

Accelerated Computation of Regularized Field Map Estimates

Michael J Allison¹, and Jeffrey A Fessler¹

¹Electrical Engineering and Computer Science, University of Michigan, Ann Arbor, Michigan, United States

INTRODUCTION

Field inhomogeneity affects magnetic resonance (MR) imaging techniques that use long readout times (e.g., spiral pulse sequences and echo-planer imaging (EPI)). To correct for reconstruction artifacts related to the inhomogeneity, one must have an accurate estimate of the off-resonance frequency at each voxel; i.e., a field map. The conventional method to estimate the field map is to acquire two scans with different echo times, reconstruct the corresponding images, and then compute their phase difference and divide by the difference in echo times [1]. However, such estimates are highly corrupted by noise in voxels with low signal. Instead, Funai et al. [1] proposed a statistical based estimator that enforces our a priori knowledge that the field maps should be smooth. Although highly robust, this estimator has a non-convex cost function that is complicated to minimize. To address this, a solution using the optimization transfer principle and separable quadratic surrogates was proposed [1]. However, this approach can require thousands of iterations to converge. Since field maps of a 3D volume are often estimated on a slice-by-slice basis, this cost is significant. We present a novel optimization transfer method that uses Huber's algorithm for quadratic surrogates [2] to minimize the non-convex cost function in [1] much faster.

METHODS

The field map estimator presented in [1] minimizes the following non-convex cost function: $\Psi(\omega) = \Phi(\omega) + \beta \|C\omega\|_2^2$ where $\Phi(\omega) = \sum_{j=1}^N \sum_{m=1}^L \sum_{p=1}^L \varphi_{jmp}(\omega_j)$ and

$$\varphi_{jmp}(\omega_j) = |y_j^m y_j^p| w_j^{m,p} \left[1 - \cos(\omega_j (\Delta_p - \Delta_m) + \angle y_j^m - \angle y_j^p) \right] \quad \text{with} \quad w_j^{m,p} \approx |y_j^m| |y_j^p| / \sqrt{\sum_{l=1}^L |y_j^l|^2}.$$

Note that ω_j is the field inhomogeneity at the j^{th} voxel, $\omega = (\omega_1, \omega_2, \dots, \omega_N)$ is the unknown field map, N is the number of voxels, L is the number of scans, y_j^n is the j^{th} voxel of image n , Δ_n is the echo time for scan n , β is a regularization parameter, and C is a finite differencing matrix. This cost function is non-convex; however, it is differentiable and can therefore be minimized using optimization transfer methods [2]. The following quadratic surrogate for the data fit term, $\varphi_{jmp}(\omega_j)$, was proposed in [1]:

$$q_{jmp}(\omega; \omega_j^{(n)}) = \varphi_{jmp}(\omega_j^{(n)}) + \hat{\varphi}_{jmp}(\omega_j^{(n)}) (\omega - \omega_j^{(n)}) + \frac{1}{2} \kappa_{jmp}(s_{jmp}^{(n)}) (\omega - \omega_j^{(n)})^2 \quad \text{where} \quad \kappa_{jmp}(s) = |y_j^m y_j^p| w_j^{m,p} (\Delta_p - \Delta_m)^2 \frac{\sin(s)}{s}$$

and $s_{jmp}^{(n)} = (\omega_j^{(n)} (\Delta_p - \Delta_m) + \angle y_j^m - \angle y_j^p) \bmod \pi$. Substituting this surrogate into the cost function and vectorizing yields

$$\Theta(\omega; \omega^{(n)}) = \mathbf{b} + \nabla \Phi(\omega^{(n)}) (\omega - \omega^{(n)}) + \frac{1}{2} (\omega - \omega^{(n)})^H \mathbf{D}^{(n)} (\omega - \omega^{(n)}) + \frac{1}{2} \omega^H \mathbf{C}^H \mathbf{C} \omega$$

