

## Smooth Optimization Using Global and Local Low-Rank Regularizers\*

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**Abstract.** Many inverse problems and signal processing problems involve low-rank regularizers based on the nuclear norm. Commonly, proximal gradient methods (PGMs) are adopted to solve this type of non-smooth problems, as they can offer fast and guaranteed convergence. However, PGM methods cannot be simply applied in settings where low-rank models are imposed locally on overlapping patches; therefore, heuristic approaches have been proposed that lack convergence guarantees. In this work we propose to replace the nuclear norm with a smooth approximation in which a Huber-type function is applied to each singular value. By providing a theoretical framework based on singular value function theory, we show that important properties can be established for the proposed regularizer, such as convexity, differentiability, and Lipschitz continuity of the gradient. Moreover, we provide a closed-form expression for the regularizer gradient, enabling the use of standard iterative gradient-based optimization algorithms (e.g., the nonlinear conjugate gradient) that can easily address the case of overlapping patches and have well-known convergence guarantees. In addition, we provide a novel step-size selection strategy based on a quadratic majorizer of the line search function that leverages the Huber characteristics of the proposed regularizer. Finally, we assess the proposed optimization framework by providing empirical results in dynamic magnetic resonance imaging (MRI) reconstruction in the context of local low-rank models with overlapping patches.

**Key words.** low-rank modeling, singular value functions, smooth optimization, majorize-minimize algorithms

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**1. Introduction.** Some inverse problems and many signal processing problems involving low-rank models are formulated as optimization problems using global low-rank regularizers as follows:<sup>1</sup>

$$(1.1) \quad \hat{\mathbf{X}} \in \arg \min_{\mathbf{X} \in \mathbb{C}^{M \times N}} f(\mathbf{X}) + \beta \|\mathbf{X}\|_*,$$

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<sup>1</sup>Throughout the paper we use upper and lower case bold letters to denote matrices and vectors, respectively. The entry in the  $k$ th row and  $l$ th column of a matrix  $\mathbf{X}$  is denoted either as  $X_{kl}$  or  $[\mathbf{X}]_{kl}$ . Analogously, the  $k$ th entry of a vector  $\mathbf{x}$  is denoted either as  $x_k$  or  $[\mathbf{x}]_k$ . Besides the nuclear norm, in later derivations we also use the Frobenius norm for matrices, denoted by  $\|\cdot\|_F$ . For vectors, we denote by  $\|\cdot\|$  and  $\|\cdot\|_\infty$  the  $\ell_2$ -norm and the  $\ell_\infty$ -norm, respectively.

where  $f$  typically corresponds to a function that measures data consistency;  $\|\cdot\|_*$  denotes the nuclear norm; and  $\beta$  is a regularization parameter. Example applications include robust PCA [4], matrix completion [9, 15], parallel MRI [28, 72, 73], low-rank + sparse models for dynamic MRI [51], among many others [8, 10, 22, 27, 65, 67, 70].

In the usual case where  $f$  has a Lipschitz continuous gradient, these optimization problems are solved easily using proximal gradient methods like the proximal optimized gradient method (POGM) [31, 61], as the nuclear norm is nonsmooth but “prox friendly.” Empirical results in dynamic MRI show that POGM converges faster than FISTA for such cost functions [41]. Nevertheless, these first-order (accelerated) proximal gradient methods have worst-case  $O(1/k^2)$  convergence rates that can be slow compared to optimization methods for smooth problems [18, 54]. Augmented Lagrangian and ADMM algorithms are another option, but these require additional tuning parameters and do not have faster convergence rates. Additional limitations for proximal gradient methods are encountered when using local low-rank models, where low-rank characteristics are promoted locally on overlapping patches constructed from  $\mathbf{X}$ . Applications include matrix approximation [33], MRI reconstruction [57, 64], dynamic MRI [14, 30, 63], multicontrast and quantitative MRI [60, 68], MR denoising [46], fMRI denoising [13, 47, 66], fMRI dynamic image reconstruction [17], and MR motion correction [7]. As an example, in dynamic MRI reconstruction the optimization problem has the form<sup>2</sup>

