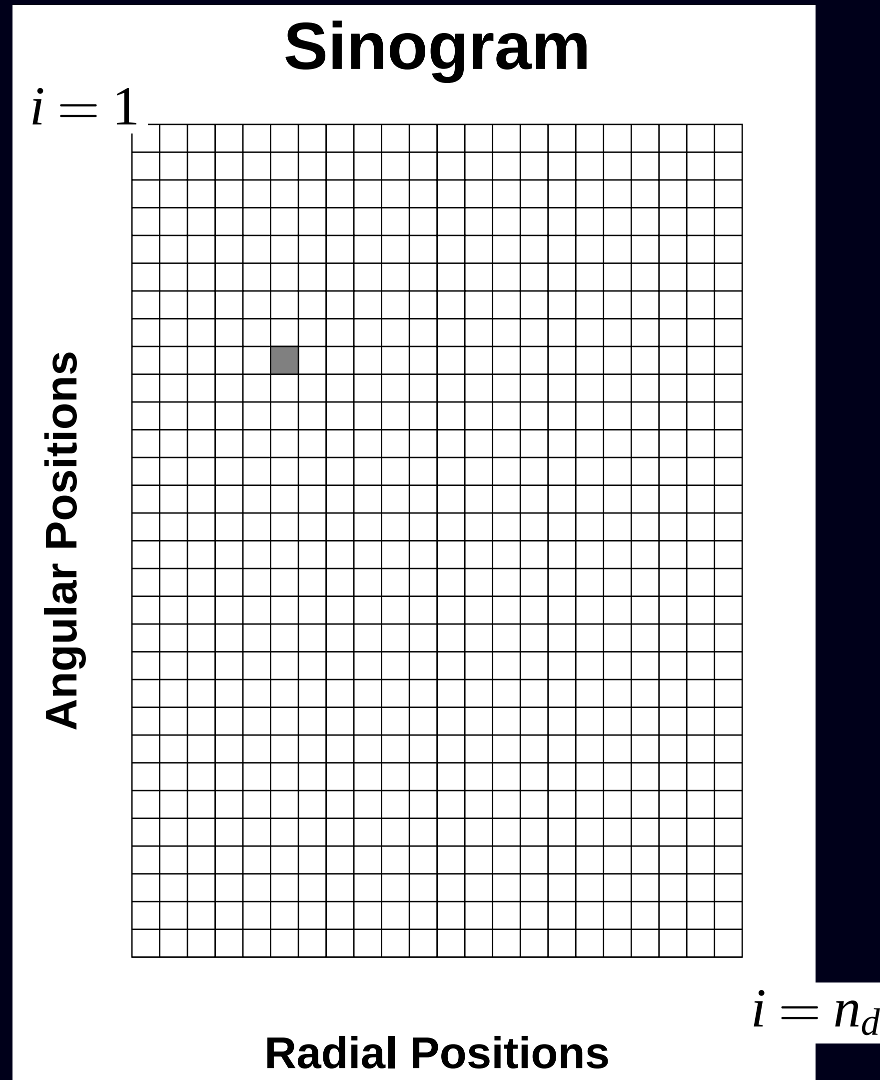




$$n_d \le (n_{\text{crystals}} - 1) \cdot n_{\text{crystals}}/2$$

SPECT Example





$$n_d = n_{\text{radial\_bins}} \cdot n_{\text{angular\_steps}}$$

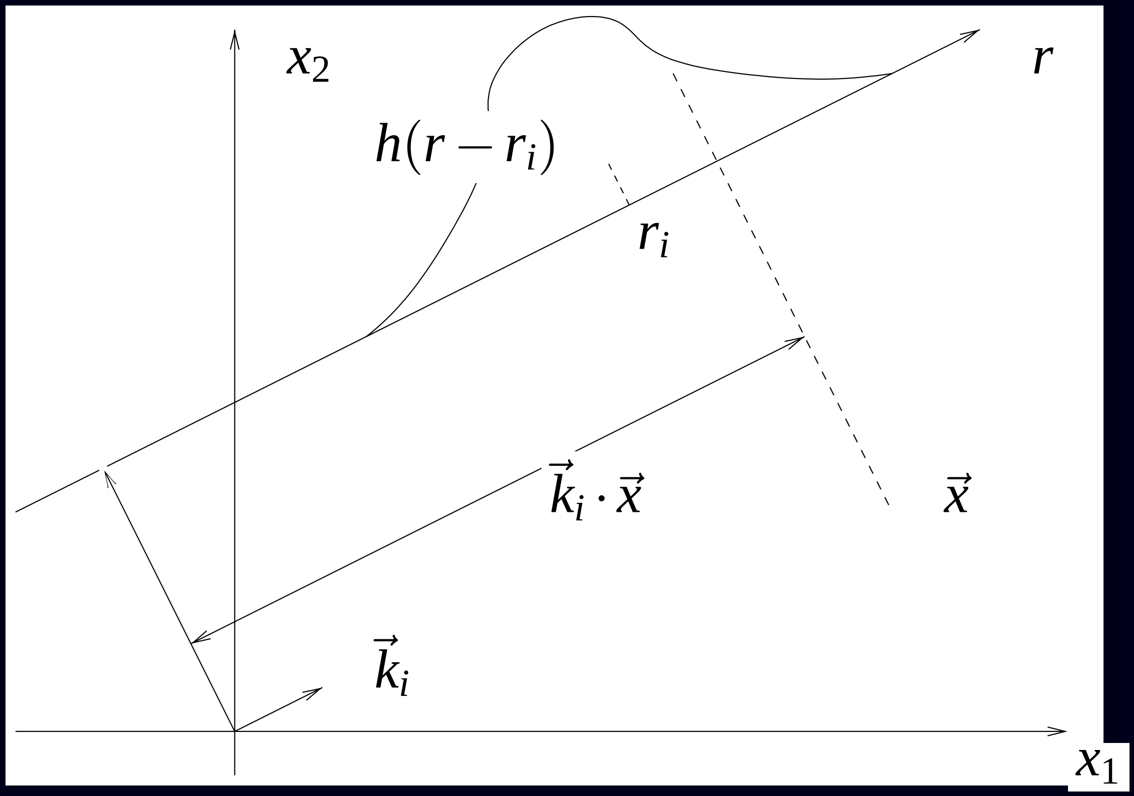
