Parallelizable algorithms for image recovery problems

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Executive Summary

- In a wide variety of estimation problems, one estimates an unknown parameter vector $\mathbf{x}^{\text{true}}$ by minimizing a *partially separable* cost function:

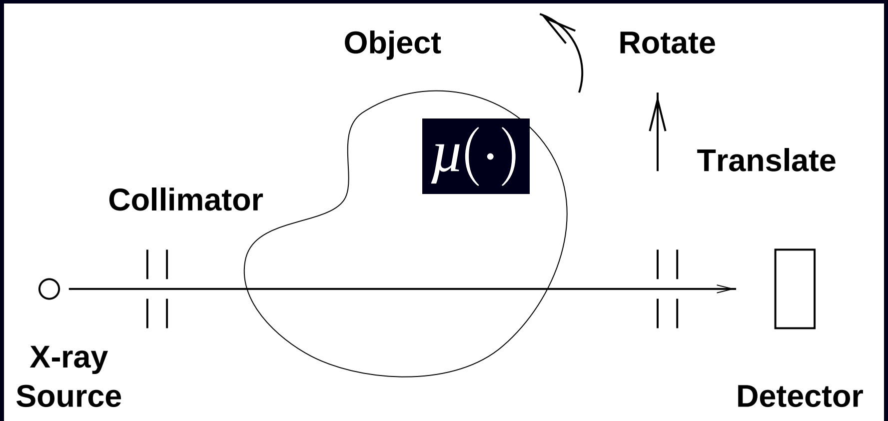
$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x} \geq \mathbf{0}} \Phi(\mathbf{x}), \qquad \Phi(\mathbf{x}) = \sum_i \psi_i([\mathbf{B}\mathbf{x} - \mathbf{c}]_i), \qquad [\mathbf{B}\mathbf{x} - \mathbf{c}]_i = \sum_j b_{ij} x_j - c_i.$$

- Fast methods for estimating $\mathbf{x}^{\text{true}}$ by minimizing $\Phi(\mathbf{x})$ are essential for successful routine use in applications such as medical tomography.
- We have developed fast converging algorithms for minimizing $\Phi(\mathbf{x})$.
- One algorithm has the fast convergence of coordinate descent, yet is parallelizable.
- The new algorithms converge faster than general-purpose minimization methods.

Outline

- Motivating applications and cost functions
- Edge-preserving regularization
- Unified cost function
- Minimization algorithms
  - Optimization transfer
  - Separable paraboloidal surrogates (SPS) algorithm
  - Paraboloidal surrogate coordinate descent (PSCD) algorithm
  - Parallelizable coordinate descent algorithm
- Representative results
- Summary and future work

Application: X-ray Computed Tomography



Statistical model: $$Y_i \sim \text{Poisson}\{b_i \exp(-[\mathbf{Ax}]_i) + r_i\}$$

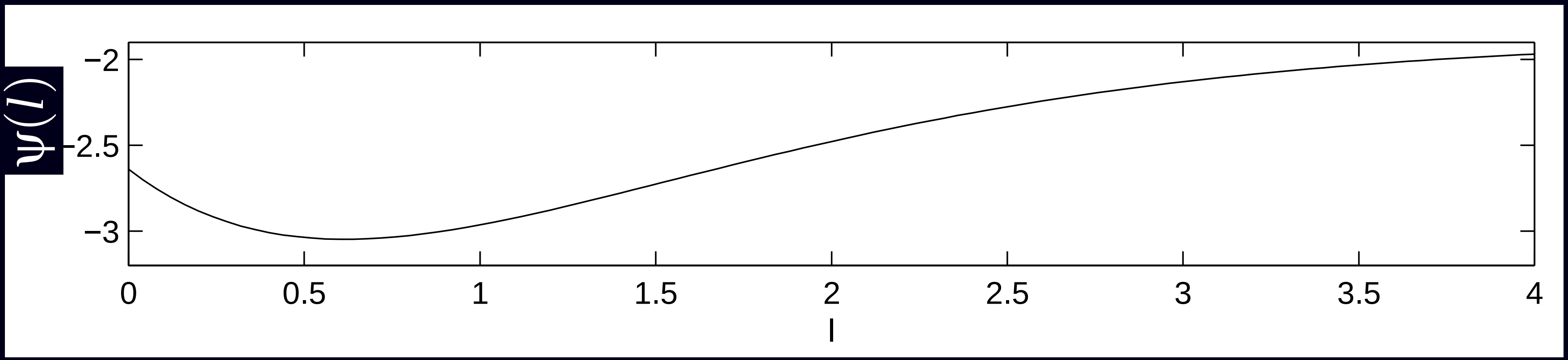
- Y_i: measurement along ith ray (statistically independent), $i = 1, \ldots, n_d$
- x_j: unknown attenuation coefficient in the jth voxel
- b_i: mean number of transmitted photons along ith ray
- a_{ij}: Radon projection matrix
- r_i: random coincidences and scatter
- Beer's Law for photon survival probability: $e^{-\int \mu(\cdot)\, dl}$
- $[\mathbf{Ax}]_i$: discrete approximation to line integral along ith ray

X-ray CT Statistical Image Reconstruction

It is natural to estimate the attenuation image $\mathbf{x}$ by finding the "best fit" to the sinogram data, as measured by the log-likelihood:

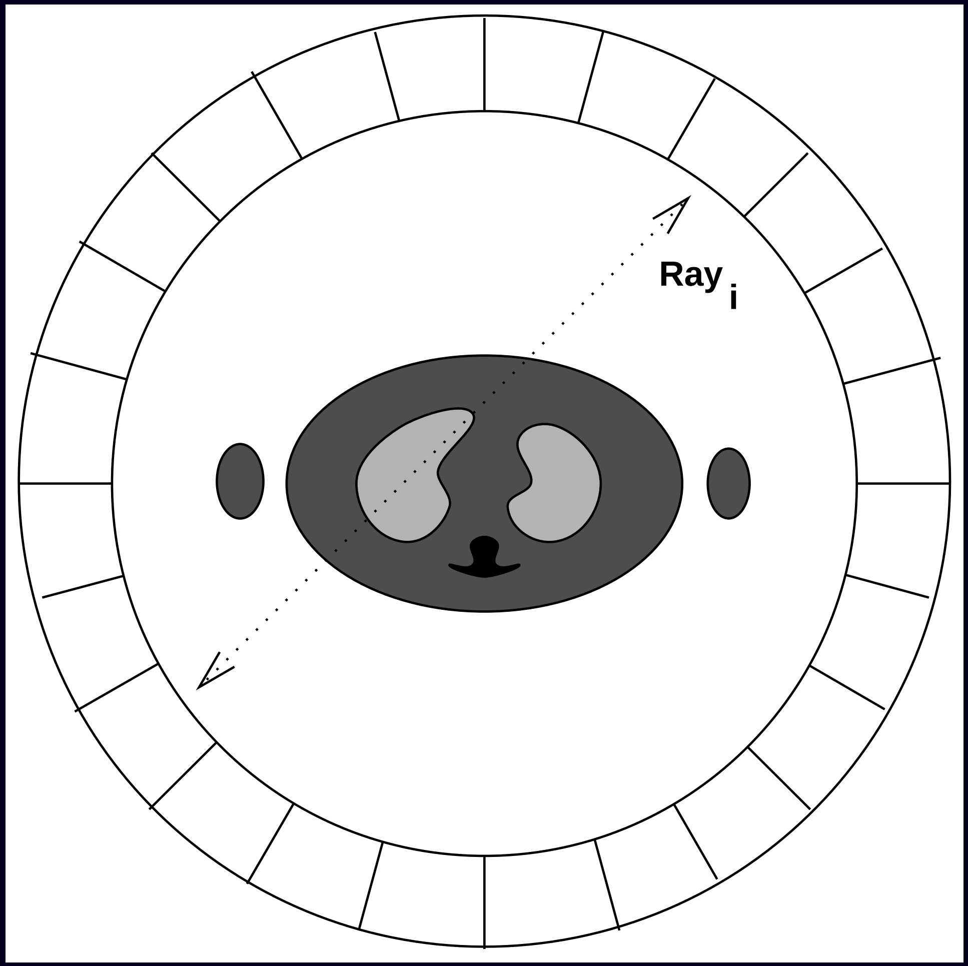
$$\hat{\mathbf{x}}_{\mathrm{ML}} \triangleq \arg\min_{\mathbf{x} \geq \mathbf{0}} \Phi^{\mathrm{data}}(\mathbf{x}) \quad \text{where} \quad \Phi^{\mathrm{data}}(\mathbf{x}) = \sum_{i=1}^{n_d} \psi_i([\mathbf{A}\mathbf{x}]_i)$$

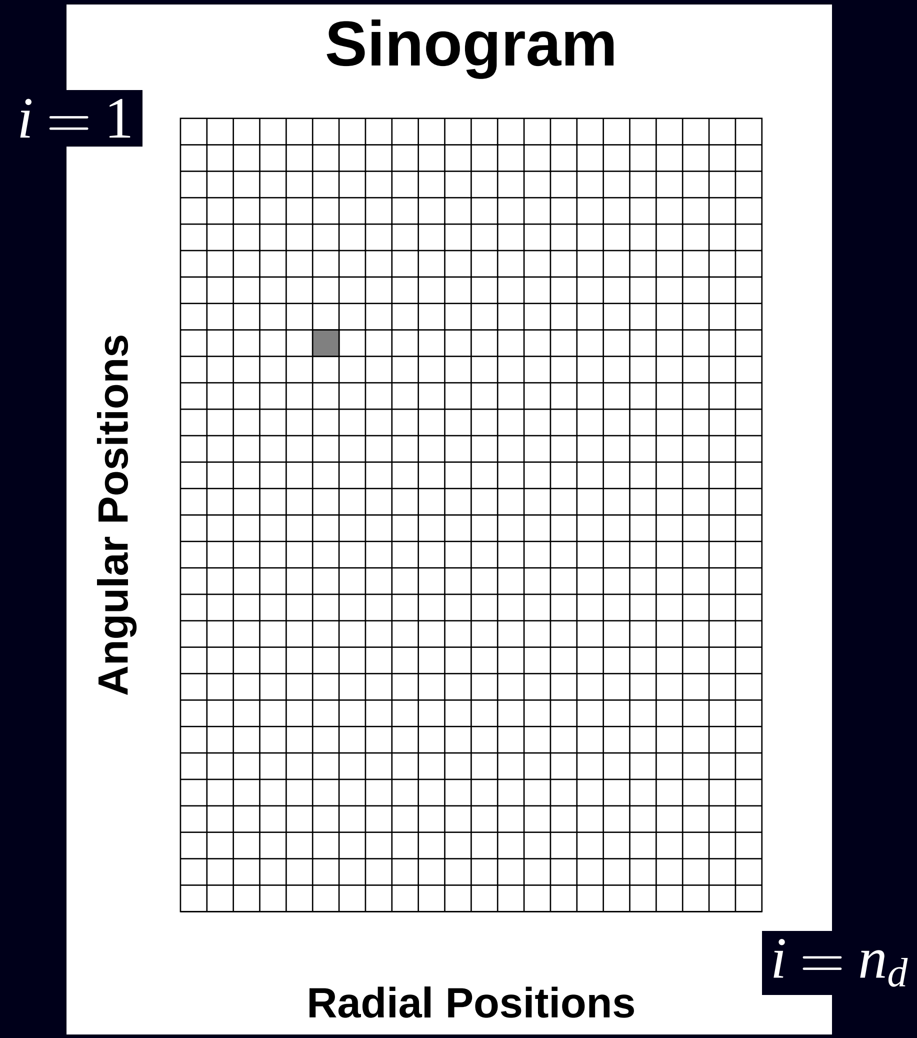
$$\psi_i(l) \triangleq \left(b_i e^{-l} + r_i\right) - Y_i \log\left(b_i e^{-l} + r_i\right).$$



- Summation form due to independence of recorded photon counts.
- Inner products $[\mathbf{A}\mathbf{x}]_i$ due to Beer's law and line integrals
- ψ_i's determined by Poisson negative log-likelihood

Application: PET Image Reconstruction





$$n_d \approx (n_{\text{crystals}})^2$$

PET Image Reconstruction

Most statistical methods for PET image reconstruction are based on the following Poisson statistical model.

