Statistical X-ray Computed Tomography Image Reconstruction with Beam Hardening Correction

Idris A. Elbakri and Jeffrey A. Fessler

EECS Department
The University of Michigan

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Outline

- X-ray transmission CT
  - FBP reconstruction
  - Statistical reconstruction
- Poly-energetic CT
  - Beam hardening physics and artifacts
  - Poly-energetic transmission CT model
  - Statistical reconstruction algorithm
- Results and comparisons
- Future work

X-ray Computed Tomography

- Goal: reconstruct $\mu(x,y,z,E)$ from sinogram measurements $\{Y_i\}$, where

$$E[Y_i] = \int I_i(E) \exp\left\{ - \int_{L_i} \mu(x,y,z,E)\, dl \right\} dE, \quad i = 1,\ldots,N$$

- Filtered Back Projection
  - Fast and deterministic
  - Properties well understood
  - FBP ignores statistics of the data
  - Metallic implants cause streak artifacts.
  - Not ideal for cone-beam and helical scanning
- Statistical Reconstruction
  - Based on a statistical model for the data
  - Non-standard geometries: cone-beam
  - Accurate physics model: beam hardening
  - Tradeoffs: computation time, software and model complexities
  - Lower noise
  - Object constraints
  - System model (detector response)

Poly-energetic Transmission CT

- X-ray beam has a broad spectrum and $\mu(x, y, z, E)$ is energy dependent. Lower energies are preferentially attenuated.
- Log-processed sinograms are non linear with tissue thickness:
  - Attenuation coefficient reduction
  - Cupping
  - Spill over
  - Dark streaks
- Beam Hardening Correction Methods
  - Mono-energetic beam gives low SNR.
  - Dual energy requires 2 scans.
  - Soft tissue pre-processing. Does not compensate for high Z materials.
  - Post-processing for soft tissue and bone (Joseph and Spittal, 1978)
    - Accuracy
    - Mixed pixels
  - Statistical!

Poly-energetic Transmission CT Model

- Object model
  - Attenuation map consists of K non-overlapping tissues

$$\mu(x,y,E) = \sum_{k=1}^{K} m_k(E) r^k(x,y) \rho(x,y)$$

  - $\{m_k(E)\}_{k=1}^{K}$ are known mass attenuation coefficients
  - $r^k(x,y) = 1$ if $(x,y) \in$ tissue k and $r^k(x,y) = 0$ otherwise (known)
  - $\{\rho^k(x,y)\}$ are unkown tissue densities

- Definitions
  - $s_i^k(\rho) \triangleq \int_{L_i} \rho^k(x,y) r^k(x,y)\,dl$ (tissue component thickness)
  - $\rho = [\rho_1, \ldots, \rho_p]'$
  - $\underline{v}(\rho) = (s_i^1, s_i^2, \ldots, s_i^K)$
- Mean of photon flux along path L_i

$$\begin{aligned} E[Y_i] &= \int I_i(E) \exp\left\{ -\sum_{k=1}^{K} m_k(E) \int_{L_i} \rho^k(x,y) r^k(x,y)\,dl \right\} dE \\ &= \int I_i(E) \exp\left\{ -\sum_{k=1}^{K} m_k(E) s_i^k(\rho) \right\} dE \\ &\triangleq \bar{Y}_i(\underline{v}_i(\rho)) \end{aligned}$$

Penalized Maximum Likelihood Estimation

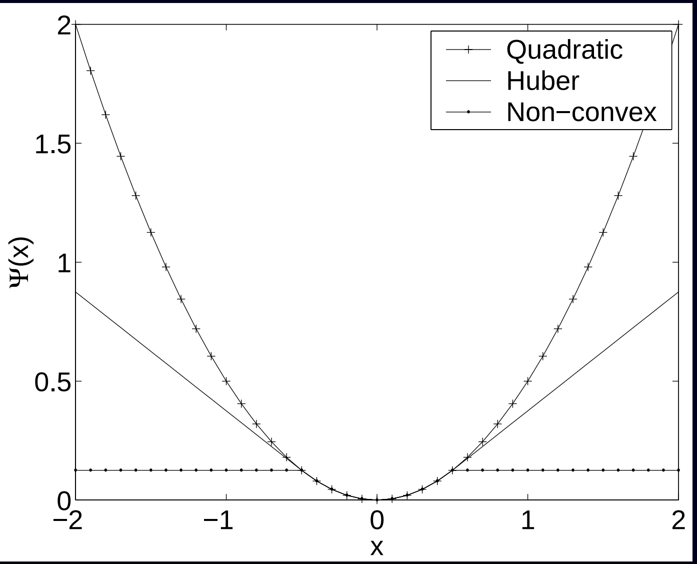
- Objective function has three components
  - Log-likelihood (data-fit term)

$$L(\rho) = \sum_{i=1}^{N} \{ Y_i \log [\bar{Y}_i(\underline{v}_i(\rho)) + r_i] - [\bar{Y}_i(\underline{v}_i(\rho)) + r_i] \}$$

  - Regularization term

$$\beta \cdot R(\rho) = \beta \sum_{j=1}^{p} \sum_{k \in N_j} w_{jk} \psi(\rho_j - \rho_k)$$

$$\psi(x;\delta) = \begin{cases} \frac{x^2}{2}, & x < \delta \\ \delta|x| - \frac{\delta^2}{2}, & x \geq \delta \end{cases}$$