where $\mathbf{D}^{(n)}$ is a diagonal matrix with elements $d_j^{(n)} = \sum_{m=1}^L \sum_{p=1}^L \kappa_{jmp}(s_{jmp}^{(n)})$ and \mathbf{b} is a vector of constants. The Hessian of this surrogate cost function is $\mathbf{H} = \mathbf{D}^{(n)} + \beta \mathbf{C}^H \mathbf{C}$. Although $\mathbf{C}^H \mathbf{C}$ is very large for typical image sizes, it has a sparse banded structure, and thus, \mathbf{H} also has this structure. We can therefore use sparse Cholesky factorization techniques (e.g., [3]) to efficiently solve a linear system of equations based on \mathbf{H} (i.e., solving $\mathbf{H}^{-1} \mathbf{x}$ for arbitrary \mathbf{x}). Exploiting this efficiency, we use Huber's algorithm for quadratic surrogates [2] to obtain an iterative algorithm for monotonically decreasing the original cost function ($\Psi(\omega)$):

$$\omega^{(n+1)} = \omega^{(n)} - \mathbf{H}^{-1} \nabla \Psi(\omega^{(n)}).$$

RESULTS

We evaluated our algorithm with a 128×128 pixel brain image and a field map acquired on a 3T GE scanner, Figure 1(a-b). We combined this field map with the brain image to create three scan images, $y^{(n)}$, with echo times $\Delta_n = 0, 0.002, 0.01$ s and $R_2 = 20 \text{ s}^{-1}$. Figure 1(c) presents the masked conventional field map estimate from the first two of these scans. We also estimated the field maps using the separable quadratic surrogate method [1] and our proposed Huber's algorithm approach, Figure 1(d-e). In both cases we used a masked version of the conventional estimate as the initialization image and set $\beta = 2^{-6}$. For each optimization transfer method, the RMS difference (RMSD) in Hz between each iteration and the corresponding converged estimate was computed within the object support. Plots of these errors versus both time and iteration are presented in Figure 2.

CONCLUSIONS

The cost function is non-convex so the two optimization transfer function methods can converge to different local minima. In this case we found that the two methods converged to solutions with an RMSD of only 0.05 Hz within the object support. Our proposed Huber's algorithm converged to an RMSD of 1 Hz in 29 steps, while the separable quadratic surrogate algorithm required approximately 15000. This was expected because our proposed method uses the exact Hessian of the surrogate $\Theta(\omega)$, whereas the algorithm in [1] uses a diagonal matrix with larger entries. Furthermore, although each step of our proposed algorithm took longer, it was also faster in time, converging in 6 s compared to 600 s for the existing algorithm. It should be noted that although sparse Cholesky factorization is efficient on typical image sizes, it is memory intensive. Thus, other approaches may be more appropriate for very large datasets such as true 3D volumes. To address this, we have explored an augmented Lagrangian based solution, similar to [4], that provides estimates at a rate between that of our proposed method and the existing method.

REFERENCES

[1] A. Funai et al., IEEE-TMI, 27(10), 2008. [2] P. Huber, Robust Statistics, pp. 184-5, 1981. [3] Y. Chen et al., TOMS, 35(3), 2008. [4] M. Allison et al., ISMRM, 2011.

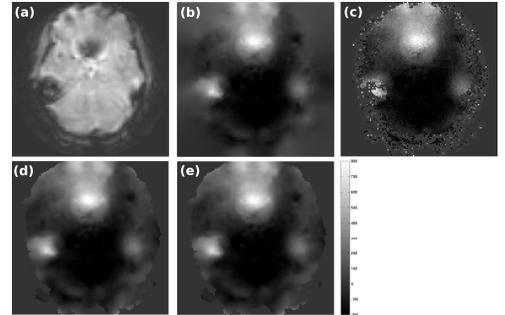


Figure 1: Magnitude of the brain image (a) and field map (b) used in the experiment. Masked estimates using the conventional method (c), our proposed method (d), and the existing separable quadratic surrogate method (e).

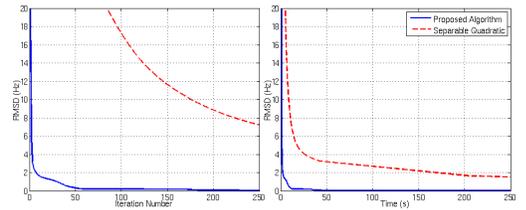


Figure 2: Plots of RMSD to the converged estimate with respect to iteration (left) and time (right) (cropped to show detail).