Standard Errors of Mean, Variance, and Standard Deviation Estimators

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I. Introduction

We often estimate the mean, variance, or standard deviation from a sample of n elements and present the estimates with standard errors or error bars (in plots) as well. A standard error of a statistic (or estimator) is the (estimated) standard deviation of the statistic. An error bar is, in a plot, a line which is centered at the estimate with length that is double the standard error. Standard errors mean the statistical fluctuation of estimators, and they are important particularly when one compares two estimates (for example, whether one quantity is higher than the other in a statistically meaningful way). In this note we review the standard errors of frequently used estimators of the mean, variance, and standard deviation.

II. Normal One Sample Problem

Let $X_1, \ldots, X_n$ be a random sample from $\mathcal{N}(\mu, \sigma^2)$ where both μ and σ are unknown parameters. Define, for convenience, two statistics (sample mean and sample variance):

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

A. Mean Estimator

The uniformly minimum variance unbiased (UMVU) estimator of μ is $\bar{X}$ [1, p. 92]. Since $\bar{X} \sim \mathcal{N}(\mu, \sigma^2/n)$, the standard error of $\bar{X}$ is

$$\sigma_{\bar{X}} = \sqrt{\mathrm{Var}\{\bar{X}\}} = \frac{\sigma}{\sqrt{n}}.$$

Hence $\hat{\sigma}_{\bar{X}} = \hat{\sigma}/\sqrt{n}$. For $\hat{\sigma}$, see Subsection II-C.

B. Variance Estimator

Note from [1, p. 92] that S^2 is UMVU for σ^2 and that

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}. \tag{1}$$

Since the chi-squared distribution with $n-1$ degrees of freedom $\left(\chi^2_{n-1}\right)$ has a variance of $2(n-1)$ [1, p. 31], the standard error of S^2 is

$$\sigma_{S^2} = \sqrt{\mathrm{Var}\{S^2\}} = \sigma^2\sqrt{\frac{2}{n-1}}.$$

Hence $\hat{\sigma}_{S^2} = S^2\sqrt{2/(n-1)}$. It is useful to note

$$\sigma_{S^2}/\sigma^2 = \hat{\sigma}_{S^2}/S^2 = \sqrt{\frac{2}{n-1}}.$$

Since σ^2 and S^2 have the square of the units of X_i, often it is preferable to report estimates of σ, as described next.

C. Standard Deviation Estimator

The UMVU estimator of σ is $K_n S$ [1, p. 92] where

$$K_n = \sqrt{\frac{n-1}{2}}\frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} = \sqrt{\frac{n-1}{2}}e^{\ln\Gamma(\frac{n-1}{2}) - \ln\Gamma(\frac{n}{2})}$$

where the second form is more numerically stable for large values of n when using the "ln gamma function." By setting $K_n = 1$, S is a common choice in practice but it is slightly biased. Since

$$\frac{\sqrt{n-1}}{\sigma}S \sim \chi_{n-1}$$

[see (1)] and the chi distribution with $n-1$ degrees of freedom (χ_{n-1}) has variance [2, p. 49: typo corrected]

$$V_n = 2\left(\frac{n-1}{2} - \frac{\Gamma^2(\frac{n}{2})}{\Gamma^2(\frac{n-1}{2})}\right),$$

the standard error of $K_n S$ is

$$\sigma_{K_n S} = \sqrt{\mathrm{Var}\{K_n S\}} = \sigma K_n \sqrt{\frac{V_n}{n-1}}.$$

To investigate the asymptotic behavior of $\sigma_{K_n S}$, we need the following approximation [3, P. 602]:

$$\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})} = \sqrt{\frac{n-1}{2}}\left(1 - \frac{1}{4(n-1)} + O\left(\frac{1}{n^2}\right)\right). \tag{2}$$

Using (2), it can be shown that

$$K_n = 1 + O\left(\frac{1}{n}\right)$$

and

$$\begin{aligned}\sigma_{K_n S} &= \frac{\sigma}{\sqrt{2(n-1)}}\left(1 + O\left(\frac{1}{n}\right)\right)\\ &= \frac{\sigma}{\sqrt{2(n-1)}} + O\left(\frac{1}{n\sqrt{n}}\right).\end{aligned}$$

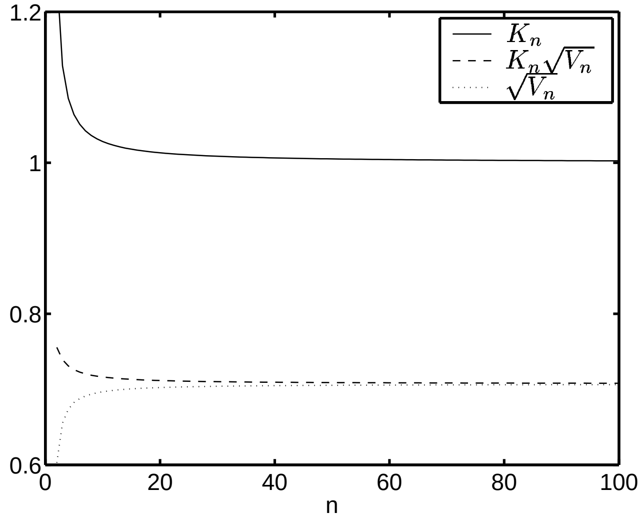

Fig. 1. This plot shows that K_n and $\sqrt{V_n}$ approach 1 and $1/\sqrt{2}$, respectively, as n increases.

To summarize,

$$\begin{aligned} \sigma_{K_n S}/\sigma &= \hat{\sigma}_{K_n S}/(K_n S) = \frac{K_n\sqrt{V_n}}{\sqrt{n-1}} \\ &\approx \frac{1}{\sqrt{2(n-1)}} \text{ for large } n. \end{aligned} \tag{3}$$

Figure 1 shows a plot of K_n, $\sqrt{V_n}$, and $K_n\sqrt{V_n}$ versus n. For $n > 10$, it seems reasonable to use $K_n = 1$ and the approximation (3) for the standard error.

REFERENCES

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