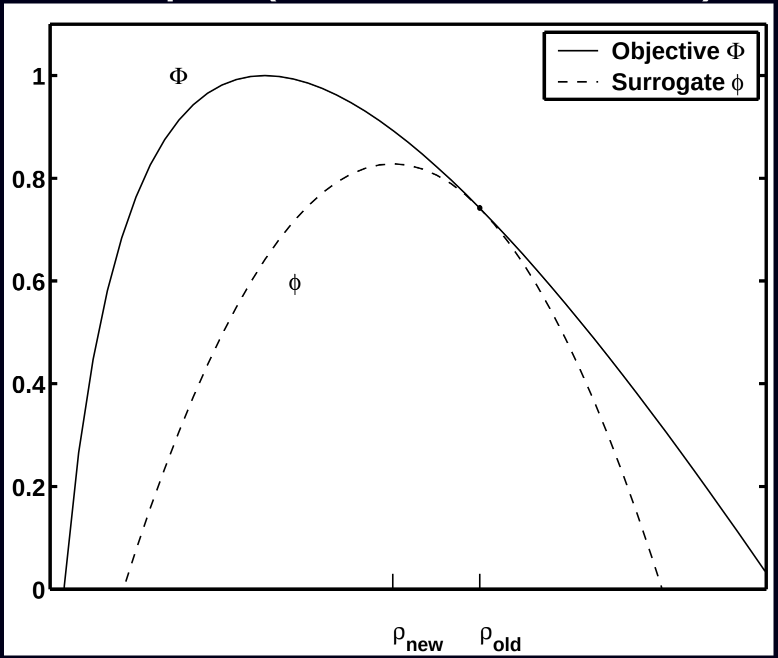
  - Constraints (nonnegativity)

Iterative Reconstruction Algorithm

$$\Phi(\rho) = L(\rho) - \beta \cdot R(\rho)$$

$$\boxed{\hat{\rho} \stackrel{\triangle}{=} \arg\max_{\rho \geq 0} \Phi(\rho)}$$

- Optimization Transfer Principle (De Pierro 93, 95)



Poly-energetic Quadratic Cost Function and Algorithm

- Negative log likelihood

$$-L(\rho) = \sum_{i=1}^{N} \{-Y_i \log[\bar{Y}_i(\underline{v}_i(\rho)) + r_i] + (\bar{Y}_i(\underline{v}_i(\rho)) + r_i)\}$$
$$= \sum_{i=1}^{N} h_i(\underline{v}_i(\rho))$$

- Expand the ray log likelihood $h_i(\underline{v}_i)$ in a second-order Taylor series around some estimate $\hat{\underline{v}}_i$:

$$h_i(\underline{v}_i) \approx h_i(\hat{\underline{v}}_i) + \nabla h_i(\hat{\underline{v}}_i)(\underline{v}_i(\rho) - \hat{\underline{v}}_i) + \frac{1}{2}(\underline{v}_i(\rho) - \hat{\underline{v}}_i)' \nabla^2 h_i(\hat{\underline{v}}_i)(\underline{v}_i(\rho) - \hat{\underline{v}}_i)$$

where

$$\nabla h_i(\hat{\underline{v}}_i) = \left(1 - \frac{Y_i}{\bar{Y}_i(\hat{\underline{v}}_i)}\right) \nabla \bar{Y}_i(\hat{\underline{v}}_i)$$
$$\nabla^2 h_i(\hat{\underline{v}}_i) \approx \frac{Y_i}{\bar{Y}_i^2(\hat{\underline{v}}_i)} \nabla' \bar{Y}_i(\hat{\underline{v}}_i) \nabla \bar{Y}_i(\hat{\underline{v}}_i)$$

- To simplify writing, define:
  - $Z_i \triangleq \sum_{k=1}^K \frac{\partial \bar{Y}_i}{\partial s_i^k}(\underline{\hat{v}}_i)\hat{s}_i^k = \nabla \bar{Y}_i(\underline{\hat{v}}_i)\underline{\hat{v}}_i$
  - $b_{ij} \triangleq \sum_{k=1}^K \frac{\partial \bar{Y}_i}{\partial s_i^k}(\underline{\hat{v}}_i) a_{ij} r_j^k$
  - $\mathbf{B} \triangleq \sum_{k=1}^K \mathbf{D}(\nabla_k \bar{Y}_i(\underline{\hat{v}}_i))\mathbf{A}\mathbf{D}(\underline{r}^k)$
- The quadratic cost function:

$$\Phi_q(\rho) = \sum_{i=1}^N \left\{ \sum_{k=1}^K \nabla_k h_i(\underline{\hat{v}}_i) \left(\sum_{j=1}^p a_{ij} r_j^k (\rho_j - \hat{\rho}_j) \right) + \frac{1}{2Y_i} \left([\mathbf{B}\rho]_i - Z_i \right)^2 \right\} + \beta R(\rho).$$

- Separable paraboloidal surrogate
  - Cost function is convex $\rightarrow$ De Pierros' trick:

$$[\mathbf{B}\rho]_i = \sum_{j=1}^p b_{ij}\rho_j = \sum_{j=1}^p \alpha_{ij} \left\{ \frac{b_{ij}}{\alpha_{ij}}(\rho_j - \rho_j^n) + [\mathbf{B}\rho^n]_i \right\}$$

where

$$\sum_{j=1}^p \alpha_{ij} = 1,\ \forall i,\ \alpha_{ij} \geq 0$$

- Move summation over pixels outside quadratic term

$$([\mathbf{B}\rho]_i - Z_i)^2 \leq \sum_{j=1}^{p} \alpha_{ij} \left(\frac{b_{ij}}{\alpha_{ij}} (\rho_j - \rho_j^n) + [\mathbf{B}\rho^n]_i - Z_i \right)^2$$

- Separable paraboloidal surrogate function:

$$\begin{aligned} Q(\rho;\rho^n) \;=\; & \sum_{j=1}^{p} \sum_{i=1}^{N} \sum_{k=1}^{K} \nabla_k h_i(\underline{\hat{\nu}}_i) a_{ij} r_j^k \rho_j - \sum_{i=1}^{N} \sum_{k=1}^{K} \nabla_k h_i(\underline{\hat{\nu}}_i) a_{ij} r_j^k \hat{\rho}_j \\ & + \sum_{j=1}^{p} \sum_{i=1}^{N} \frac{1}{2Y_i} \alpha_{ij} \left(\frac{b_{ij}}{\alpha_{ij}} (\rho_j - \rho_j^n) + [\mathbf{B}\rho^n]_i - Z_i \right)^2 + \beta S(\rho;\rho^n) \end{aligned}$$