Detector Unit Sensitivity Patterns

Spatial localization:

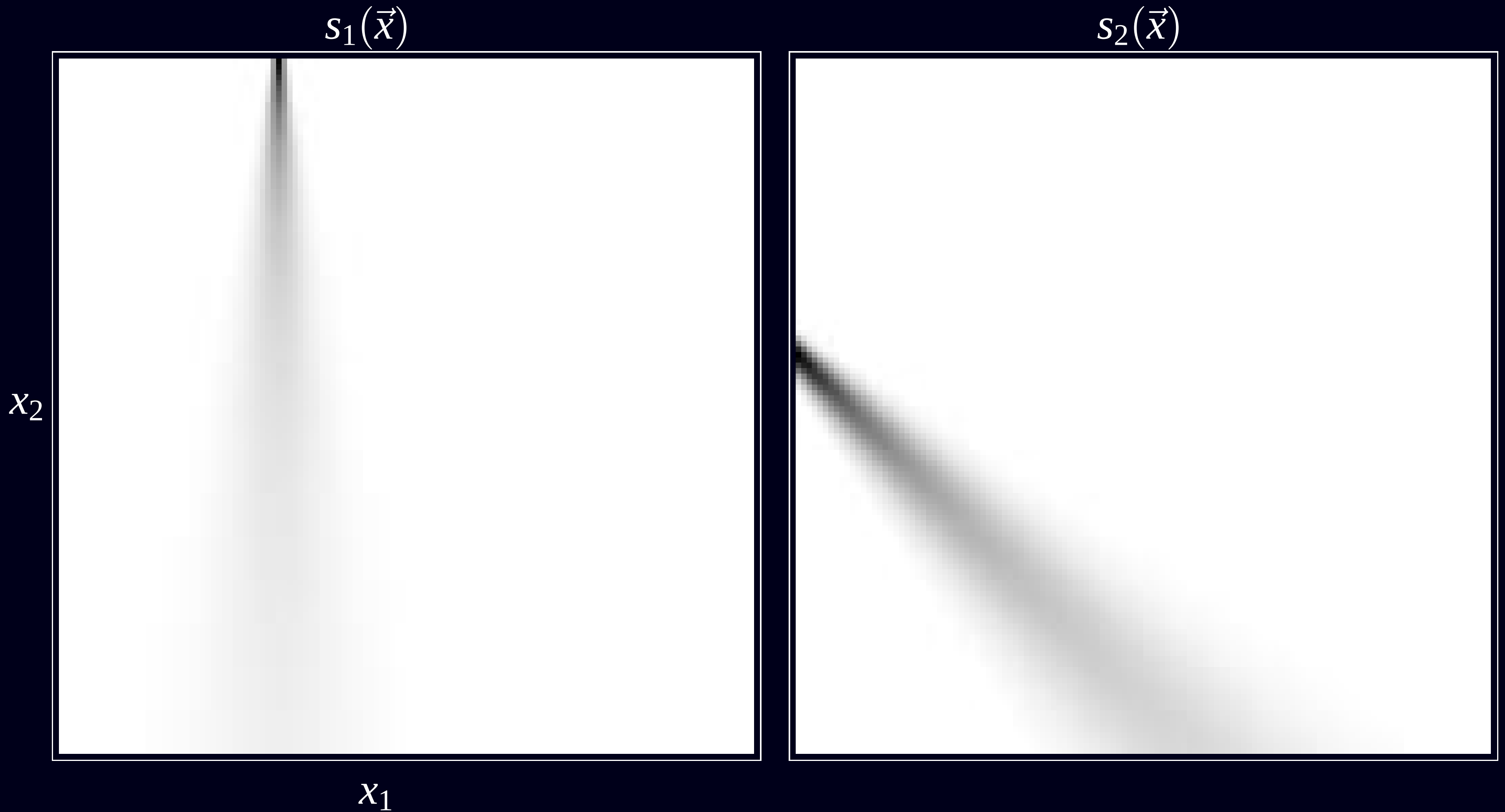
$$s_i(\vec{x}) \triangleq \text{probability that decay at } \vec{x} \text{ is recorded by } i\text{th detector unit.}$$