$$(1.2) \quad \hat{\mathbf{X}} \in \arg \min_{\mathbf{X} \in \mathbb{C}^{M \times N}} f(\mathbf{X}) + \beta \sum_{\mathbf{s} \in \Lambda} \sum_{\mathbf{p} \in \Gamma} \|\mathcal{P}_{\mathbf{p}}(\mathcal{S}_{\mathbf{s}}(\mathbf{X}))\|_*,$$

where  $\Lambda, \Gamma \subseteq \mathbb{Z}^d$  are sets of shift and location indexes, respectively, and  $d$  corresponds to the image voxel dimensions;<sup>3</sup>  $\mathcal{S}_{\mathbf{s}} : \mathbb{C}^{M \times N} \rightarrow \mathbb{C}^{M \times N}$  is a linear operator that circularly shifts the images composing the columns of  $\mathbf{X}$  according to the shift index  $\mathbf{s}$ ; and  $\mathcal{P}_{\mathbf{p}} : \mathbb{C}^{M \times N} \rightarrow \mathbb{C}^{P \times N}$  is a linear operator that constructs a matrix from a patch extracted from  $\mathbf{X}$  whose entries depend on the location index  $\mathbf{p}$ . In many applications these patches are considered overlapping (i.e., share entries of  $\mathbf{X}$ ), which makes standard proximal algorithms unsuitable for solving (1.2), as there is no known simple proximal mapping for the regularizer in this case. This challenge has led to various heuristics like “cycle spinning” [62], proximal averaging [19], and other approaches that do not have any convergence guarantees [71]. Replacing the nuclear norm with a *smooth approximation* could facilitate the use of standard iterative gradient-based optimization algorithms like nonlinear conjugate gradient (NCG), or perhaps even second-order optimization methods like quasi-Newton methods. These algorithms might lead to faster convergence while providing convergence guarantees. Moreover, they could be applied directly to the local low-rank formulation, even in the case of overlapping patches.

The nuclear norm in (1.1) is a convex relaxation of a regularizer based on the rank of  $\mathbf{X}$ , so it is already a kind of approximation to the “ideal” low-rank model. Thus, it seems quite reasonable to entertain further approximations. This work investigates the use of smooth

<sup>2</sup>Here we briefly describe the dynamic MRI reconstruction optimization problem using local low-rank regularizers. Section 6 provides more details.

<sup>3</sup>For simplicity, we consider  $d = 2$  throughout the paper (i.e., 2D images); however, our results can be easily extended to higher dimensions.

regularizers instead of nonsmooth alternatives such as the nuclear norm. Specifically, we study regularizers of the form

$$(1.3) \quad R_\psi(\mathbf{X}) \triangleq \sum_{k=1}^r \psi(\sigma_k(\mathbf{X})),$$

where  $r = \min(M, N)$ ;  $\sigma_k(\mathbf{X})$  denotes the  $k$ th singular value of  $\mathbf{X}$ , and  $\psi$  is a smooth function subject to mild regularity conditions that can be either convex or nonconvex. We study the case where  $\psi$  satisfies the so-called Huber conditions [26, p. 184], which is why we call  $R_\psi$  a *Huber-based low-rank regularizer*. Then we propose to impose global low-rank models by solving

$$(1.4) \quad \hat{\mathbf{X}} \in \arg \min_{\mathbf{X} \in \mathbb{C}^{M \times N}} \Psi_{\text{Global}}(\mathbf{X}), \quad \Psi_{\text{Global}}(\mathbf{X}) = f(\mathbf{X}) + \beta R_\psi(\mathbf{X})$$

and to impose local low-rank models by solving

$$(1.5) \quad \hat{\mathbf{X}} \in \arg \min_{\mathbf{X} \in \mathbb{C}^{M \times N}} \Psi_{\text{Local}}(\mathbf{X}), \quad \Psi_{\text{Local}}(\mathbf{X}) = f(\mathbf{X}) + \beta \sum_{s \in \Lambda} \sum_{p \in \Gamma} R_\psi(\mathcal{P}_p(\mathcal{S}_s(\mathbf{X}))).$$

The local case with overlapping patches is our primary motivation, since that is the case where the regularizer is not “prox friendly.”