- Iterative beam hardening algorithm
  - Take first derivative of Q and set equal to zero:

$$\rho_j^{n+1} = \left[\rho_j^n - \frac{\left.\dfrac{\partial Q(\rho;\rho^n)}{\partial \rho_j}\right|_{\rho=\rho^n}}{\left.\dfrac{\partial^2 Q(\rho;\rho^n)}{\partial \rho_j^2}\right|_{\rho=\rho^n}} \right]_+ , \quad j = 1, \ldots, p.$$

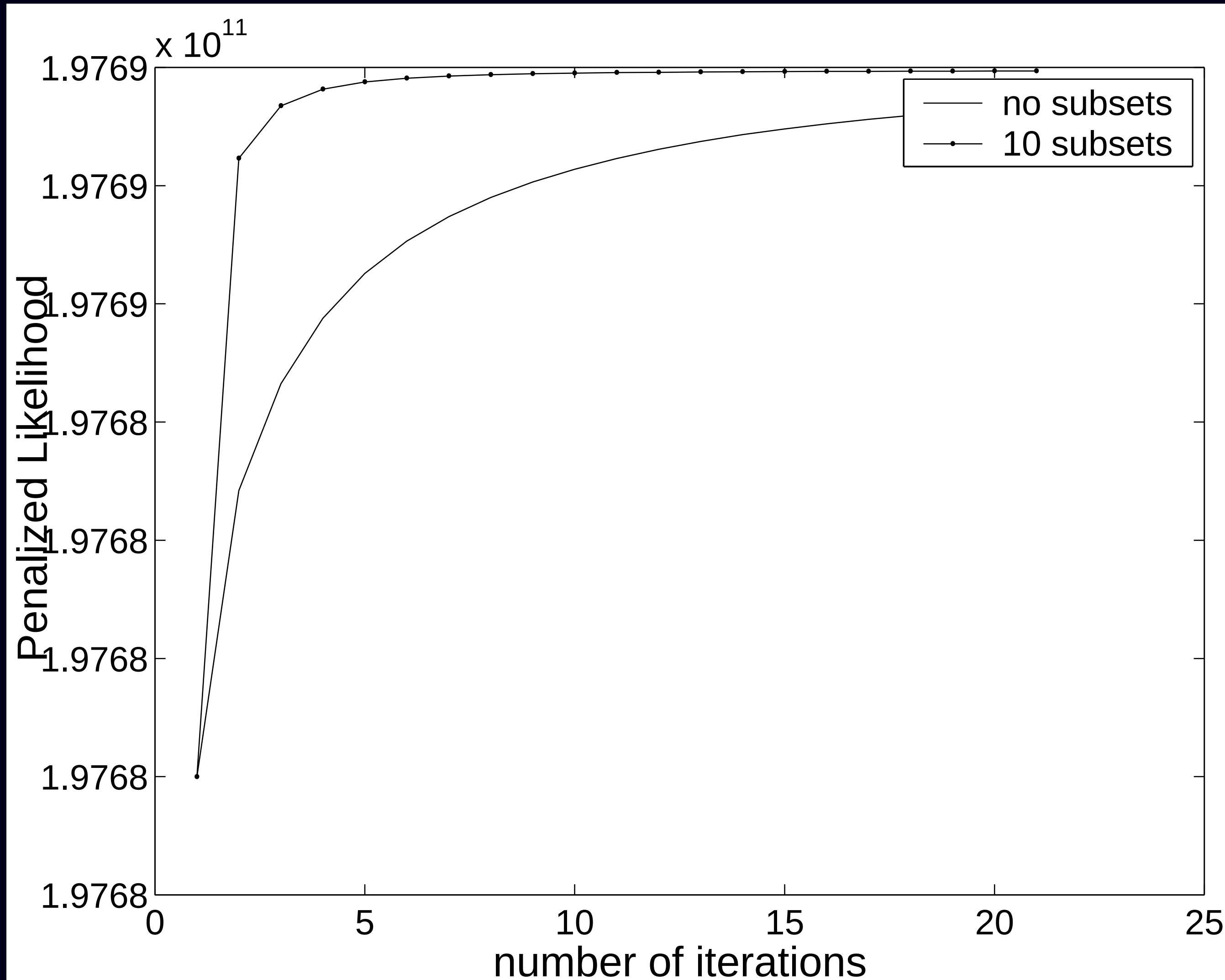
where

$$\left.\frac{\partial Q(\rho;\rho^n)}{\partial \rho_j}\right|_{\rho=\rho^n} = \sum_{i=1}^{N} \sum_{k=1}^{K} a_{ij} r_j^k \nabla_k h_i(\underline{\hat{v}}_i) + \beta \left.\frac{\partial S}{\partial \mu_j}\right|_{\rho=\rho^n}$$

$$\left.\frac{\partial^2 Q(\rho;\rho^n)}{\partial \rho_j^2}\right|_{\rho=\rho^n} = \sum_{i=1}^{N} \frac{1}{Y_i} \frac{{b_{ij}}^2}{\alpha_{ij}} + \beta \left.\frac{\partial^2 S}{\partial \mu_j^2}\right|_{\rho=\rho^n}$$