Idealized Example. Shift-invariant PSF: $s_i(\vec{x}) = h(\vec{k}_i \cdot \vec{x} - r_i)$

- r_i is the radial position of ith ray
- $\vec{k}_i$ is the unit vector orthogonal to ith parallel ray
- $h(\cdot)$ is the shift-invariant radial PSF (*e.g.*, Gaussian bell or rectangular function)

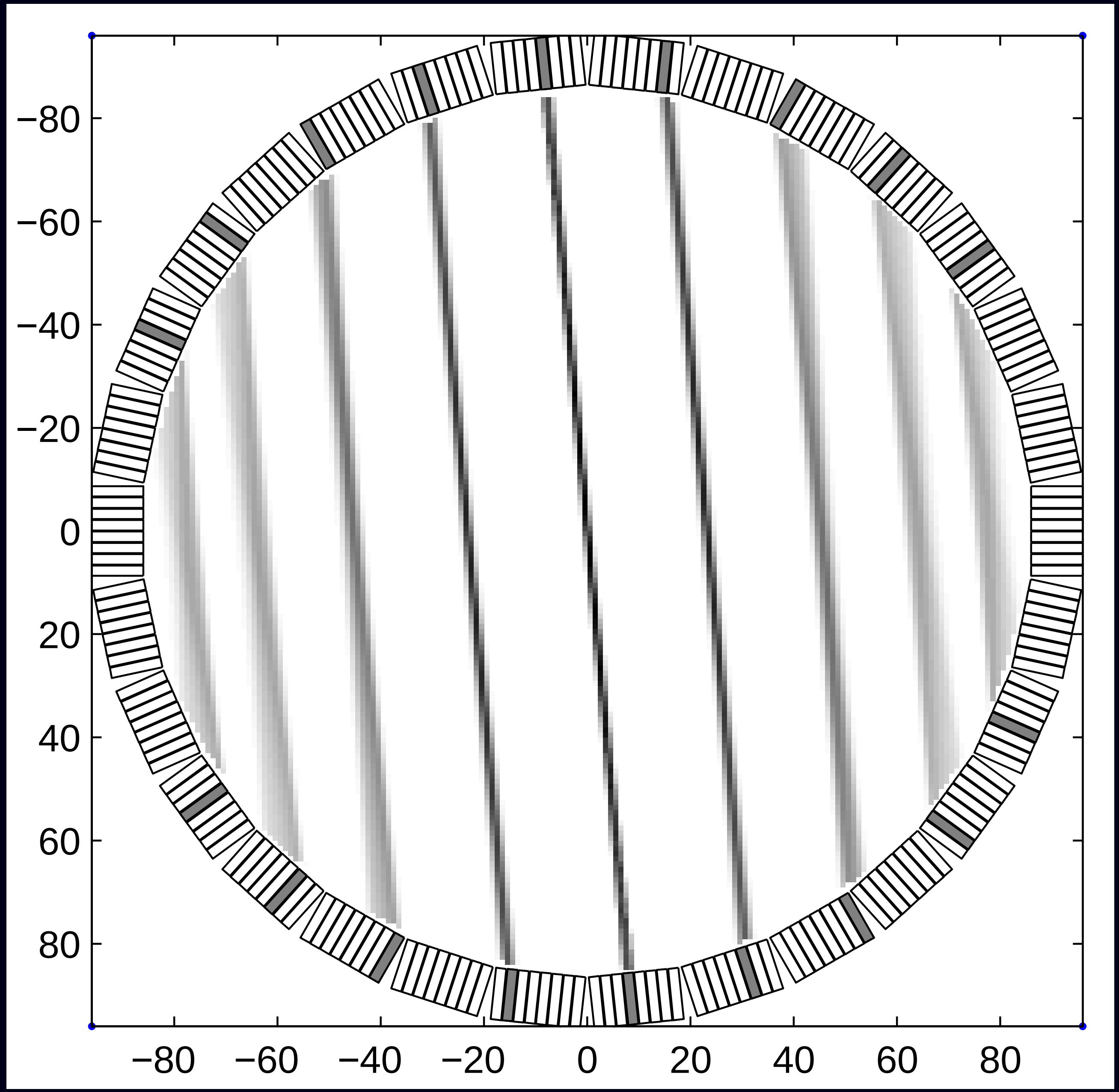


Example: SPECT Detector-Unit Sensitivity Patterns