Similar low-rank regularizers to (1.3) have been proposed in the global and local cases in [45] and [42], respectively. The resulting optimization problem is solved by approximating the regularizer using a first-order expansion that produces a nonconvex and nonsmooth cost function. This setting allows one to establish algorithms with convergence guarantees as shown in [45] and [42]. However, in the local low-rank case, the algorithm proposed in [42] introduces tuning parameters that affect convergence speed, and the total number of these parameters grows with the number of patches. In contexts like dynamic MRI reconstruction, the cost function (1.5) might include hundreds of patches, hindering the applicability of algorithms like the one in [42]. Nonconvex Schatten- $p$  quasi norms with  $0 < p < 1$  have also been investigated as beneficial alternatives to the nuclear norm (see, e.g., [50]). Other nonconvex functions of the singular values have also been investigated (see, e.g., [56, 6, 21, 49]).

The regularizer proposed in (1.3) corresponds to a unitarily invariant function [34], and therefore it only depends on the singular values. Functions with these characteristics are known as singular value functions [38, 39], and their properties (e.g., convexity, differentiability, Lipschitz continuity of the gradient) have been extensively investigated over the last two decades [34, 35, 36, 37, 38, 39, 53, 58, 59]. However, to the best of our knowledge, the application of these properties has not been explored in inverse problems using low-rank models. In this work we show that relevant properties for the optimization of  $R_\psi$  are directly related to the analogous properties of  $\psi$ . Using these derivations, we show that it is possible to use standard gradient-based iterative optimization algorithms (e.g., NCG, quasi-Newton methods) to solve (1.4) and (1.5) that offer convergence guarantees without needing to introduce tuning parameters.

Iterative gradient-based optimization algorithms usually require two main computations: the cost function gradient, and a step size in each iteration that guarantees a decrease of the

cost function. We show that the gradient of the cost function can be calculated in closed form by relying on singular value function theory to calculate  $\nabla R_\psi(\mathbf{X})$ . Then we provide a step-size selection strategy that exploits the Huber/Lipschitz characteristics of  $\psi$ . Specifically, we propose a majorize-minimize (MM) dynamic step-size selection strategy to monotonically decrease the line search function at each iteration, where a quadratic majorizer is constructed by leveraging on the Huber conditions satisfied by  $\psi$ . This construction is based on deriving novel majorizers for the proposed regularizer  $R_\psi$ , which is one of our main theoretical contributions in this work.

In addition, we generalize our theoretical framework by considering a weighted version of the proposed regularizers that we call *weighted Huber-based low-rank regularizers*. Specifically, we study regularizers of the form

$$(1.6) \quad R_{\psi, \mathbf{w}}(\mathbf{X}) \triangleq \sum_{k=1}^r w_k \psi(\sigma_k(\mathbf{X})),$$

where  $\mathbf{w} = [w_1, \dots, w_r]^T$  corresponds to a nonzero vector of nonnegative weights, and  $(\cdot)^T$  denotes the transpose operation. To properly impose low-rank models,  $R_{\psi, \mathbf{w}}$  should penalize more small singular values than large singular values, as the former are usually associated with noise in real applications. This property suggests that the weights should be nondecreasing, i.e.,  $0 \leq w_1 \leq w_2 \leq \dots \leq w_r$ . However, we show that under this setting  $R_{\psi, \mathbf{w}}$  is nonconvex, which is relevant from an optimization perspective. A similar analysis was provided by Chen, Dong, and Chan [6] for the adaptive nuclear norm; we extend their results to the family of weighted Huber-based low-rank regularizers. Interestingly, if  $\psi$  is selected as the hyperbola function, the regularizer in (1.6) can be interpreted as a smooth version of the adaptive nuclear norm [6]. Analogously to the analysis we provide for  $R_\psi$ , we study the convexity and differentiability of  $R_{\psi, \mathbf{w}}$  and the Lipschitz continuity of its gradient. We show that the proposed optimization framework can be extended such that  $R_{\psi, \mathbf{w}}$  can be used instead of  $R_\psi$ .

The explicit gradient calculation of the cost function, jointly with the step-size selection strategy provided in this work, can be seen as the main steps of a smooth optimization framework for inverse problems using low-rank models. Importantly, by enabling standard gradient-based iterative optimization algorithms, it is possible to solve (1.5) in the case of overlapping patches while also providing convergence guarantees. Moreover, the same optimization algorithm can be used for different choices of  $\psi$  when Huber's conditions are satisfied. This offers users a convenient mechanism to explore different regularizers; however, this flexibility might decrease computational speed compared to algorithms that are designed for a particular nonsmooth regularizer. We show in the context of dynamic MRI reconstruction that our proposed framework can converge faster and provide better image quality than state-of-the-art algorithms (e.g., POGM) that are tailored to commonly used nonsmooth regularizers.