$$Y_i \sim \text{Poisson}\left\{ \varepsilon_i s_i \sum_j g_{ij} x_j + r_i \right\}, \; i = 1, \ldots, n_d.$$

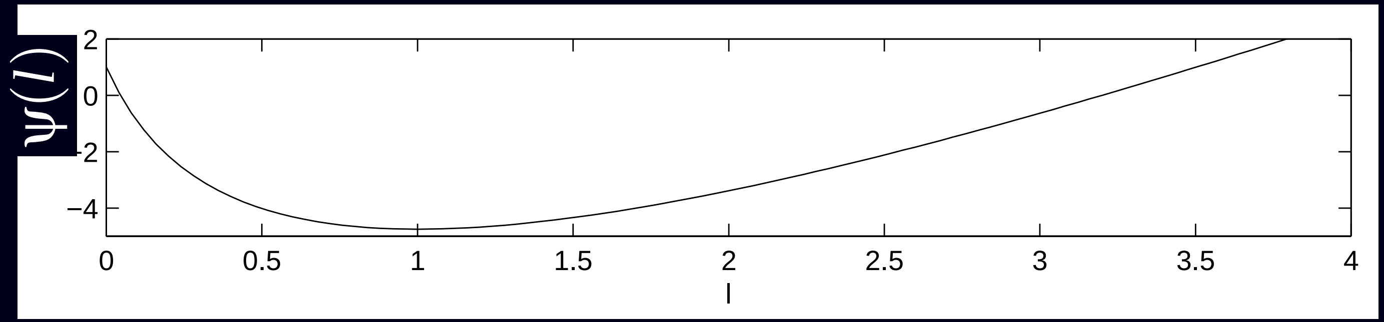
- Y_i: measured counts in sinogram bins (statistically independent)
- x_j: unknown radiotracer concentration in the jth voxel
- ε_i: ith detector efficiency
- s_i: photon survival probability along ith ray (attenuation)
- g_{ij}: projection matrix
- r_i: random coincidences and scatter
- n_d: number of detector pairs

Maximum-Likelihood PET Image Reconstruction

If the Poisson model is valid, it is natural to estimate the emission image $\mathbf{x}$ by finding the "best fit" to the sinogram data, as measured by the log-likelihood:

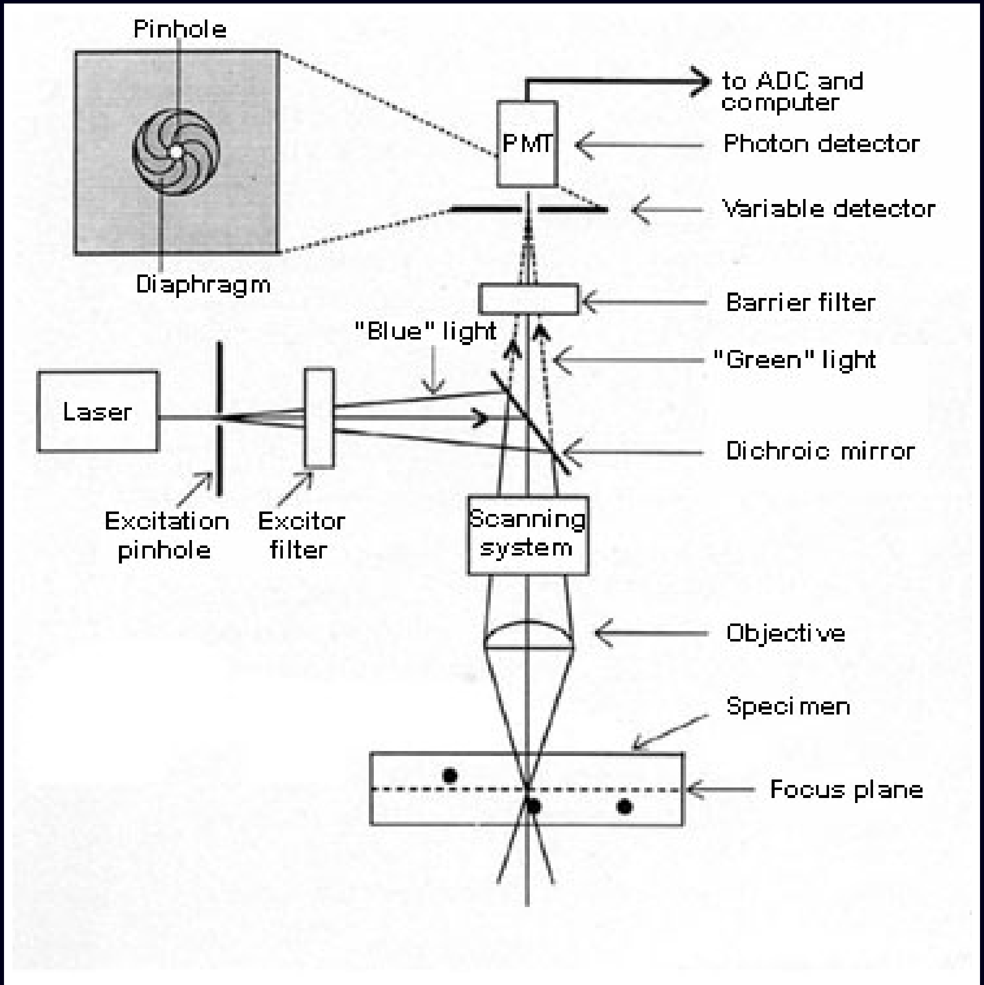
$$\hat{\mathbf{x}}_{\mathrm{ML}} \triangleq \arg\min_{\mathbf{x}\geq\mathbf{0}} \Phi^{\mathrm{data}}(\mathbf{x}) \quad \text{where} \quad \Phi^{\mathrm{data}}(\mathbf{x}) = \sum_{i=1}^{n_d} \psi_i([\mathbf{A}\mathbf{x}]_i)$$

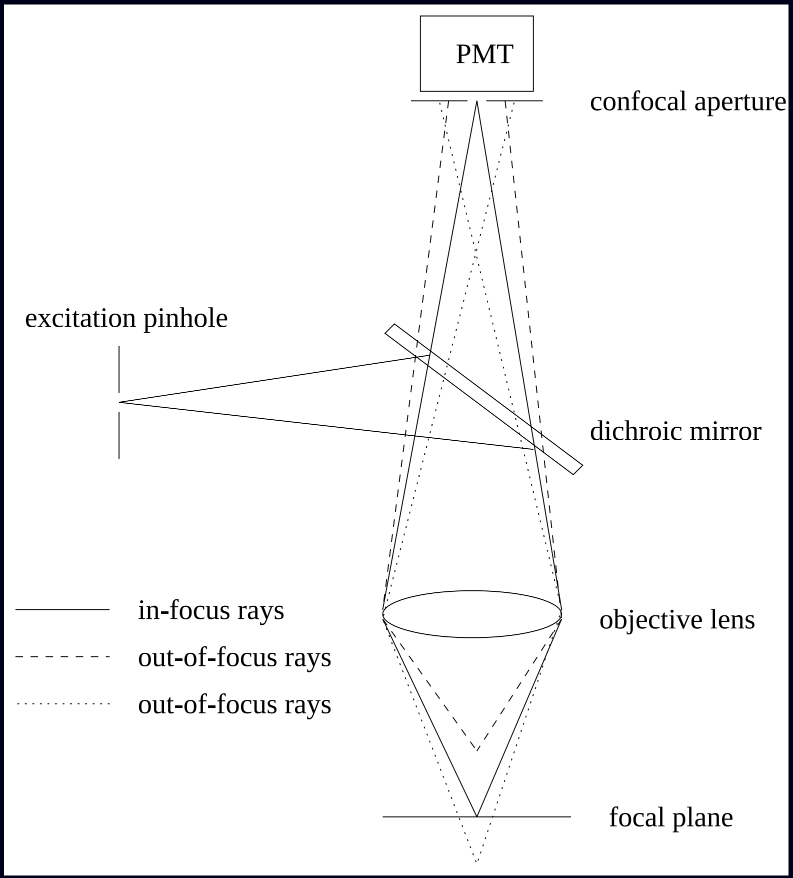
$$\psi_i(l) \triangleq (l + r_i) - Y_i \log(l + r_i), \quad a_{ij} \triangleq \varepsilon_i s_i g_{ij}.$$



- Summation form due to independence of recorded photon counts.
- Inner products $[\mathbf{A}\mathbf{x}]_i$ due to Radon tomographic projection
- ψ_i's determined by Poisson negative log-likelihood

Application: Confocal Microscopy 3D Image Restoration





Cost function is comparable to that of PET / SPECT.

Application: Robust Multiuser Detection

Wang and Poor, Feb. 1999 IEEE Tr. Sig. Proc.
"Robust multi-user detection in non-Gaussian channels"

Model for direct-sequence code-division multiple access (CDMA):

$$Y_i = \sum_{j=1}^{K} a_{ij} x_j + N_i, \; i = 1, \ldots, n_d$$

- Y_i: sampled output of chip-matched filter
- x_j: jth information bit scaled by received amplitude
- N_i: possibly non-Gaussian noise
- a_{ij}: signature sequence of jth user

Robust bit estimator (using, *e.g.*, the Huber function):

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \Phi(\mathbf{x}), \qquad \Phi(\mathbf{x}) = \sum_{i=1}^{n_d} \psi(Y_i - [\mathbf{A}\mathbf{x}]_i)$$

Application: Physics-based MR image reconstruction

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \text{noise}$$

- $\mathbf{y}$: samples in spatial frequency space
- $\mathbf{x}$: object transverse magnetization
- $\mathbf{A}$: Fourier transform modified by magnetic field inhomogeneity

$$Y_i = \sum_{j=1}^{n_p} x_j \exp\left(\sqrt{-1}2\pi\left[\underline{k}_i \cdot \underline{r}_j + \Delta_j t_i\right]\right)$$

- $\underline{k}_i$: frequency space location of ith sample
- $\underline{r}_j$: coordinates of jth voxel
- Δ_j: field inhomogeneity induced off-resonance frequency for jth voxel
- t_i: time of ith sample

Gaussian noise, so $\psi_i(t) = t^2/2$ (least squares estimation)

Edge-preserving Regularization

Minimizing Φ^{data} alone is inadequate for ill-conditioned inverse problems.

Generic prior "knowledge" of piece-wise smoothness:

- $x_j - x_{j-1} \approx 0$ (piece-wise constant)
- $x_{j-1} - 2x_j + x_{j+1} \approx 0$ (piece-wise linear)
- $x_j \approx 0, \; j \in J \subset \{1, \ldots, n_p\}$ (support constraints)
- $\ldots$