Two representative $s_i(\vec{x})$ functions for a collimated Anger camera.

Example: PET Detector-Unit Sensitivity Patterns



Detector Unit Sensitivity Patterns

$s_i(\vec{x})$ can include the effects of

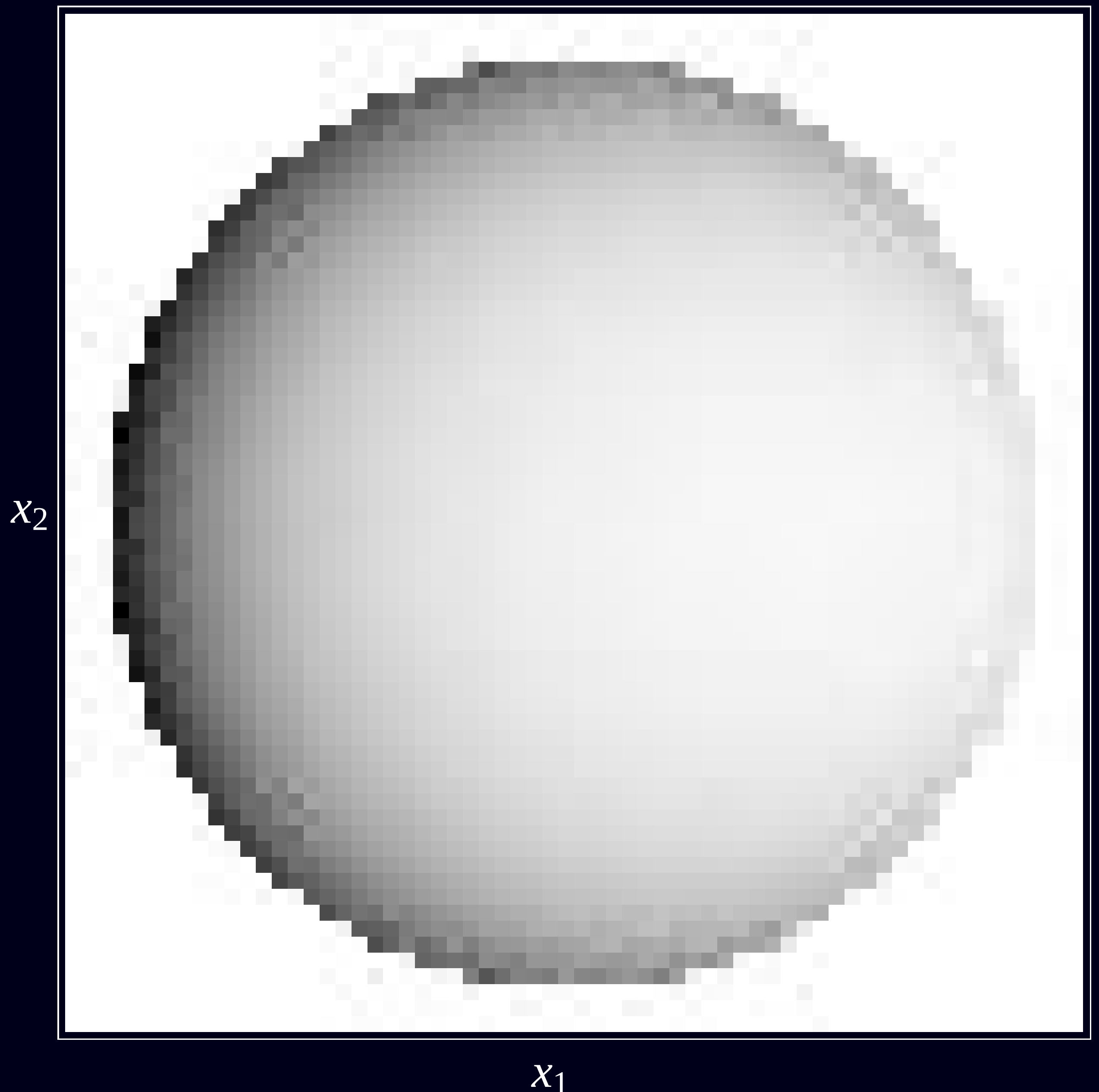
- geometry / solid angle
- collimation
- scatter
- attenuation
- detector response / scan geometry
- duty cycle (dwell time at each angle)
- detector efficiency
- positron range, noncollinearity
- ...