Ordered Subsets

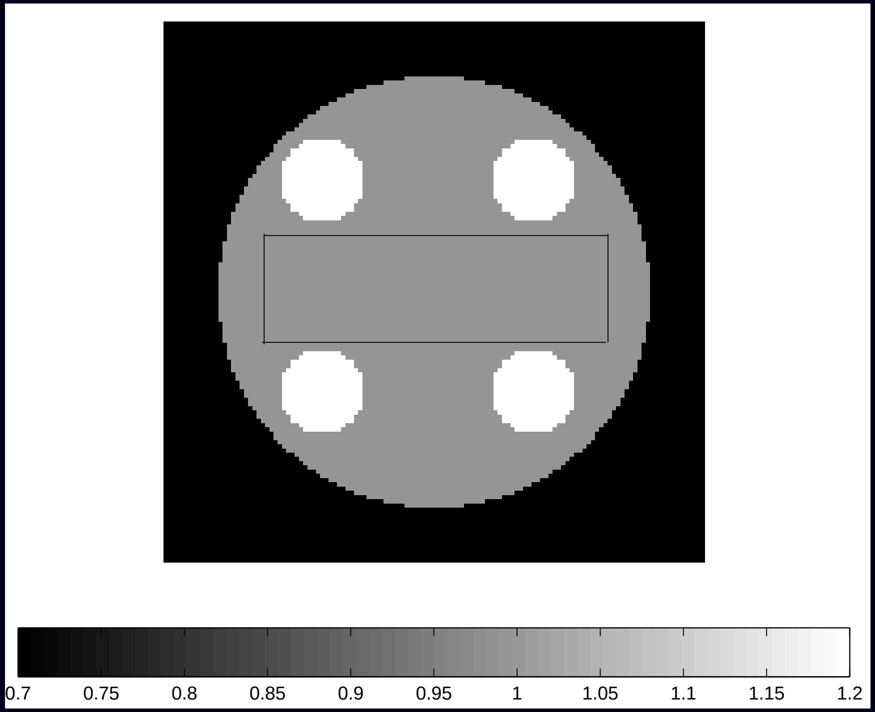
- Each $\sum_{i=1}^{N}$ is a backprojection
- Replace “full” backprojections with partial backprojections
- Partial backprojection based on angular subsampling
- Cycle through subsets of projection angles
- Pros
  - Significantly accelerates “convergence”
  - Very simple to implement
  - Reasonable images in a few iterations
  - Regularization easily incorporated
- Cons
  - Does not converge to true maximizer
  - Makes analysis of properties difficult



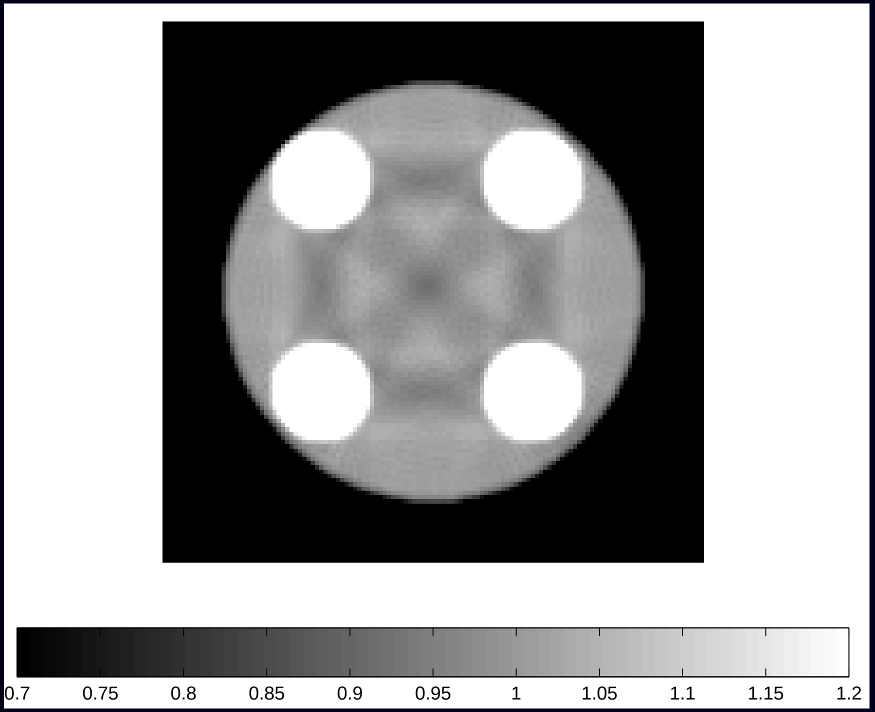
x 10^11
1.9769
1.9769
1.9769
1.9768
1.9768
1.9768
1.9768
1.9768
Penalized Likelihood
0
5
10
15
20
25
number of iterations
no subsets
10 subsets

Beam Hardening Correction

- 128×128 Bone/water test phantom
- FOV 50 cm, 100 cm source-detector distance, 150×150 sinogram, parallel beam geometry