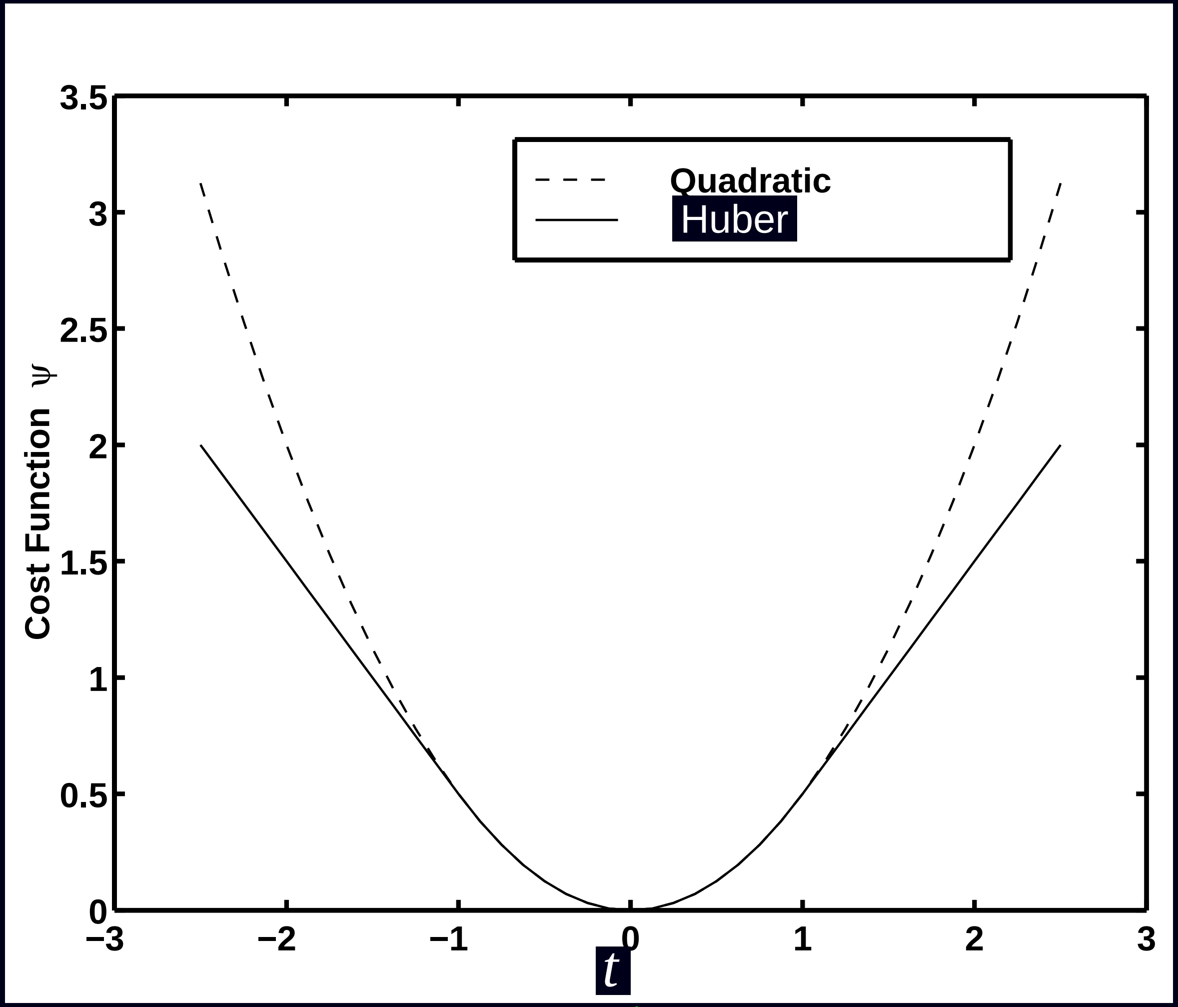
Combining: $\mathbf{C}\mathbf{x} \approx \mathbf{z}$ (where typically $\mathbf{z} = \mathbf{0}$).

Expressed as penalty function:

$$\Phi^{\text{penalty}}(\mathbf{x}) = \sum_k \psi_k^{\text{penalty}}?([\mathbf{C}\mathbf{x} - \mathbf{z}]_k).$$

To "preserve" edges, ψ_k^{penalty} should be nonquadratic.

Example of edge-preserving potential function

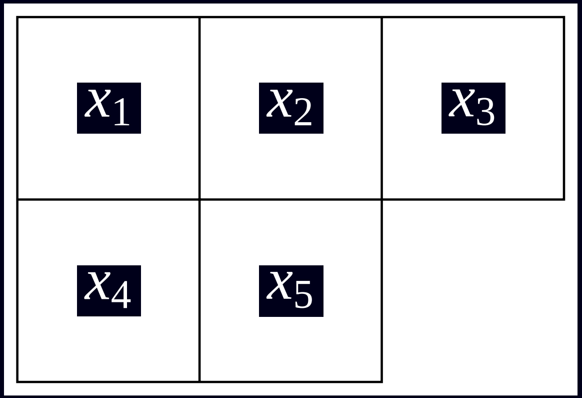


Huber function: $\psi(t) = \begin{cases} t^2/2, & |t| \leq \delta, \\ \delta|t| - \delta^2/2, & |t| > \delta \end{cases}$

Penalty Function: General Form

$$\Phi^{\text{penalty}}(\mathbf{x}) = \sum_k \psi_k([\mathbf{C}\mathbf{x}]_k), \quad \text{where} \quad [\mathbf{C}\mathbf{x}]_k = \sum_j c_{kj} x_j$$

Example:



$$\mathbf{C}\mathbf{x} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_5 - x_4 \\ x_4 - x_1 \\ x_5 - x_2 \end{bmatrix}$$

Unified Cost Function

$$\boxed{\Phi(\mathbf{x}) = \sum_{i=1}^{N} \psi_i([\mathbf{B}\mathbf{x} - \mathbf{c}]_i)}$$ "partially separable"

Regularized edge-preserving cost function is a special case:

$$\Phi(\mathbf{x}) = \Phi^{\text{data}}(\mathbf{x}) + \Phi^{\text{penalty}}(\mathbf{x}), \quad \mathbf{B} = \begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix}$$

Optimization problem:

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \Phi(\mathbf{x}) \quad \text{or} \quad \boxed{\hat{\mathbf{x}} = \arg\min_{\mathbf{x} \geq \mathbf{0}} \Phi(\mathbf{x}).}$$

This formulation encompasses a wide variety of inverse problems.

Challenges: nonnegativity constraint, nonquadratic ψ_i's, size of $\mathbf{B}$.

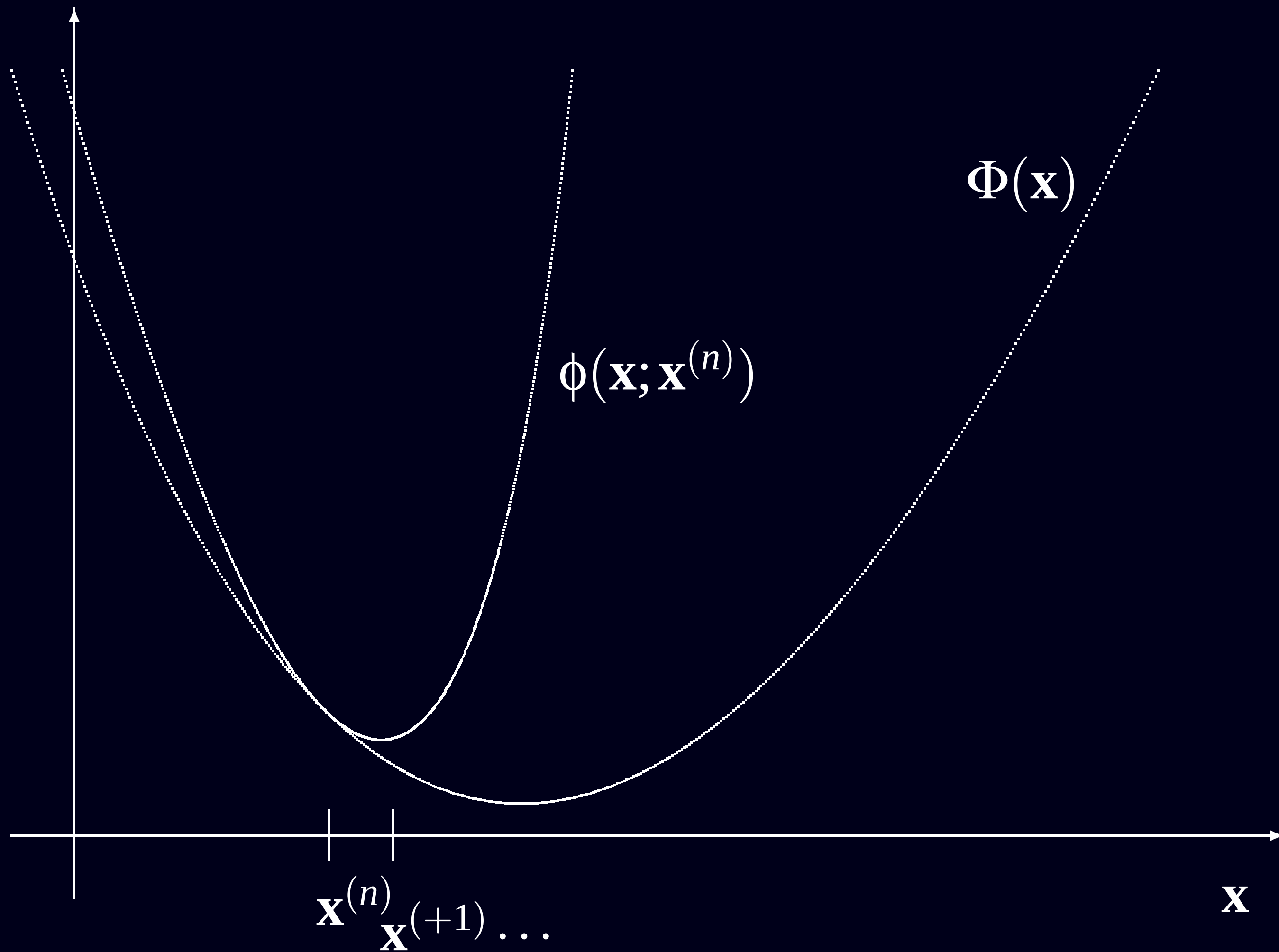
Ideal Algorithm

$$\hat{\mathbf{x}} \triangleq \arg\min_{\mathbf{x} \geq \mathbf{0}} \Phi(\mathbf{x}) \quad \text{(global minimizer)}$$

stable and convergent	$\{\mathbf{x}^{(n)}\}$ converges to $\hat{\mathbf{x}}$ if run indefinitely
converges quickly	$\{\mathbf{x}^{(n)}\}$ gets "close" to $\hat{\mathbf{x}}$ in just a few iterations
globally convergent	$\lim_n \mathbf{x}^{(n)}$ independent of starting image
fast	requires minimal computation per iteration
robust	insensitive to finite numerical precision
user friendly	nothing to adjust (*e.g.* acceleration factors)
monotonic	$\Phi(\mathbf{x}^{(n)})$ increases every iteration
parallelizable	(when necessary)
simple	easy to program and debug
flexible	accommodates any type of system model (matrix stored by row or column or projector/backprojector)

Choices: forgo one or more of the above

Optimization Transfer (1D illustration)



Optimization Transfer

(cf EM Algorithm)

- E-step: choose surrogate function $\phi(\mathbf{x};\mathbf{x}^{(n)})$
- M-step: minimize surrogate function

$$\mathbf{x}^{(n+1)} = \arg\min_{\mathbf{x} \geq \mathbf{0}} \phi(\mathbf{x};\mathbf{x}^{(n)})$$

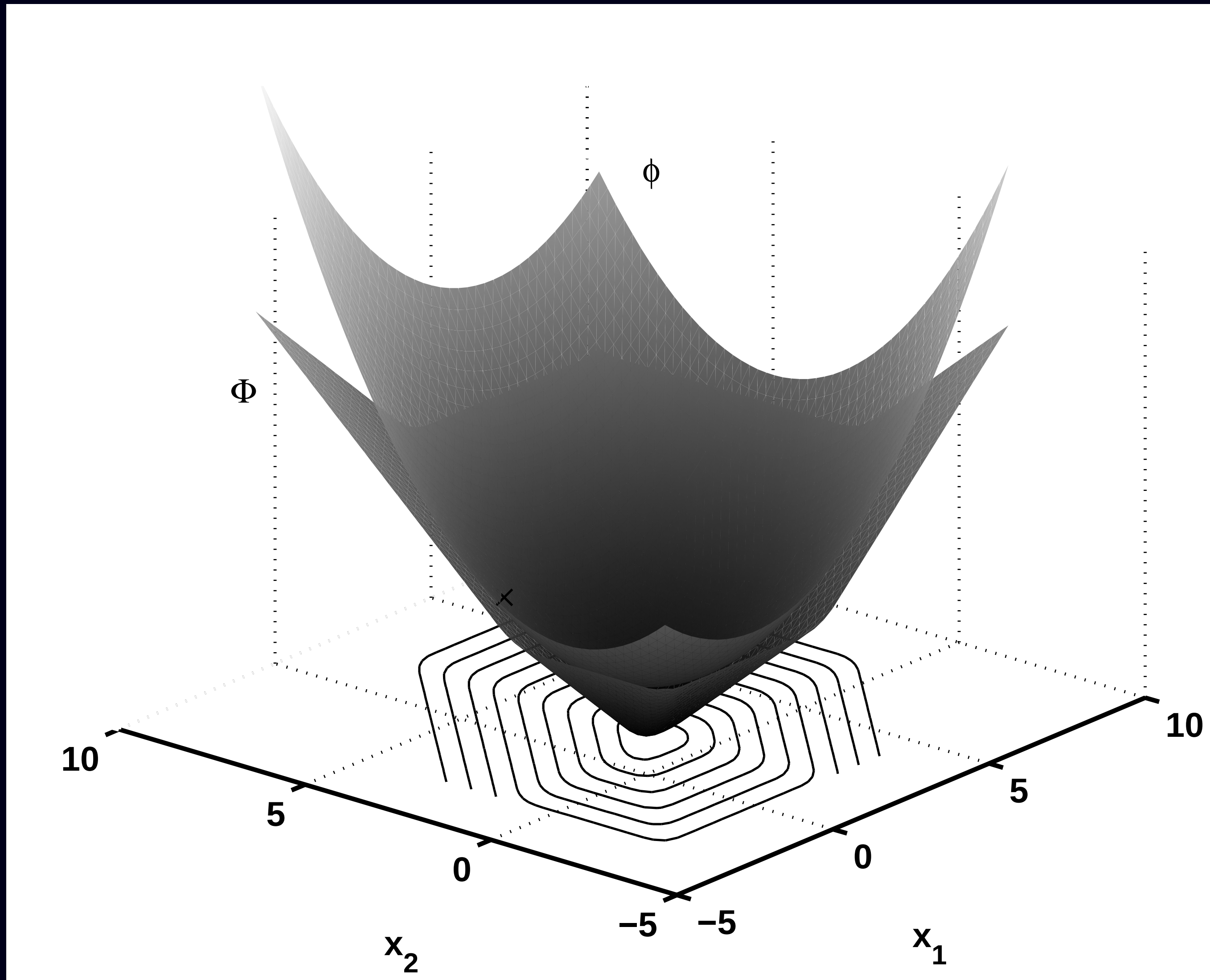
Surrogate design goals:

