Analytical Approach to Regularization Design for Isotropic Spatial Resolution

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Motivation

True Phantom

"FBP" Noiseless

Conventional Regul.

Proposed Regul. (QPWLS–PCG)
History

- 1994 MIC, Fessler and Rogers
  - Uniform quadratic penalties cause nonuniform image resolution
  - Simple “certainty-based” correction for shift-invariant systems
- 1998 ICIP, Stayman and Fessler
  - Improved regularization design for shift-invariant systems, compensating for anisotropy of local PSF
- 1999 Fully 3D
  - Qi and Leahy: design for uniform pixel contrast
  - Stayman and Fessler: design for 3D shift-invariant systems
- 2001 MIC, Stayman and Fessler
  - Improved (but complicated) design allowing negative weights
- 2002 MIC
  - Stayman and Fessler: faster method for space varying systems
  - Nuyts and Fessler: simplified design

All based on matrix analysis!
Local Impulse Response

- Noisy measurement vector \( y = Ax + \text{noise} \)
  \( y \): measured projection data
  \( A \): system matrix
  \( x \): unknown image pixel values to reconstruct

- General image reconstruction method: \( \hat{x} = \hat{x}(y) \)

- Local impulse response for \( j \)th pixel:
  \[
  l^j = \lim_{\delta \to 0} \frac{\hat{x}(y + \delta Ae^j) - \hat{x}(y)}{\delta}
  \]

  \( e^j \) = point source in \( j \)th pixel
  “How does a small impulse in the \( j \)th pixel affect other pixels?”

- Useful for design of regularized reconstruction methods

Goal. Design the estimator \( \hat{x} \) to have good noise properties and spatial resolution properties that are isotropic and uniform, or ...
Penalized-Likelihood Reconstruction

Regularized estimator:

\[ \hat{x} = \arg \min_x L(Ax, y) + R(x) \]

- \( x \): unknown image pixel values to reconstruct
- \( y \): measured projection data
- \( A \): system matrix
- \( L \): negative log-likelihood (e.g., Poisson statistical model)
- \( R(x) \): quadratic regularizing roughness penalty 
  \[ R(x) = \frac{1}{2} x' R x \]
  \( R \) is the Hessian of the penalty function \( R(x) \)

Local impulse response:

\[ l^j = [A' W A + R]^{-1} A' W A e^j \]

\( W \) depends on the log-likelihood and \( y \), e.g., 
\[ W = \text{diag}\{1/y_i\} \]

This matrix form has been the foundation of most previous methods!
Local Discrete Fourier Approximations

Let \( Q \) denote the DFT matrix for image domain.

Local system frequency response:

\[
A'W Ae^j \approx Q \text{diag}\{\lambda_k^j\} Q'e^j,
\]

\[
\lambda^j = \text{FFT} \{A'W Ae^j\}
\]

Local regularization frequency response:

\[
Re^j \approx Q \text{diag}\{\omega_k^j\} Q'e^j,
\]

\[
\omega^j = \text{FFT} \{Re^j\}
\]

Local impulse response with local Fourier approximation:

\[
I^j = [A'WA + R]^{-1} A'W Ae^j \approx Q \text{diag}\left\{\frac{\lambda_k^j}{\lambda_k^j + \omega_k^j}\right\} Q'e^j
\]

Useful for design of the regularizer \( R \), but requires FFTs for every pixel.
(And forward- / back-projections for each pixel for shift varying systems.)
Position-Dependent Regularization

\[ R(f) = \sum_{n,m} \begin{align*}
    r_1^{(n,m)} & \left| f[n,m] - f[n - 1, m - 0] \right|^2 + \\
    r_2^{(n,m)} & \left| f[n,m] - f[n - 1, m + 1] \right|^2 + \\
    r_3^{(n,m)} & \left| f[n,m] - f[n + 0, m + 1] \right|^2 + \\
    r_4^{(n,m)} & \left| f[n,m] - f[n + 1, m - 1] \right|^2
\end{align*} \]

\( r^j = (r_1, \ldots, r_4) \): 4 penalty coefficients per pixel.
Conventional regularizer: \( r_1 = r_3 = 1, \; r_2 = r_4 = 1 / \sqrt{2}. \)
Linearized Regularization Design

Goal: choose \( R \) (i.e., \( \{ r^j \} \)) such that the resulting local impulse response \( l^j \) approximates some desired target PSF.

Natural target PSF is from unweighted penalized least-squares:

\[
l^j = \left[ A'WA + R \right]^{-1} A'WAe^j \approx \left[ A'_0A_0 + R_0 \right]^{-1} A'_0A_0e^j.
\]

Local impulse resp. \hspace{1cm} Target PSF

Nonlinear in \( R \) \( \Rightarrow \) complicated design.

Linearize by “cross multiplying:”

\[
\left[ A'_0A_0 + R_0 \right] A'WAe^j \approx \left[ A'WA + R \right] A'_0A_0e^j.
\]

Simplify using “local shift invariance” approximations:

\[
R_0A'WAe^j \approx RA'_0A_0e^j.
\]

“Linearized regularization design” (still with matrices):

\[
\min_{R \in \mathcal{R}} \left\| R_0A'WAe^j - RA'_0A_0e^j \right\|.
\]
Analytical Regularization Design

Matrix approach:  \[
\min_{R \in \mathcal{R}} \| R_0 A' W A e^j - R A_0' A_0 e^j \| \]

Key idea: replace 4 matrices with analytical Fourier approximations.