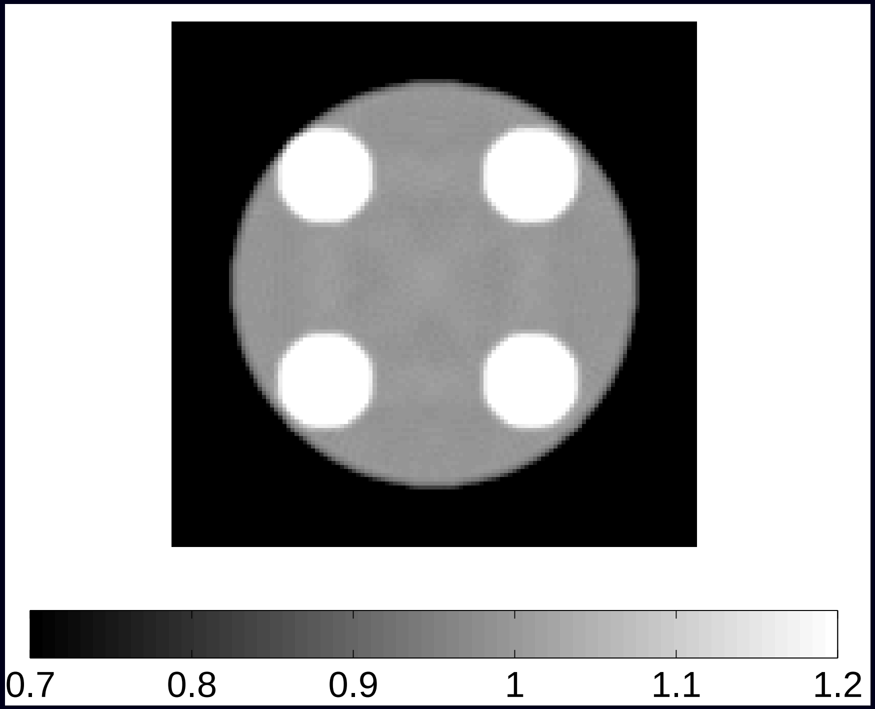


True Phantom

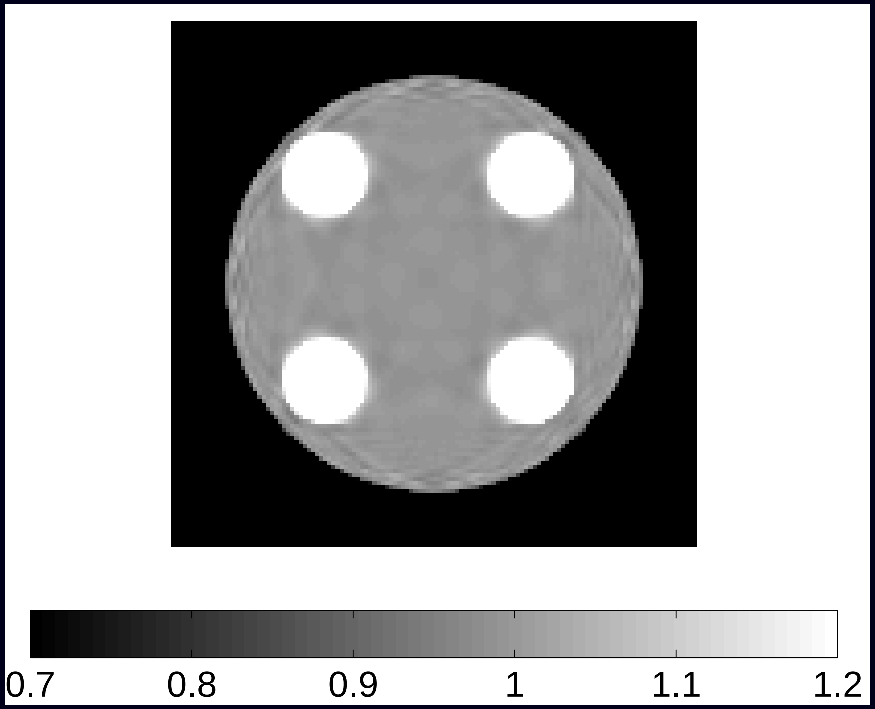


Soft Tissue Correction

Noise-free Results

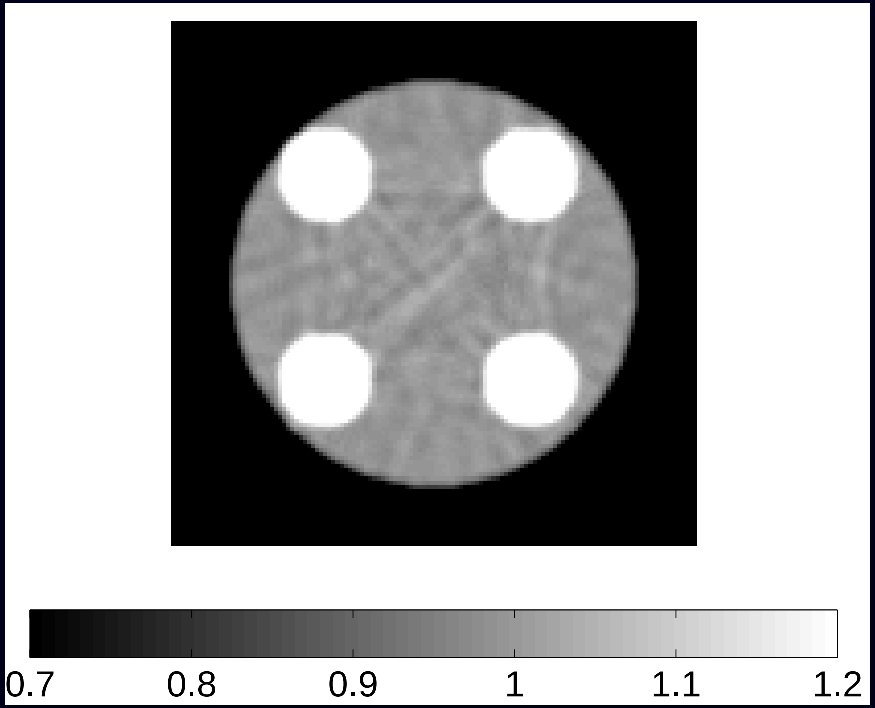


Joseph & Spittal
Standard Deviation 0.0026

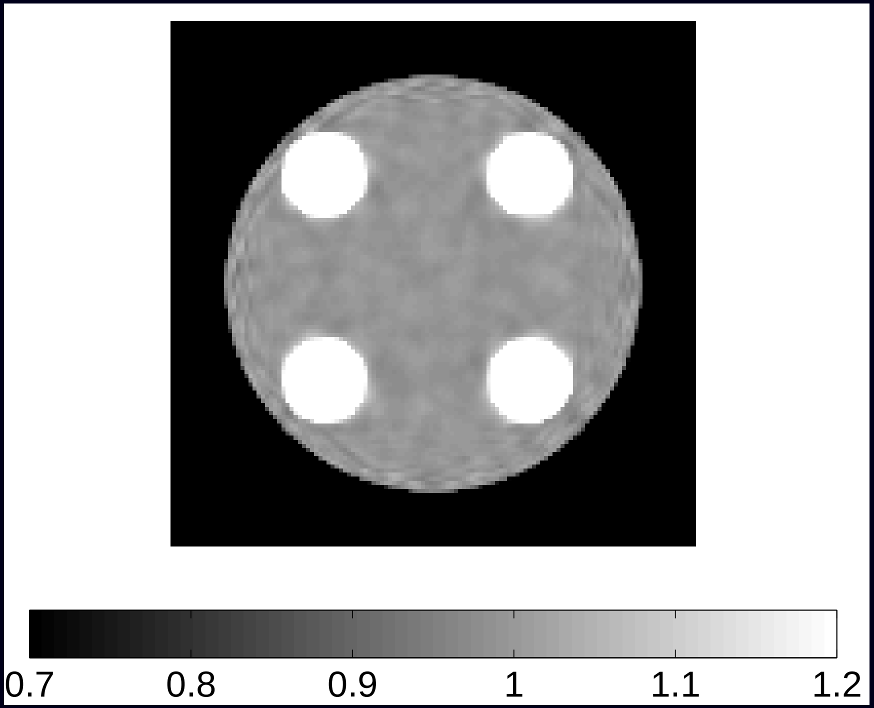


Statistical Reconstruction
Standard Deviation 0.0027

Noisy Results: 1.3 $\times$ 10^{10} counts