- Easy to "compute"
- Easy to minimize
- Fast convergence rate
- Monotone convergence

$$\Phi(\mathbf{x}^{(n)}) - \Phi(\mathbf{x}) \geq \phi(\mathbf{x}^{(n)};\mathbf{x}^{(n)}) - \phi(\mathbf{x};\mathbf{x}^{(n)})$$

Often mutually incompatible goals $\therefore$ compromises

Optimization Transfer in 2D



Exploiting Partial Separability (E-step)

Cost Function — Paraboloidal Surrogate Function

$$\Phi(\mathbf{x}) = \sum_{i=1}^{N} \psi_i([\mathbf{B}\mathbf{x} - \mathbf{c}]_i) \;\leq\; \phi(\mathbf{x}; \mathbf{x}^{(n)}) \triangleq \sum_{i=1}^{N} q_i([\mathbf{B}\mathbf{x} - \mathbf{c}]_i; t_i^{(n)}),$$

where $t_i^{(n)} \triangleq [\mathbf{B}\mathbf{x}^{(n)} - \mathbf{c}]_i)$.

1-D tangent parabola surrogate:

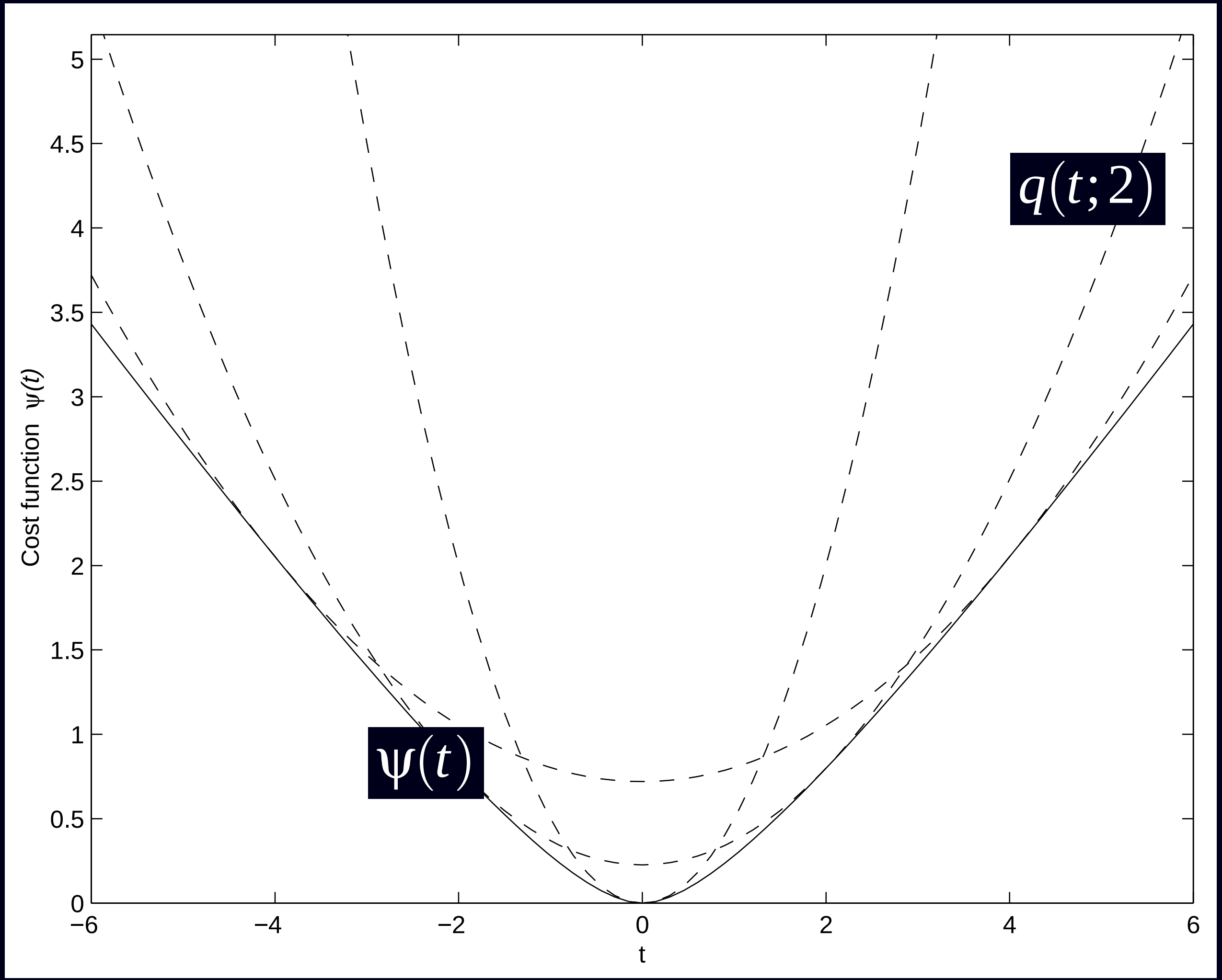
$$\psi_i(t) \leq q_i(t; t_i^{(n)}), \quad q_i(t; s) \triangleq \psi_i(s) + \dot{\psi}_i(s)(t - s) + \kappa_i(s)\frac{(t-s)^2}{2}.$$

Optimal parabola curvature (for fastest convergence rate):

$$\kappa_i(s) \triangleq \min\{\kappa \geq 0 \,:\, q_i(t; s) \geq \psi_i(t) \;\forall t\}.$$

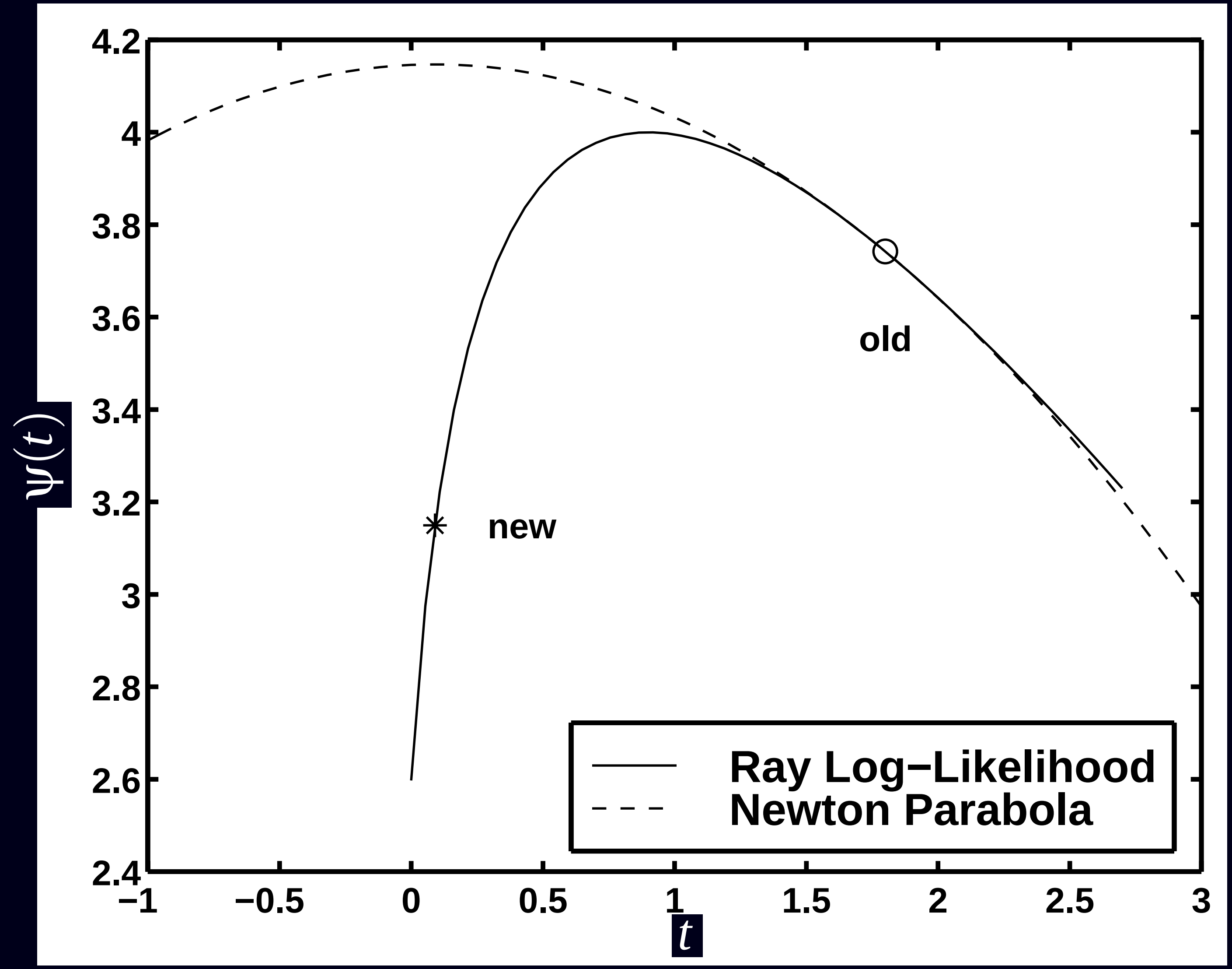
For Huber-like functions: $\kappa_i(s) = \dot{\psi}_i(s)/s \triangleq \omega_i(s)$.
For emission and transmission tomography, optimal κ_i derived by Erdoğan (Tr. Med. Im., 1999)

Tangent Parabolas



$\omega_\psi(t_0)$ is the curvature of the parabola that is tangent at t_0

Nonmonotonicity of Newton-Raphson



Minimizing the Paraboloidal Surrogate (M-step)

$$\phi(\mathbf{x};\mathbf{x}^{(n)}) = c_0 + \nabla\Phi(\mathbf{x}^{(n)})(\mathbf{x}-\mathbf{x}^{(n)}) - \frac{1}{2}(\mathbf{x}-\mathbf{x}^{(n)})'\mathbf{B}'\,\mathrm{diag}\left\{\kappa_i^{(n)}\right\}\mathbf{B}(\mathbf{x}-\mathbf{x}^{(n)}),$$

where the tangent parabola curvatures are:

$$\kappa_i^{(n)} = \kappa_i(t_i^{(n)}) = \kappa_i([\mathbf{B}\mathbf{x}^{(n)} - \mathbf{c}]_i).$$

M-step: Minimize $\phi(\mathbf{x};\mathbf{x}^{(n)})$ using any iterative “least squares” algorithm that accommodates nonnegativity constraints.