System sensitivity pattern:

$$s(\vec{x}) \triangleq \sum_{i=1}^{n_d} s_i(\vec{x}) = 1 - s_0(\vec{x}) \leq 1$$

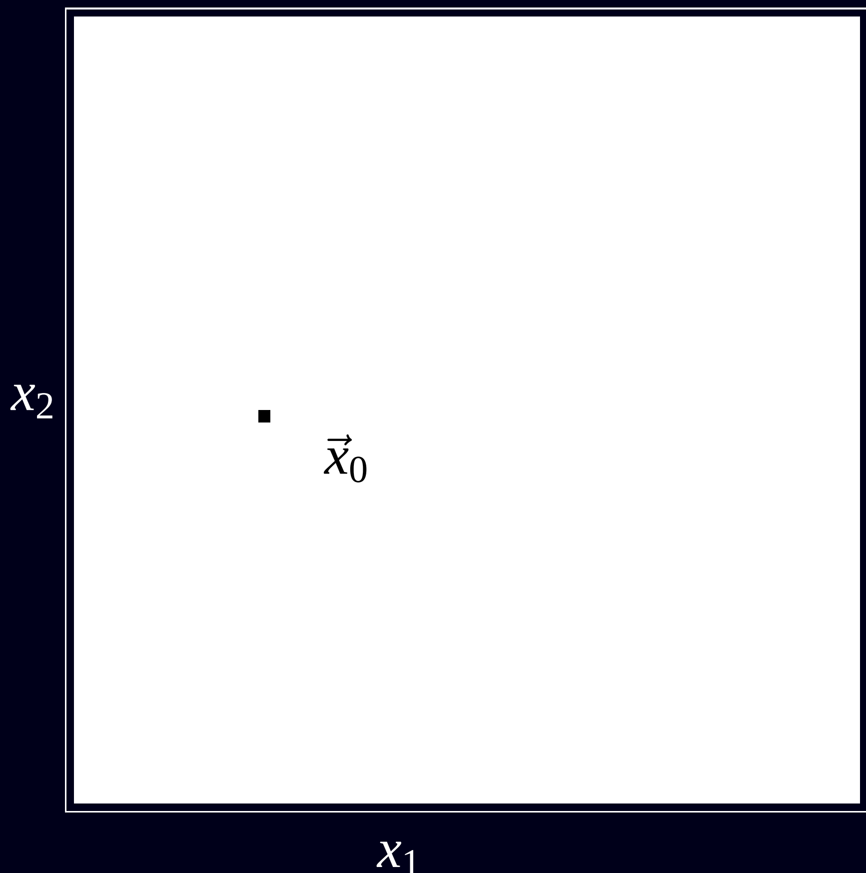
(probability that decay at location $\vec{x}$ will be detected *at all* by system)