1. Nominal system transfer function

\[
A_0' A_0 \equiv \frac{|B(\rho)|^2}{\rho}
\]

- \((\rho, \varphi)\): polar coordinates in frequency space
- \(B(\rho)\): “typical” detector frequency response

2. Weighted system transfer function

\[
A' W A \equiv \frac{w^j(\varphi) |B^j_\varphi(\rho)|^2}{\rho}
\]

- \(B^j_\varphi(\rho)\): detector response at projection angle \(\varphi\) for \(j\)th pixel
- \(w^j(\varphi)\): angular weighting (certainty) for \(j\)th pixel (from \(W\))
Analytical Regularization Design

3. Isotropic 1st-order roughness: \( R_0(f) = \int \| \nabla f \|^2 \)

\[ R_0 \equiv |2\pi \rho|^2 \]

4. Local roughness penalty (simplified)

\[ R(f) = \sum_{n,m} \sum_{l=1}^{L} r_l \frac{1}{2} | f[n,m] - f[n-n_l,m-m_l] |^2 \]

Penalty coefficients \( r^j = (r_1, \ldots, r_L) \) to be designed (for each pixel).

After some Fourier analysis…:

\[ R \equiv (2\pi \rho)^2 \sum_{l=1}^{L} r_l \cos^2 (\phi - \phi_l), \quad \phi_l \triangleq \tan^{-1} \frac{m_l}{n_l} \]

\( \phi_l = (0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}) \) for \( L = 4 \)

(Each penalty coefficient influences PSF shape along some direction.)
Analytical Regularization Design

Rewrite the “matrix” minimization using the 4 Fourier approximations. Simplifying yields the following matrix-free design criterion:

\[ r^j = \arg \min_{r \geq 0} \int_0^\pi \left| w^j(\phi) - \sum_{l=1}^{L} r_l \cos^2(\phi - \phi_l) \right|^2 d\phi \]

\( w^j(\phi) \): angular “certainty” weighting for \( j \)th pixel, from data statistics.
\( \cos^2(\phi - \phi_l) \): angular contribution for \( l \)th penalty direction.

No matrix inverses (cf. analytical \( 1/\rho \)).

For 2nd-order neighborhood \( (L = 4) \), exact closed-form solution. (No NNLS iterations needed.)

Solution requires just three sums (over projection angle) per pixel:

\[
\begin{bmatrix}
    d_1^j \\
    d_2^j \\
    d_3^j
\end{bmatrix} =
\begin{bmatrix}
    \frac{1}{\pi} \int_0^\pi w^j(\phi) d\phi \\
    \frac{1}{\pi} \int_0^\pi w^j(\phi) \cos(2\phi) d\phi \\
    \frac{1}{\pi} \int_0^\pi w^j(\phi) \sin(2\phi) d\phi
\end{bmatrix}
\]

“average”
“0 and \( \pi/2 \)"
“\( \pi/4 \) and \( 3\pi/4 \)”
Eight-fold symmetry

\[ \frac{d_3}{d_1} = \frac{2}{3} \frac{d_2}{d_1} - \frac{1}{3} \]
Analytical solution

Four penalty coefficients per pixel for 2nd-order neighborhood:

1. \[ r_1 = \frac{4}{3} (d_1 + d_2), \quad r_2 = r_3 = r_4 = 0 \]

2. \[ r_1 = \frac{8}{5} \left[ \frac{1}{2} d_1 + \frac{3}{2} d_2 - d_3 \right], \quad r_3 = \frac{12}{5} \left[ d_3 - \left( \frac{2}{3} d_2 - \frac{1}{3} d_1 \right) \right], \quad r_2 = r_4 = 0 \]

3. \[ r_1 = 4d_2, \quad r_2 = 0, \quad r_3 = d_1 - 2d_2 + 2d_3, \quad r_4 = 2 \left[ \frac{1}{2} d_1 - (d_2 + d_3) \right] \]

4. \[ r_1 = 2 \left( \frac{1}{4} d_1 + d_2 \right), \quad r_2 = 2 \left( \frac{1}{4} d_1 - d_2 \right) \]
\[ r_3 = 2 \left( \frac{1}{4} d_1 + d_3 \right), \quad r_4 = 2 \left( \frac{1}{4} d_1 - d_3 \right) \]
Example

Local impulse response functions at \((20,10)^{400}\)

- Fourier–NNLS
- Nuyts

Contours

Target, Standard, Certainty, Nuyts, Fourier–NNLS
Comparison

Target

Conventional

Certainty

Nuyts

Proposed2

Proposed4

(QPWLS–PCG)
Ring Profiles

![Graph showing ring profiles with different intensity levels and angular positions.]
Summary

- Simple, fast, effective regularization design for uniform, isotropic spatial resolution
- Analogy to FBP: solve first, discretize second. (cf. Fourier \((1/\rho)^{-1} = \rho\) versus matrix \([A' A_0]^{-1}\))
- Recommendation: combine modest regularization with post-filtering
- Extends to 3D and to shift-variant systems. Requires somewhat more computation for designing the regularizer, but is still more practical than alternatives.
- Analytical approximations also applicable to variance/autocorrelation predictions.
- Non-quadratic edge-preserving regularizers for transmission case?
- Matlab tomography toolbox:
  
  http://www.eecs.umich.edu/~fessler
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