Reasonable choices of algorithms

- PSCD Paraboloidal surrogates coordinate descent:
  fast converging, but non-parallelizable
- SPS (separable paraboloidal surrogates):
  slow converging, but fully parallelizable
- PPCD (partitioned-separable paraboloidal surrogate coordinate descent)
  best of both worlds?

Paraboloidal surrogates coordinate descent (PSCD)

- Update one pixel at a time, w.r.t. the surrogate, holding other pixels fixed:

$$x_j^{(n+1)} = \arg\min_{x_j \geq 0} \phi(x_1^{(n+1)}, \ldots, x_{j-1}^{(n+1)}, x_j, x_{j+1}^{(n)}, \ldots, x_{n_p}^{(n)}; \mathbf{x}^{(n)}).$$

- Cycle through all pixels, then update the paraboloidal surrogate ($\kappa_i^{(n)}$'s).

Advantages:

- Intrinsically monotonic, global convergence (for a broad family of ψ_i's)
- Fast converging (from good initial image)
- Nonnegativity constraint trivial

Disadvantages:

- Requires column access of system matrix
- Poorly parallelizable

Separable paraboloid surrogate

One can use the convexity of the paraboloidal surrogate ϕ to define a second surrogate function that is **separable**:

$$\phi(\mathbf{x};\mathbf{x}^{(n)}) \leq \phi^{SP}(\mathbf{x};\mathbf{x}^{(n)}) \triangleq \sum_{j=1}^{n_p} \phi_j(x_j - x_j^{(n)};\mathbf{x}^{(n)})$$

where

$$\phi_j(t;\mathbf{x}^{(n)}) \triangleq \sum_{i=1}^{N} \pi_{ij}\kappa_i^{(n)} \frac{1}{2}\left(\frac{b_{ij}}{\pi_{ij}}t + [\mathbf{B}\mathbf{x}^{(n)} - \mathbf{c}]_i\right)^2,$$

$$\pi_{ij} = \frac{|b_{ij}|}{\sum_{j=1}^{n_p} |b_{ik}|}.$$

Minimizing the separable paraboloid ϕ^{SP} is trivial, especially compared to minimizing a paraboloid.

$$\mathbf{x}^{(n+1)} = \arg\min_{\mathbf{x}\geq\mathbf{0}} \phi(\mathbf{x};\mathbf{x}^{(n)}) \quad \Rightarrow \quad x_j^{(n+1)} = \arg\min_{x_j\geq\mathbf{0}} \phi_j(x_j - x_j^{(n)};\mathbf{x}^{(n)}), \; j = 1,\ldots,n_p.$$

Separable paraboloid surrogate (SPS) algorithm

$$x_j^{(n+1)} = \left[x_j^{(n)} - \frac{\dot{q}_j(0; \mathbf{x}^{(n)})}{\ddot{q}_j(0; \mathbf{x}^{(n)})} \right]_+ \quad \Rightarrow \quad \mathbf{x}^{(n+1)} = \left[\mathbf{x}^{(n)} - \mathrm{diag}\left\{ \frac{1}{\ddot{q}_j(0; \mathbf{x}^{(n)})} \right\} \nabla \Phi(\mathbf{x}^{(n)}) \right]_+$$

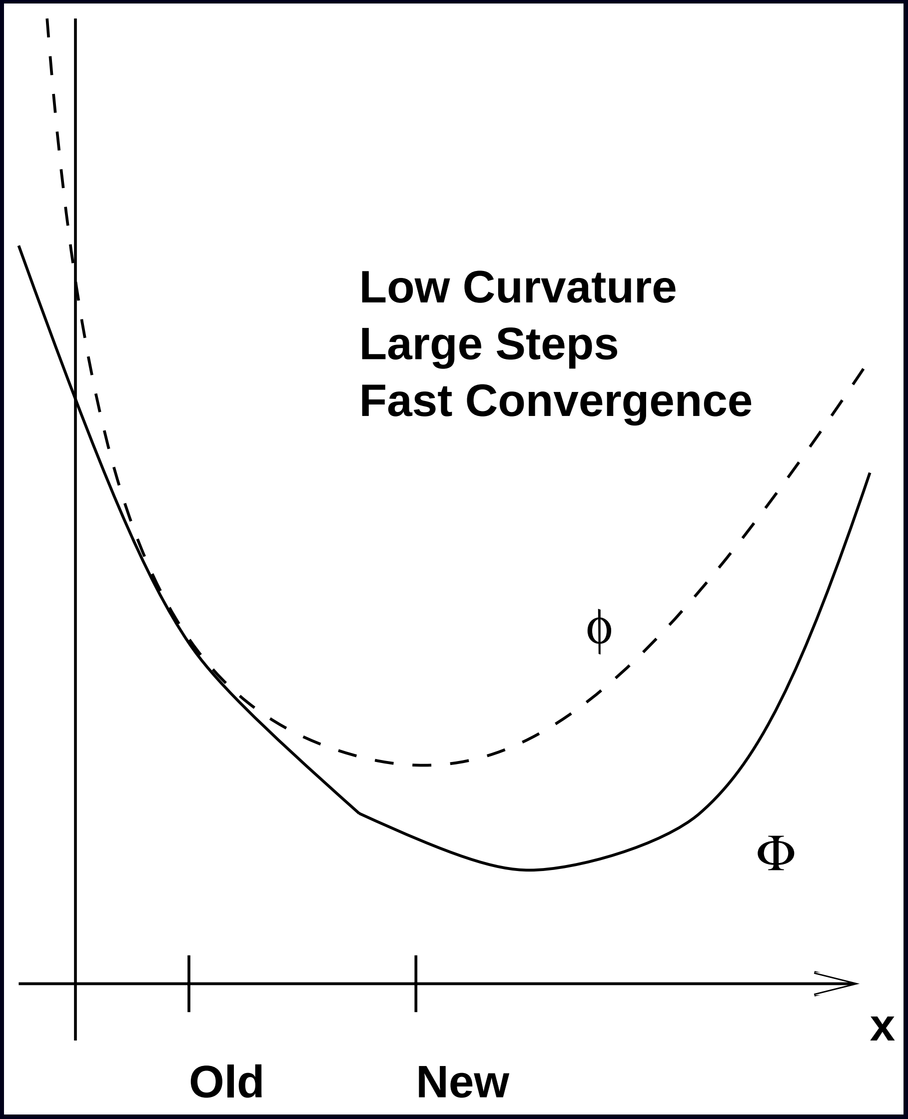
Advantages:

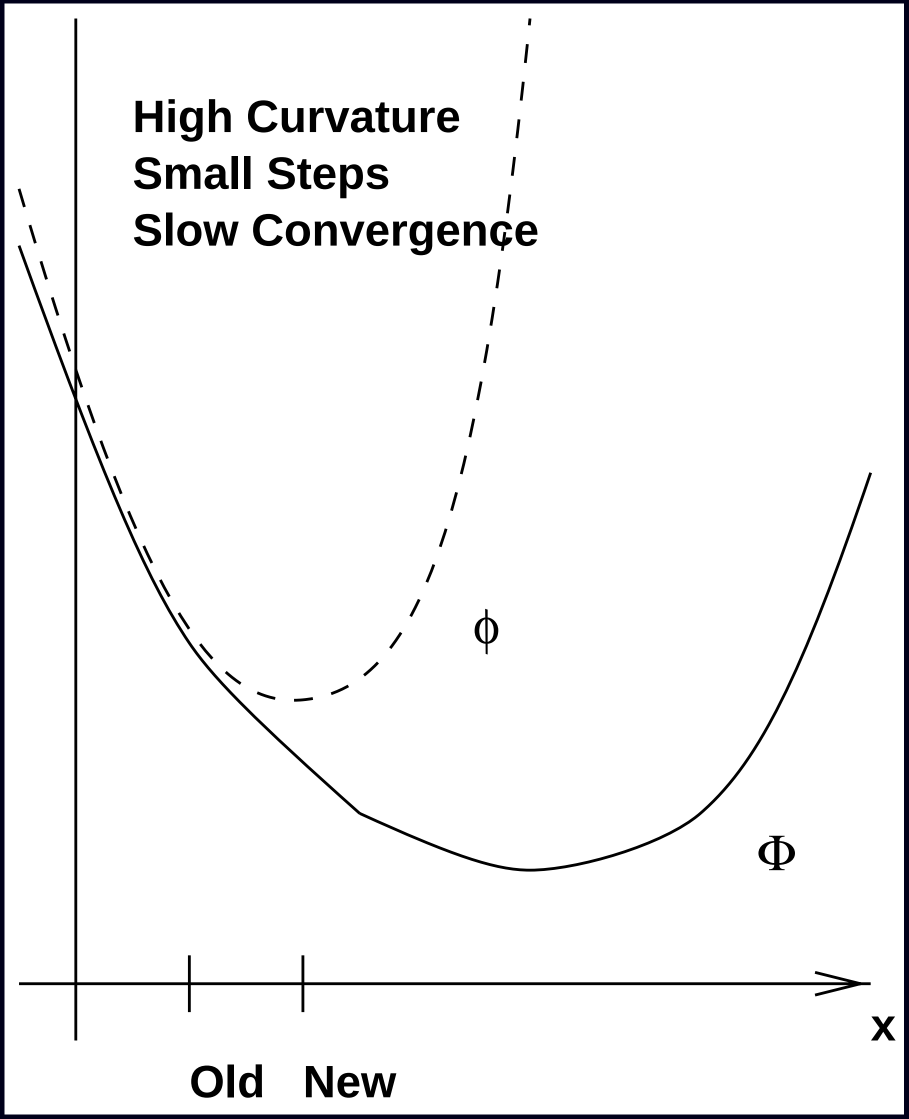
- Monotonically decreases Φ
- Converges globally to unique minimizer (for broad family of convex ψ_i's)
- No matrix inversion required
- Easily enforces nonnegativity constraint
- Completely parallelizable (all pixels updated simultaneously)

Disadvantages:

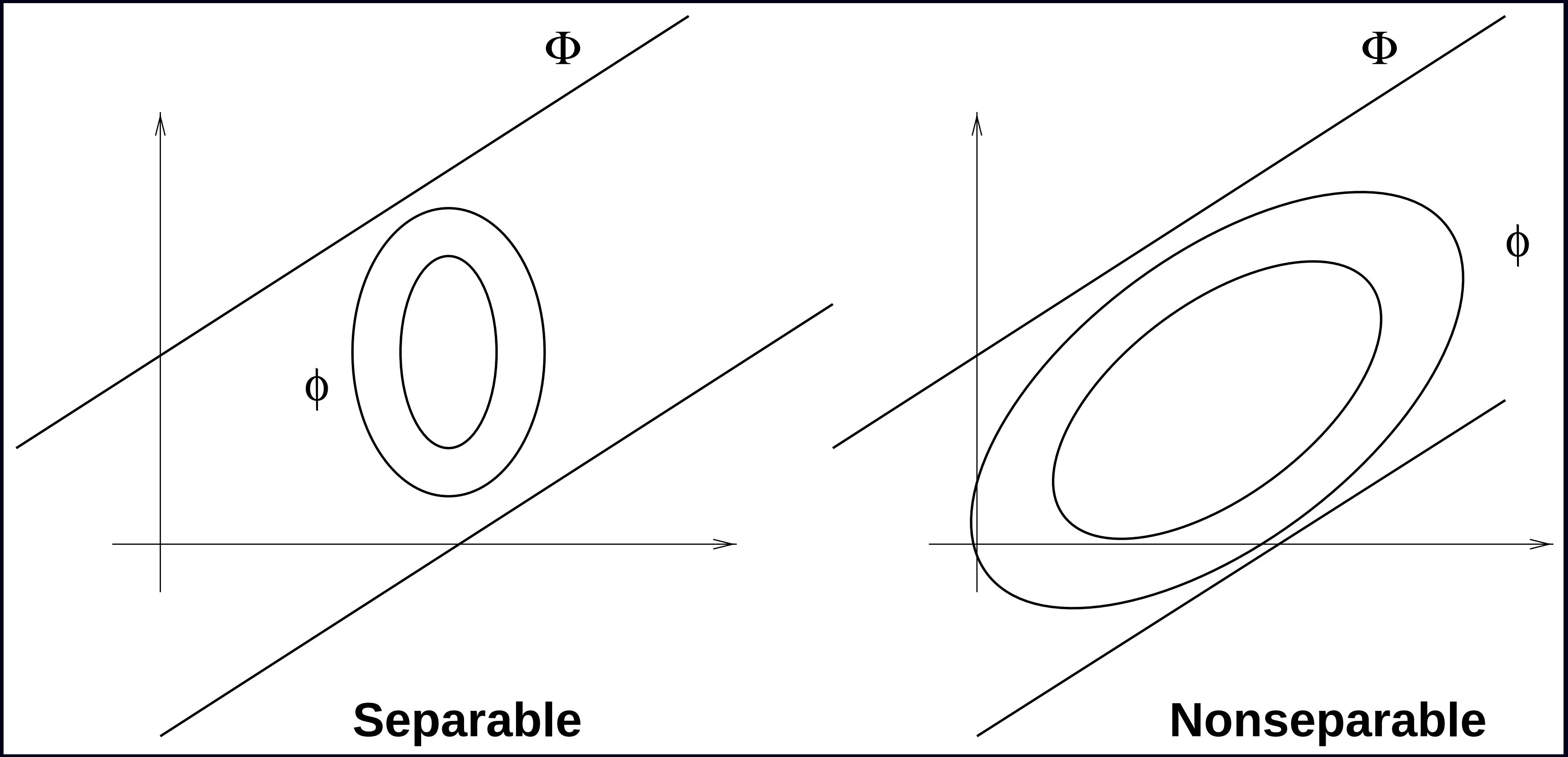
- Very slow convergence (ala EM algorithm)