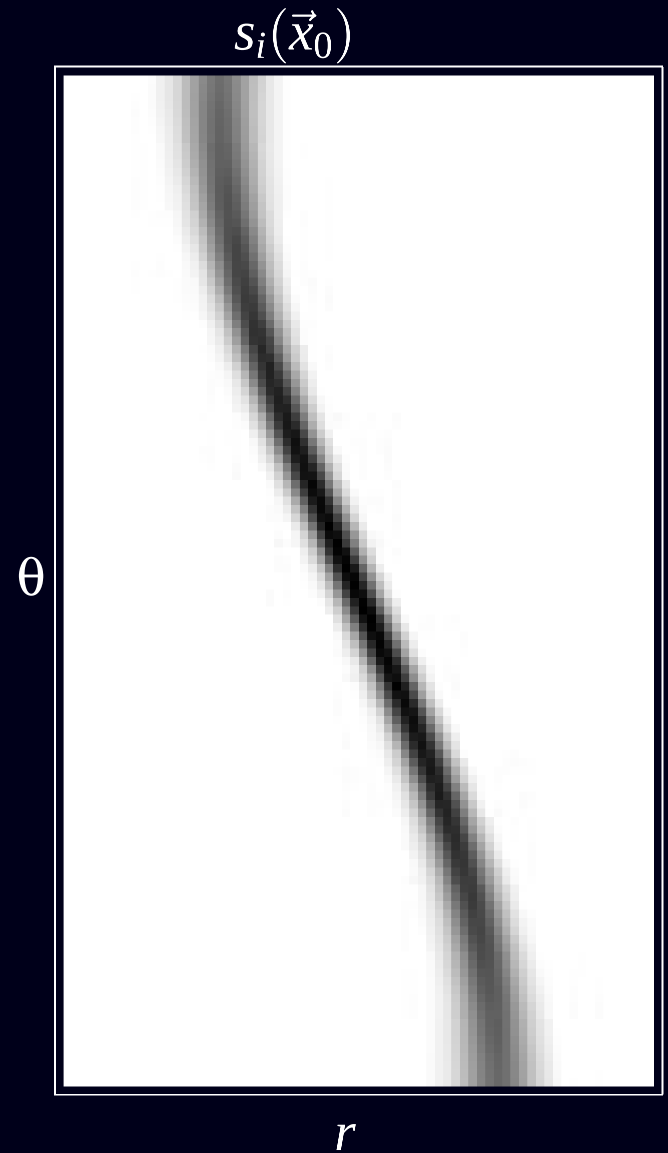
System Sensitivity Pattern $s(\vec{x})$



Example: collimated 180° SPECT system with uniform attenuation.

Detection Probabilities $s_i(\vec{x}_0)$ (vs det. unit index i)





Summary of Random Phenomena

- Number of tracer atoms injected N
- Spatial locations of tracer atoms $\{\vec{X}_k\}_{k=1}^{N}$
- Time of decay of tracer atoms $\{T_k\}_{k=1}^{N}$
- Detection of photon $[S_k \neq 0]$
- Recording detector unit $\{S_k\}_{i=1}^{n_d}$

Emission Scan

Record events in each detector unit for $t_1 \le t \le t_2$.

$Y_i \triangleq$ number of events recorded by ith detector unit during scan, for $i = 1, \ldots, n_d$.

$$Y_i \triangleq \sum_{k=1}^{N} 1_{\{S_k = i,\ T_k \in [t_1, t_2]\}}.$$

The collection $\{Y_i : i = 1, \ldots, n_d\}$ is our *sinogram*. Note $0 \le Y_i \le N$.

Fact. Under Assumptions 1-6 above,

$$Y_i \sim \text{Poisson}\left\{ \int s_i(\vec{x}) \lambda(\vec{x}) \, d\vec{x} \right\}$$ (cf “line integral”)

and Y_i’s are statistically independent random variables,
where the *emission density* is given by

$$\lambda(\vec{x}) = \mu_N \int_{t_1}^{t_2} \frac{1}{\mu_T} e^{-t/\mu_T} f_{\vec{X}(t)}(\vec{x}) \, dt.$$

(Local number of decays per unit volume during scan.)

Ingredients:

- dose (injected)
- duration of scan
- decay of radionuclide
- distribution of radiotracer

Poisson Statistical Model (Emission)

Actual measured counts = “foreground” counts + “background” counts.

Sources of background counts:

- cosmic radiation / room background
- random coincidences (PET)
- scatter not account for in $s_i(\vec{x})$
- “crosstalk” from transmission sources in simultaneous T/E scans
- anything else not accounted for by $\int s_i(\vec{x})\lambda(\vec{x})\,\mathrm{d}\vec{x}$

Assumption 7.
The background counts also have independent Poisson distributions.

Statistical model (continuous to discrete)

$$Y_i \sim \text{Poisson}\left\{\int s_i(\vec{x})\lambda(\vec{x})\,\mathrm{d}\vec{x} + r_i\right\}, \qquad i = 1,\ldots,n_d$$

r_i : mean number of “background” counts recorded by ith detector unit.

Emission Reconstruction Problem

Estimate the emission density $\lambda(\cdot)$ using (something like) this model:

$$Y_i \sim \text{Poisson}\left\{ \int s_i(\vec{x})\lambda(\vec{x})\,\mathrm{d}\vec{x} + r_i \right\}, \qquad i = 1, \ldots, n_d.$$

Knowns:

- $\{Y_i = y_i\}_{i=1}^{n_d}$: observed counts from each detector unit
- $s_i(\vec{x})$ sensitivity patterns (determined by system models)
- r_i's : background contributions (determined separately)

Unknown: $\lambda(\vec{x})$

List-mode acquisitions

Recall that conventional sinogram is temporally binned:

$$Y_i \triangleq \sum_{k=1}^{N} 1_{\{S_k = i,\; T_k \in [t_1, t_2]\}}.$$

This binning discards temporal information.