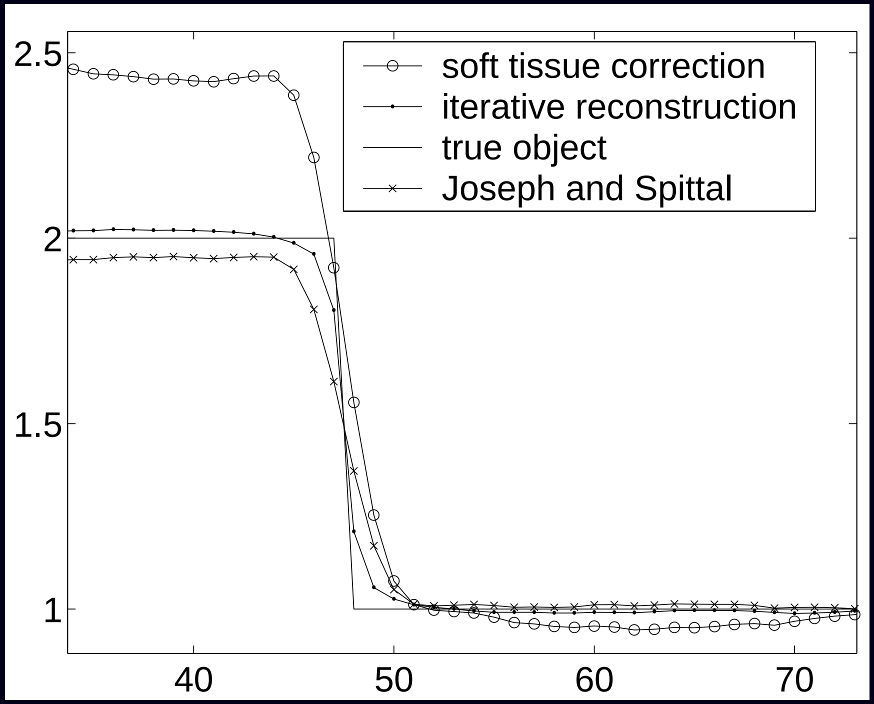


Joseph & Spittal
Standard Deviation 0.0040

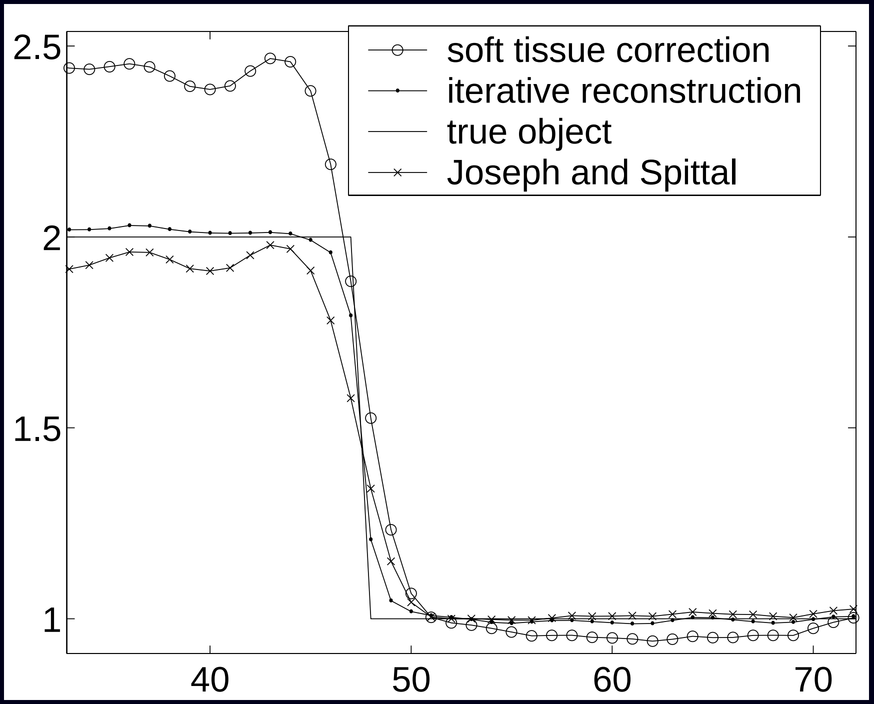


Statistical Reconstruction
Standard Deviation 0.0026

Profile Plots

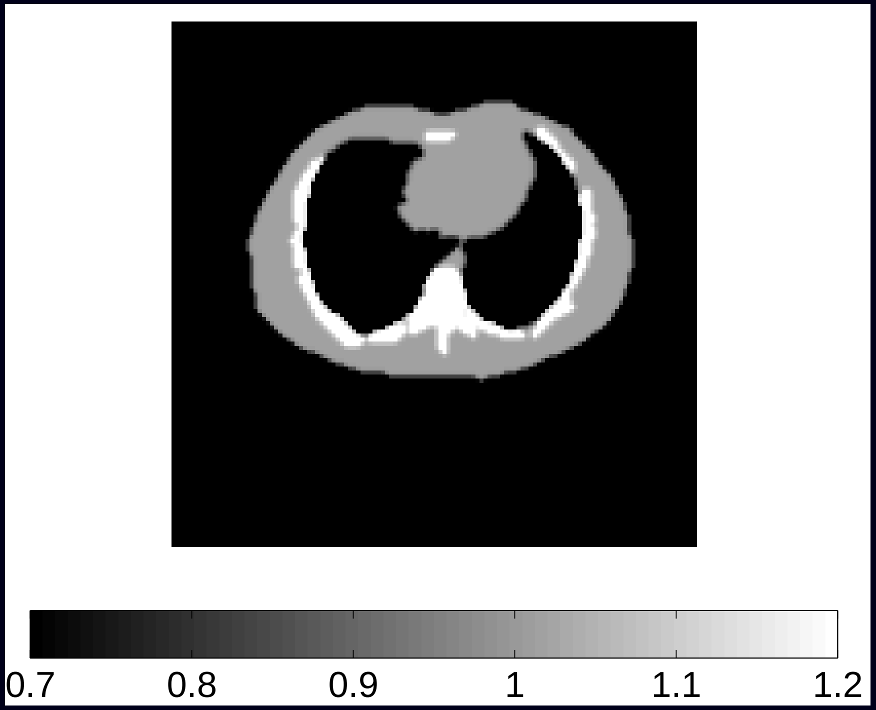


Noise-free Data

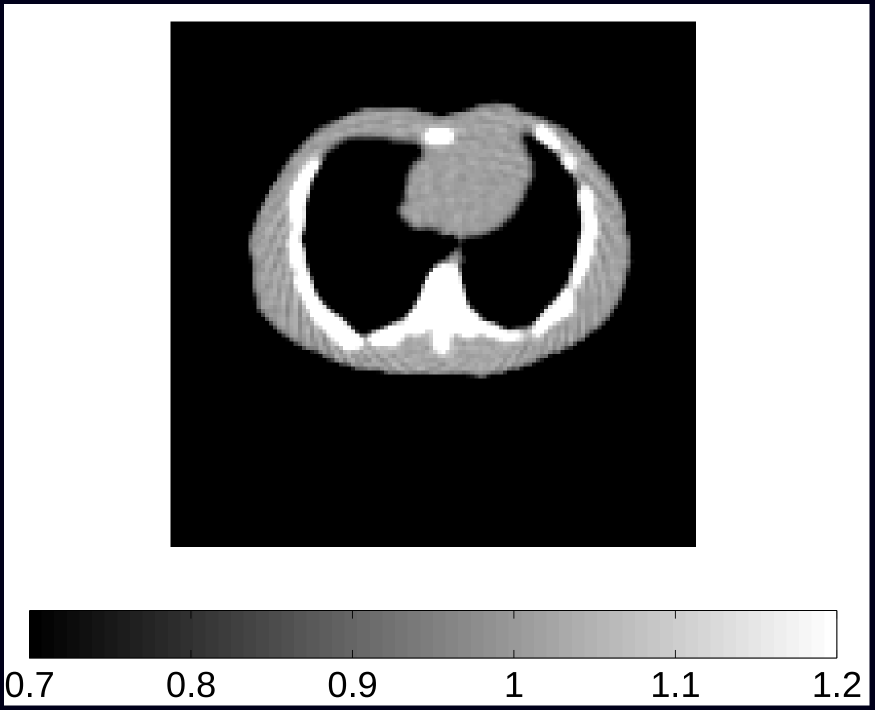


Noisy Data

Thorax Phantom

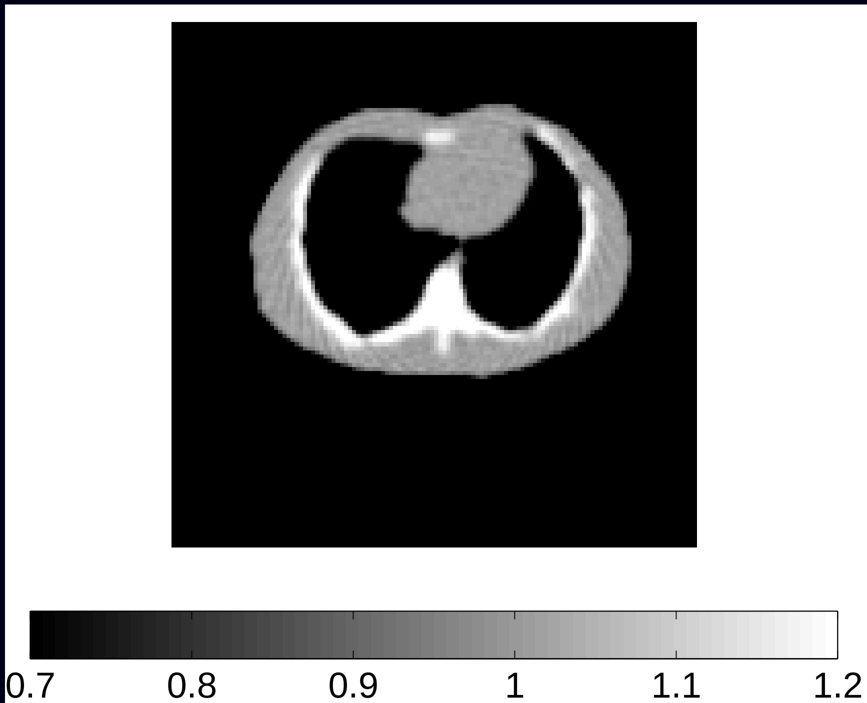


True Phantom