Convergence Rate / Surrogate Curvature



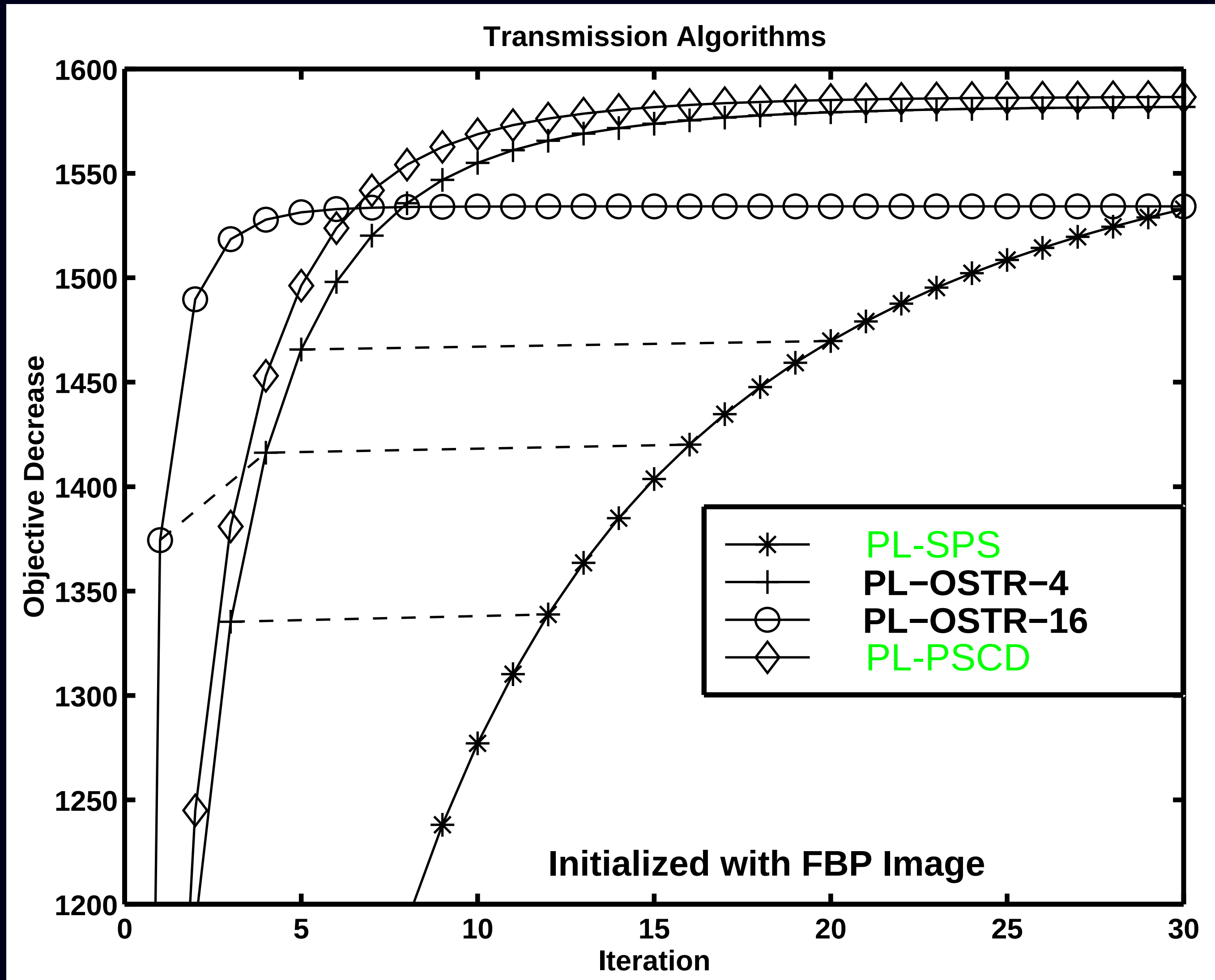


Separable vs Nonseparable Surrogates



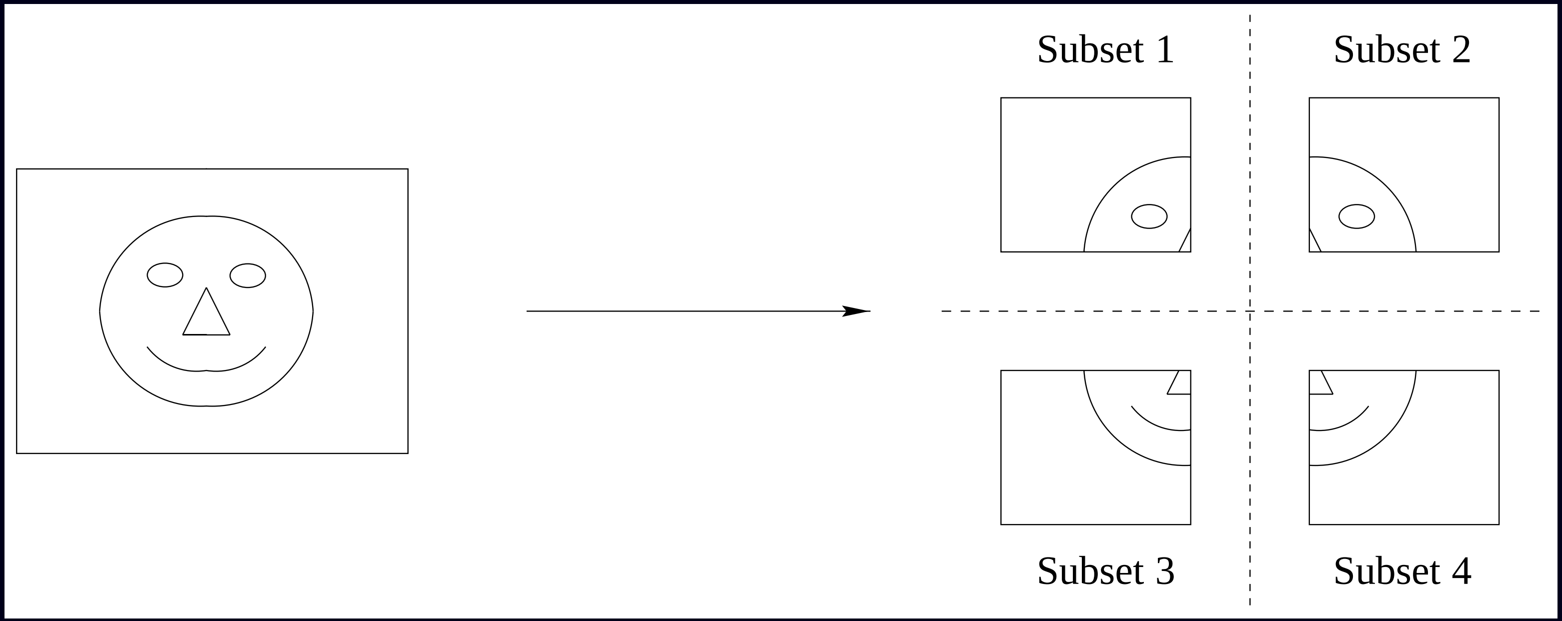
Separable surrogates (*e.g.* EM) have high curvature ∴ slow convergence.
Nonseparable surrogates can have lower curvature ∴ faster convergence.
Harder to minimize? Use paraboloids (quadratic surrogates).

PSCD vs SPS Algorithm



Naive Parallelizable Coordinate Descent

- Goal: fast convergence of coordinate descent, yet parallelizable
- Suitable for coarse-grain parallelization



- Each processor applies coordinate descent independently to its block
- Not guaranteed to be monotonic!

Partitioned-separable paraboloidal surrogate

Partion pixels into K subsets indexed by J_k, where $\bigcup_{k=1}^{K} J_k = \{1, 2, \ldots, n_p\}$.

E-step: Form a surrogate function that is separable *between blocks*:

$$\Phi(\mathbf{x}) \leq \phi(\mathbf{x}; \mathbf{x}^{(n)}) \leq Q(\mathbf{x}; \mathbf{x}^{(n)}) = \sum_{k=1}^{K} Q_k(\mathbf{x}_{J_k}; \mathbf{x}^{(n)}).$$

By construction, decreasing this new surrogate $Q(\mathbf{x}; \mathbf{x}^{(n)})$ will monotonically decrease the cost function Φ:

$$\Phi(\mathbf{x}^{(n)}) - \Phi(\mathbf{x}) \geq Q(\mathbf{x}^{(n)}; \mathbf{x}^{(n)}) - Q(\mathbf{x}; \mathbf{x}^{(n)}).$$

M-step: Partitioned-separable form allows processor parallelization:

$$\mathbf{x}^{(n+1)} = \arg\min_{\mathbf{x} \geq \mathbf{0}} Q(\mathbf{x}; \mathbf{x}^{(n)}) \quad \Rightarrow \quad \mathbf{x}_{J_k}^{(n+1)} = \arg\min_{\mathbf{x}_{J_k} \geq \mathbf{0}} Q_k(\mathbf{x}_{J_k}; \mathbf{x}^{(n)}), \; k = 1, \ldots, K.$$

PPCD Derivation

Adaptation of De Pierro's convexity trick for modified EM algorithm:

$$[\mathbf{B}\mathbf{x}-\mathbf{c}]_i = \sum_{j=1}^{n_p} b_{ij}x_j - c_i = \sum_{k=1}^{K} \pi_{ik}\left(\frac{s_{ik}^{(n)}(\mathbf{x}_{J_k})}{\pi_{ik}} + t_i^{(n)}\right)$$

$$\text{where } \pi_{ik} = \frac{\sum_{j\in J_k}|b_{ij}|}{\sum_{j=1}^{n_p}|b_{ij}|} \geq 0 \text{ and } \sum_{k=1}^{K}\pi_{ik} = 1,$$

$$s_{ik}^{(n)} = [\mathbf{B}_{J_k}(\mathbf{x}_{J_k} - \mathbf{x}_{J_k}^{(n)}) - \mathbf{c}]_i = \sum_{j\in J_k} b_{ij}(x_j - x_j^{(n)}) - c_i.$$

When q is a quadratic (and hence convex function):

$$q([\mathbf{B}\mathbf{x}-\mathbf{c}]_i) = q\left(\sum_{k=1}^{K}\pi_{ik}\left(\frac{s_{ik}^{(n)}(\mathbf{x}_{J_k})}{\pi_{ik}} + t_i^{(n)}\right)\right) \leq \sum_{k=1}^{K}\pi_{ik}\, q\left(\frac{s_{ik}^{(n)}(\mathbf{x}_{J_k})}{\pi_{ik}} + t_i^{(n)}\right).$$

The latter term is the foundation for Q_k, being partitioned separable.