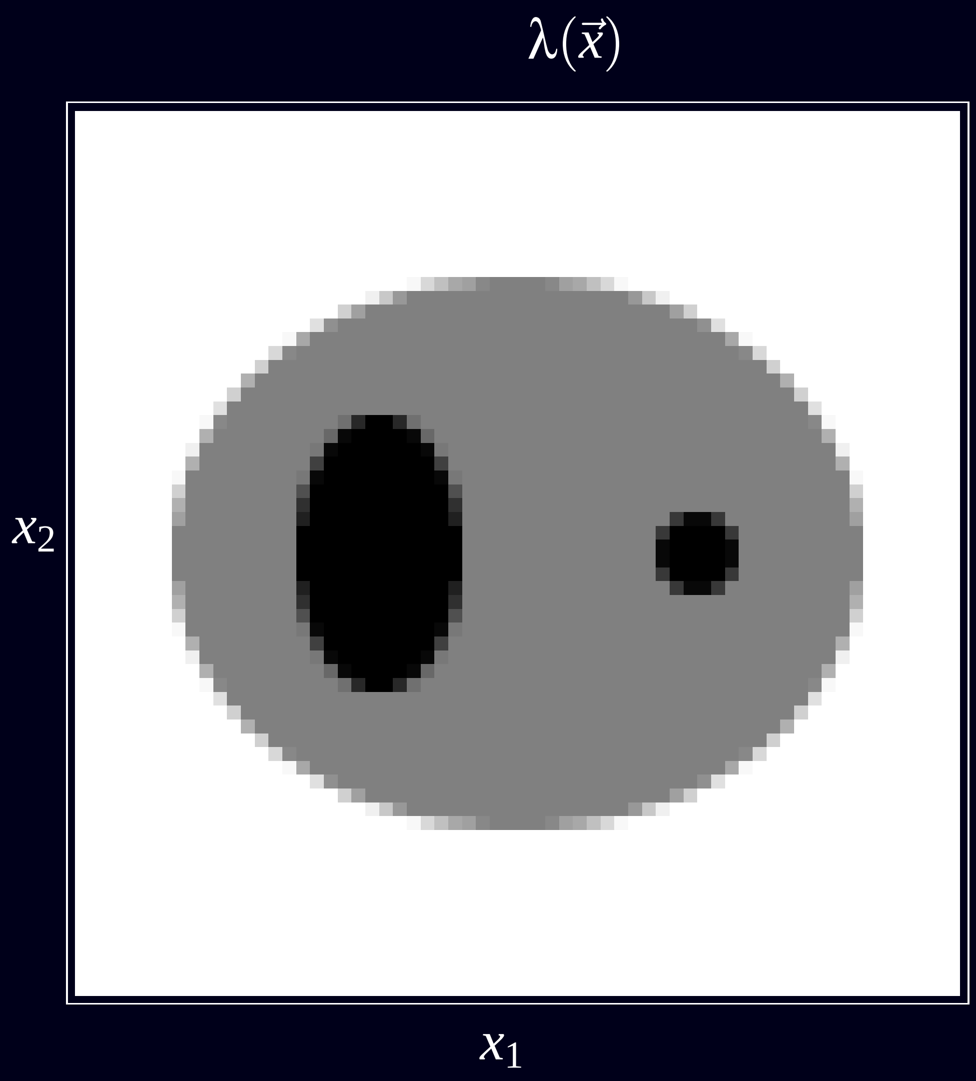
List-mode measurements: record all (detector,time) pairs in a list, *i.e.*,