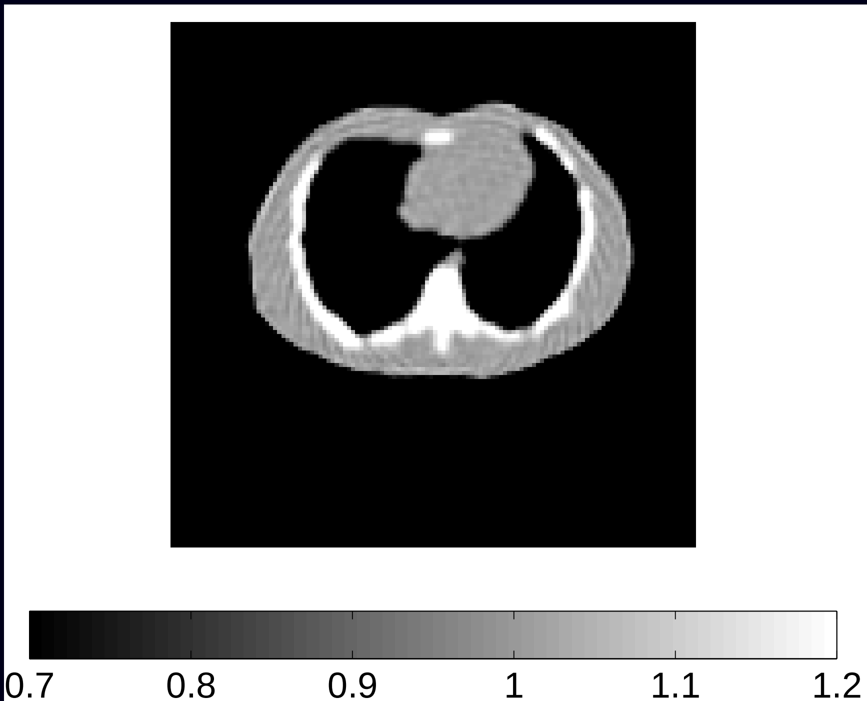


Soft Tissue Correction

Noisy Data: 1.55 $\times$ 10^{10} counts

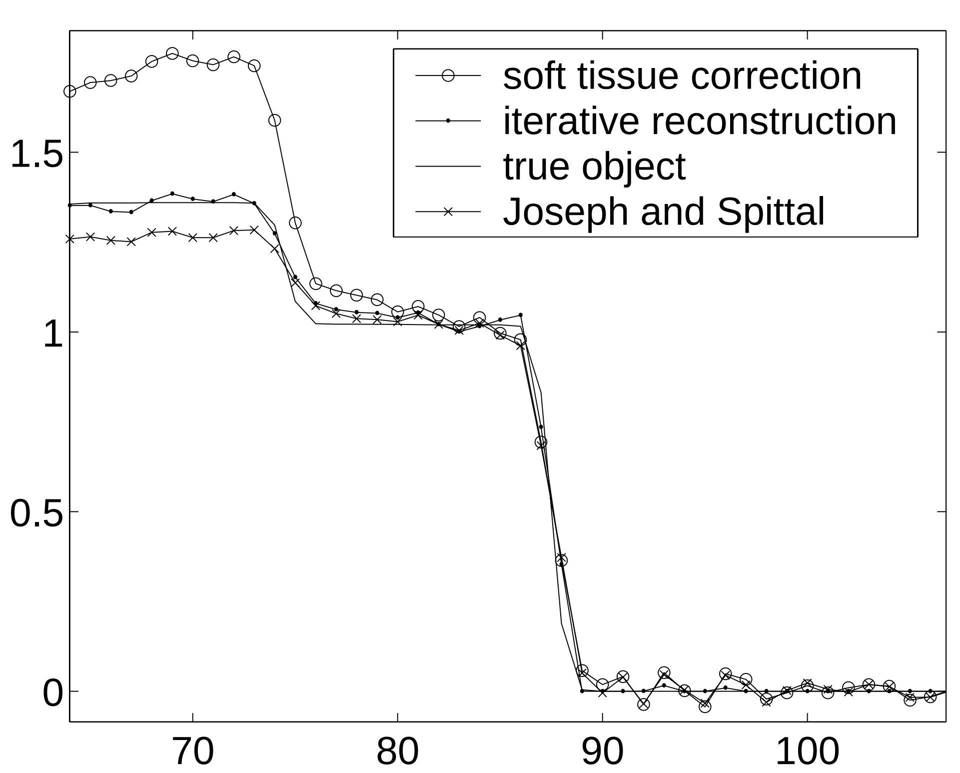


Joseph & Spittal
Standard Deviation 0.110



Statistical Reconstruction
Standard Deviation 0.113

Profile Plots



Density Profiles

Future Work

- Model accuracy
  - Improve the approximation to the log likelihood by using an actual surrogate
  - Implement 3-substance model (Iodine contrast agent)
  - Mixture models $r^k(x,y) \in [0,1]$
  - Joint density estimation and classification
- Regularization
  - Non-subjective choice for penalty parameters
  - Joint penalties
- Comparisons and experimental validation
  - Compare with FBP (bias-variance)
  - Real data
- Computation time
  - Software and hardware
  - Algorithm design