Partitioned separable Paraboloidal-surrogate Coordinate Descent (PPCD) Algorithm

E-step

- Form paraboloidal surrogate $\phi(\mathbf{x};\mathbf{x}^{(n)})$ from cost function $\Phi(\mathbf{x})$
- Form partitioned separable surrogate $Q(\mathbf{x};\mathbf{x}^{(n)}) = \sum_{k=1}^{K} Q_k(\mathbf{x}_{J_k};\mathbf{x}^{(n)})$

M-step

- K processors independently "minimize" (or decrease) $\{Q_k(\mathbf{x}_{J_k};\mathbf{x}^{(n)})\}$, using any nonnegative least-squares method such as coordinate descent

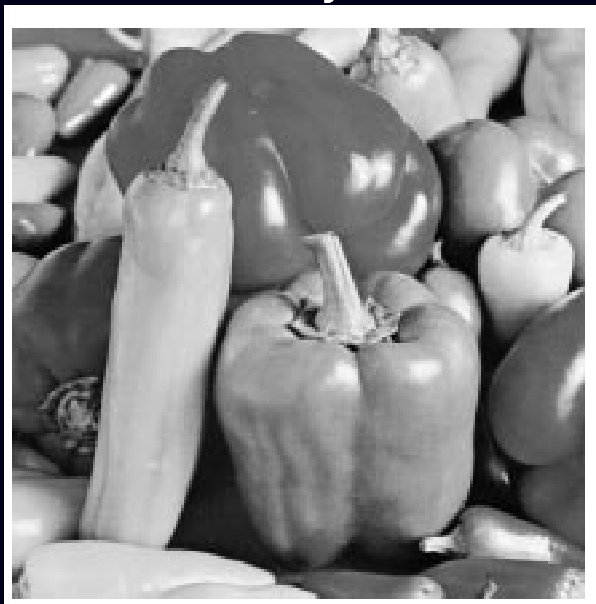
$$\mathbf{x}_{J_k}^{(n+1)} = \arg\min_{\mathbf{x}_{J_k} \geq \mathbf{0}} Q_k(\mathbf{x}_{J_k};\mathbf{x}^{(n)}), \; k = 1,\ldots,K.$$

- Broadcast $\mathbf{x}_{J_k}^{(n+1)}$ to other processors

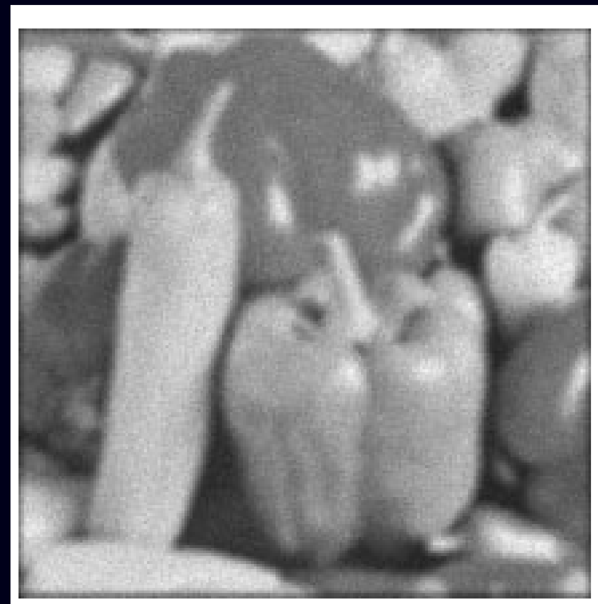
For a broad class of convex ψ_i's, global convergence follows from SAGE convergence proof, Fessler and Hero, 1995 (IEEE T-SP).

Representative 2D Simulation Results

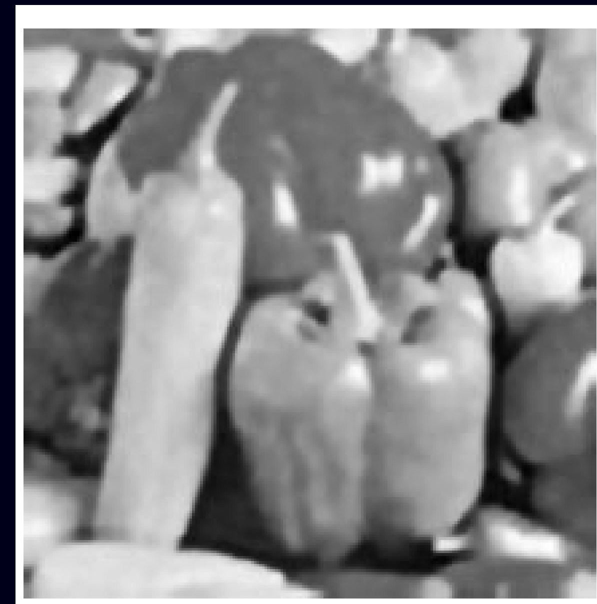
Object



Measurement



Restoration

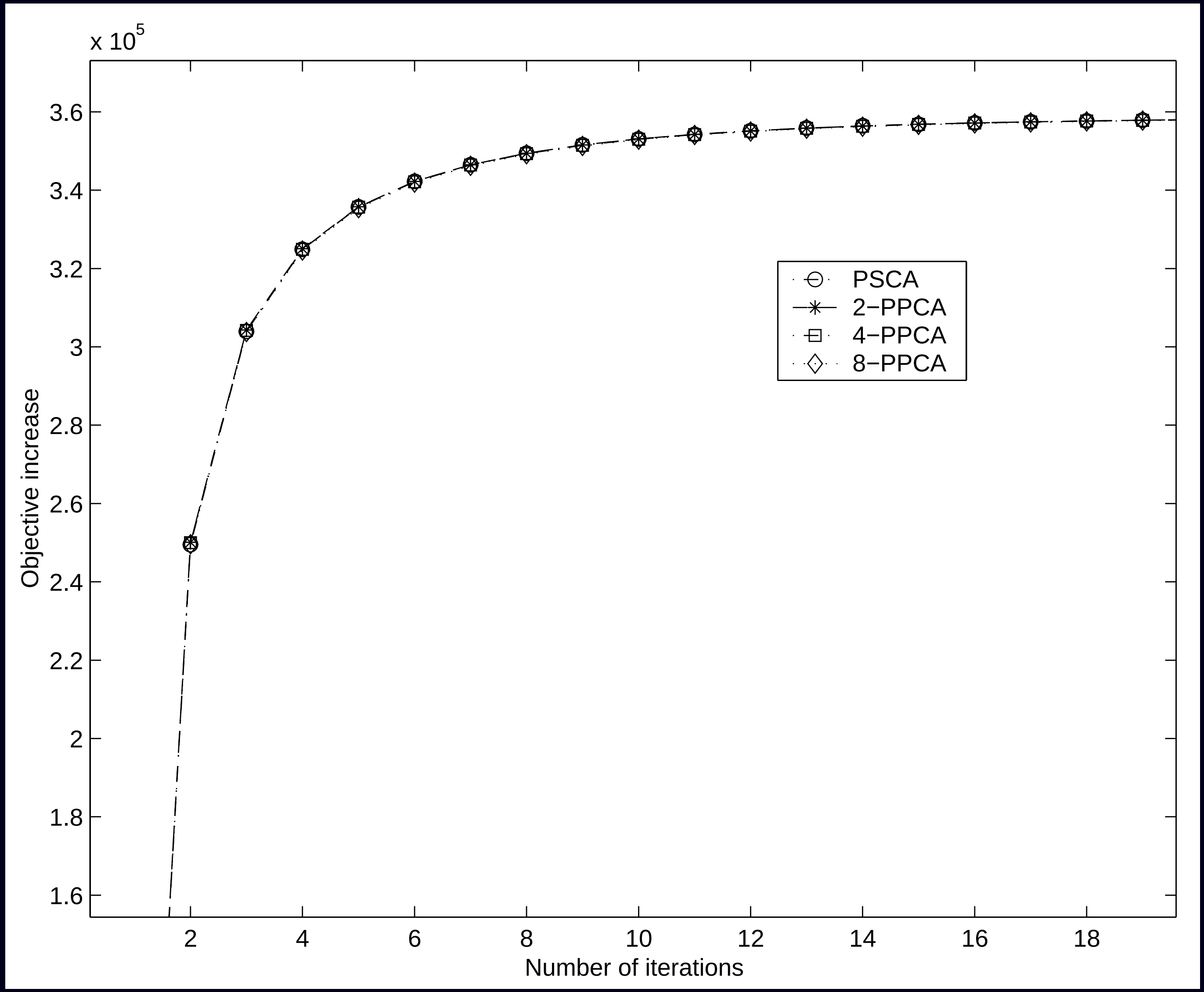


Poisson measurement noise with PSNR = 25 dB

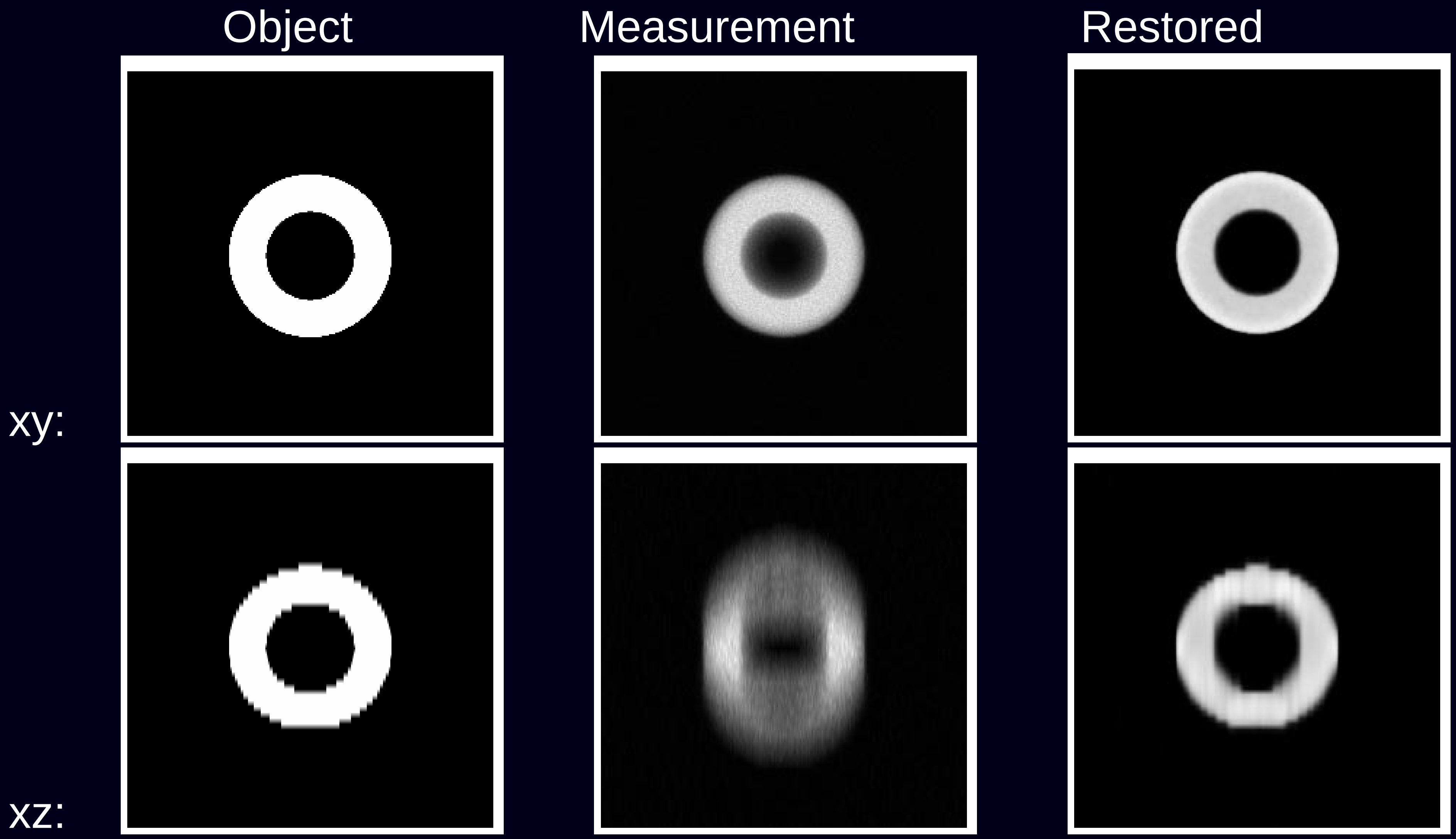
Restored by maximum penalized likelihood
using nonquadratic edge-preserving penalty function:

$$\psi(t) = \delta^2 \left[\left| \frac{t}{\delta} \right| - \log\left(1 + \left| \frac{t}{\delta} \right| \right) \right], \quad \text{with } \delta = 1.5.$$