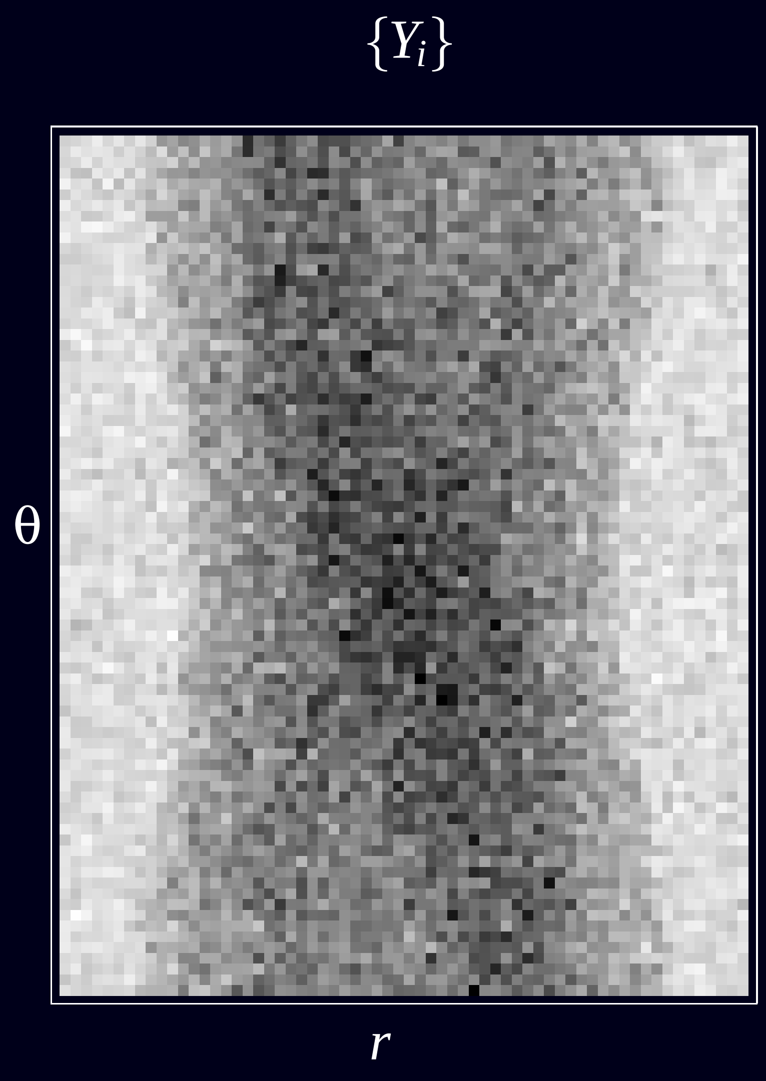
$$\{(S_k, T_k) : k = 1, \ldots, N\}.$$

List-mode dynamic reconstruction problem:

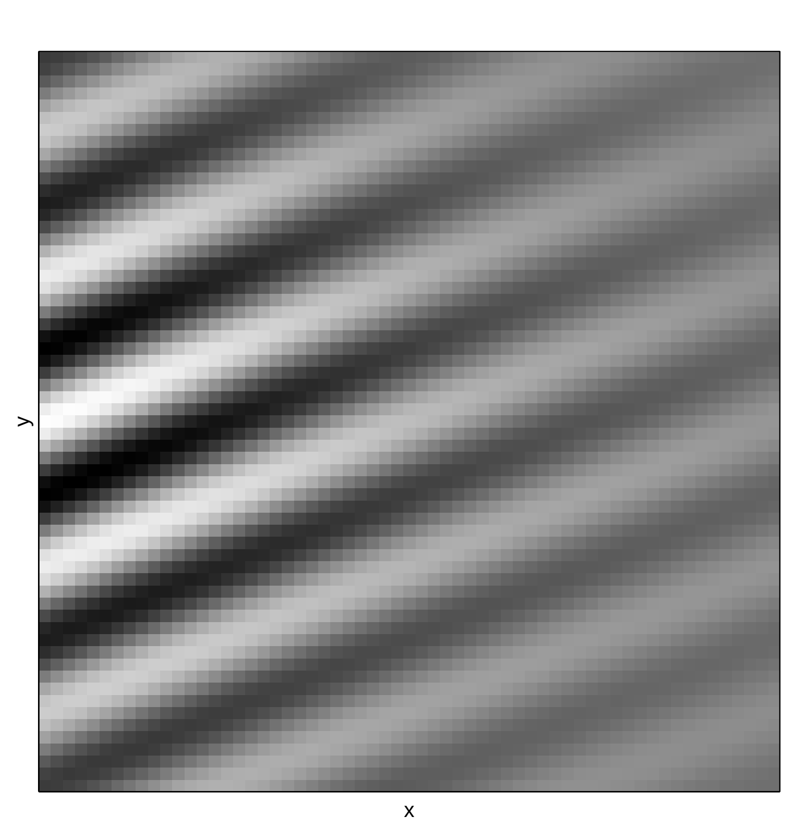
Estimate $\lambda(\vec{x}, t)$ given $\{(S_k, T_k)\}$.

Emission Reconstruction Problem - Illustration





Example: MRI "Sensitivity Pattern"



Each "k-space sample" corresponds to a sinusoidal pattern weighted by:

- RF receive coil sensitivity pattern
- phase effects of field inhomogeneity
- spin relaxation effects.

$$y_i = \int f(\vec{x})\, c_{\mathrm{RF}}(\vec{x}) \exp(-\imath\omega(\vec{x})t_i) \exp(-t_i/T_2(\vec{x})) \exp\left(-\imath 2\pi \vec{k}(t_i)\cdot\vec{x}\right) \mathrm{d}\vec{x} + \varepsilon_i$$