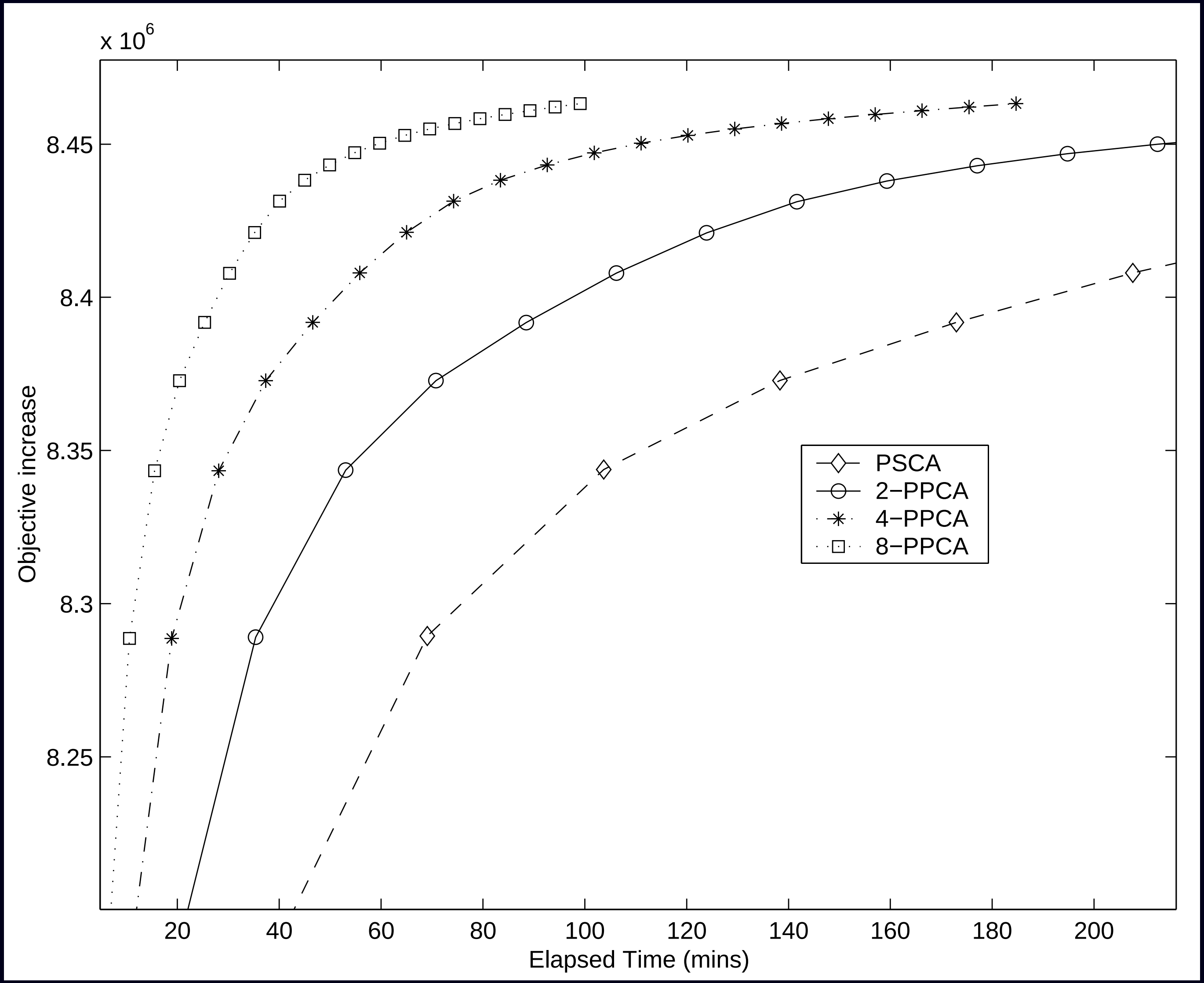
Convergence rate vs number of processors (2D)



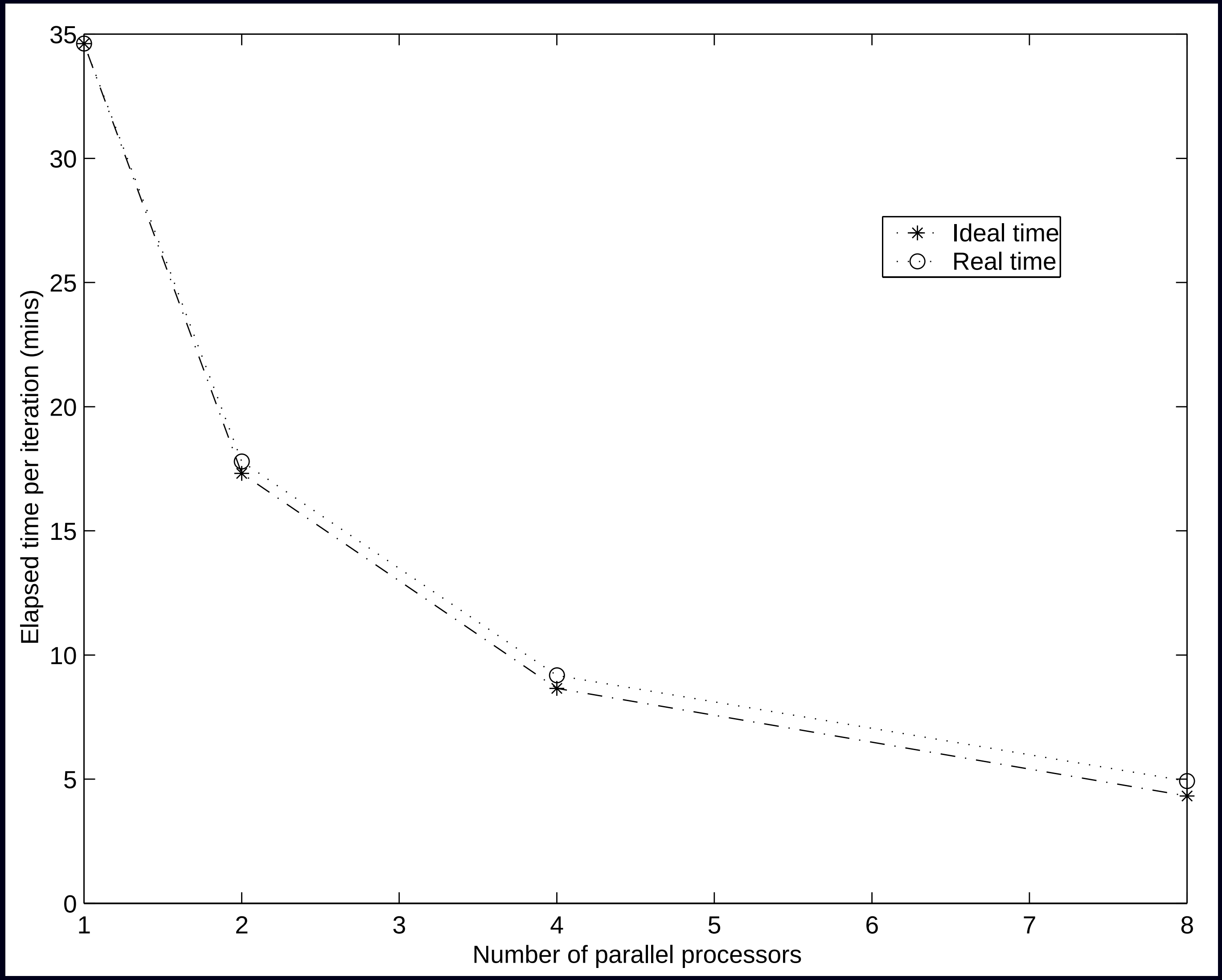
Confocal microscopy simulation (3D)



Cost function decrease vs Elapsed time



Elapsed time per iteration vs # of Processors



Summary

Optimization transfer

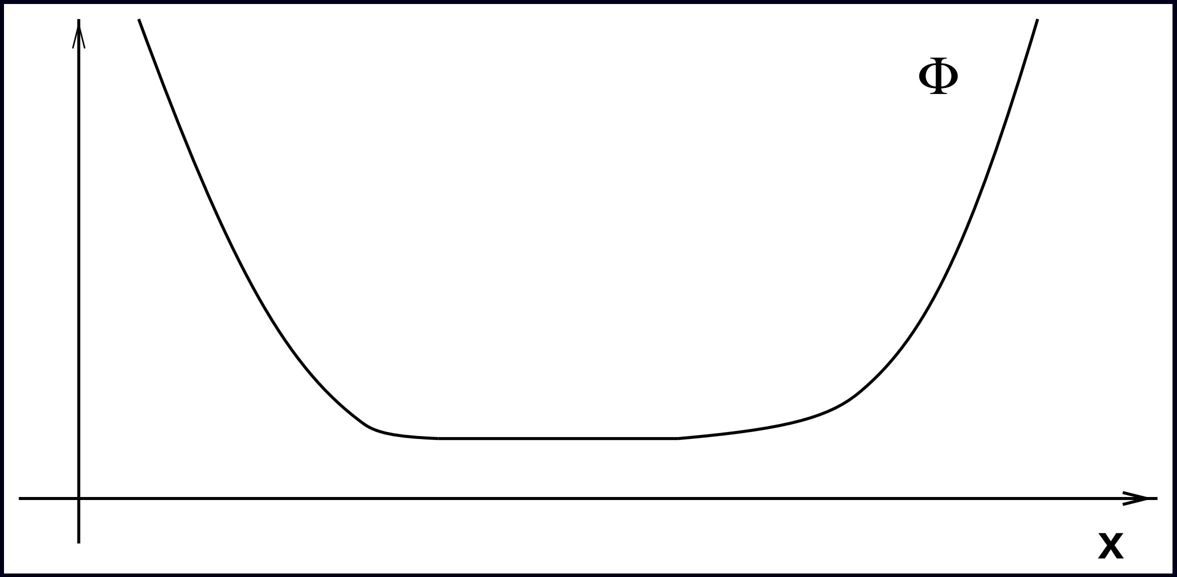
- Natural framework for algorithm development
- Exploits structure of “partially separable” cost functions

Partitioned separable paraboloidal surrogate coordinate descent algorithm

- Accommodates non-quadratic cost functions
- Monotonically decreases Φ
- Converges globally to unique minimizer (for broad class of ψ_i’s)
- Easily accommodates nonnegativity constraint
- Parallelizable
- Converges faster than a general-purpose optimization method

Future Work

Convergence proof for multiple minima:



Slides and paper available from:

`http://www.eecs.umich.edu/~fessler`

Monotone Convergence

From R. Meyer "Sufficient conditions for the convergence of monotonic mathematical programming algorithms," J. Comput. System. Sci., 1976.

Let M be a point to set mapping such that, on G,

- M is uniformly compact,
- M is upper semi-continuous,
- M is strictly monotonic with respect to the function Φ.

If $\{\mathbf{x}^{(n)}\}$ is any sequence generated by the algorithm $\mathbf{x}^{(n+1)} \in M(\mathbf{x}^{(n)})$:

- all accumulation points of $\{\mathbf{x}^{(n)}\}$ will be fixed points,
- $\Phi(\mathbf{x}^{(n)}) \to \Phi(\mathbf{x}^{\star})$ where $\mathbf{x}^{\star}$ is a fixed point,
- $\|\mathbf{x}^{(n+1)} - \mathbf{x}^{(n)}\| \to 0$, and
- either $\{\mathbf{x}^{(n)}\}$ converges or the accumulation points of $\{\mathbf{x}^{(n)}\}$ form a continuum.