The main contributions of this work correspond to the following:

- We provide a smooth optimization framework to solve inverse problems involving either global or local low-rank models, which are imposed through regularizers constructed using Huber-type smooth functions.
- We establish a theoretical framework based on singular value function theory to study relevant properties of the proposed regularizers (e.g., convexity, differentiability, Lipschitz continuity of the gradient).

- We propose a step-size selection strategy based on an MM approach that monotonically decreases the line search function. For this purpose, we provide novel majorizers for the proposed regularizers that exploit their Huber characteristics.
- We empirically show that, by using the proposed optimization framework, standard smooth optimization algorithms with convergence guarantees can be used to solve inverse problems involving local low-rank models with overlapping patches. Specifically, we show the application of the proposed smooth optimization framework in the context of dynamic MRI reconstruction.

This paper is organized as follows. Section 2 provides a detailed description of the proposed Huber-based low-rank regularizers; their convexity and differentiability properties are studied by establishing connections with singular value functions. Section 3 introduces novel majorizers for the proposed regularizers that are used to show the Lipschitz continuity of the regularizer gradient and to provide a quadratic majorizer for the line search function in later sections. Section 4 extends the proposed theoretical framework to weighted Huber-based low-rank regularizers. Section 5 provides a smooth optimization framework based on the proposed Huber-based low-rank regularizers to minimize (1.4) or (1.5) using standard iterative gradient-based optimization algorithms. Section 6 illustrates the applicability of the proposed optimization framework in the context of dynamic MRI reconstruction using local low-rank models with overlapping patches. Finally, section 7 provides discussion and conclusions.

**2. Huber-based low-rank regularizers.** Let  $\mathbf{X} \in \mathbb{C}^{M \times N}$  be a rectangular matrix with singular values  $\sigma_1(\mathbf{X}) \geq \sigma_2(\mathbf{X}) \geq \dots \geq \sigma_r(\mathbf{X}) \geq 0$  where  $r = \min(M, N)$ , and let the vector-valued function  $\boldsymbol{\sigma} : \mathbb{C}^{M \times N} \rightarrow \mathbb{R}^r$  be defined as  $\boldsymbol{\sigma}(\mathbf{X}) \triangleq [\sigma_1(\mathbf{X}), \dots, \sigma_r(\mathbf{X})]^T$ . Using this notation, we write the proposed low-rank regularizer in (1.3) as

$$(2.1) \quad R_\psi(\mathbf{X}) = (f_\psi \circ \boldsymbol{\sigma})(\mathbf{X}),$$

where  $\circ$  denotes function composition;  $f_\psi : \mathbb{R}^r \rightarrow \mathbb{R}$  is defined as  $f_\psi(\mathbf{x}) \triangleq \sum_{k=1}^r \psi(x_k)$ ; and  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  corresponds to a function satisfying the Huber conditions [26, p. 184], which we refer to as the *potential function*. The extra notation introduced to write the regularizer expression in (2.1) might seem redundant, but it facilitates the study of the properties of  $R_\psi$  in later sections. In fact, the expression in (2.1) reveals that  $R_\psi$  corresponds to a singular value function [34, 38] that involves the potential function, as indicated later in this section. Notably, relevant properties for the optimization of the regularizer highly depend on the analogous properties of the potential function. For this reason, we provide a detailed description of the latter in the following subsection.

**2.1. Huber potential functions.** The potential function  $\psi$  in (1.3) and (2.1) satisfies the following Huber conditions [26, p. 184]:

- $\psi$  is differentiable.
- $\psi$  is even, i.e.,  $\psi(t) = \psi(-t) \forall t \in \mathbb{R}$ .
- $\omega_\psi(t) \triangleq \dot{\psi}(t)/t$ , where  $\dot{\psi}$  denotes the derivative of  $\psi$ , is bounded, is nonnegative, and is monotone nonincreasing for  $t > 0$ . This function is called the *weighting function*.

This family of functions, collectively called Huber potential functions, is extensive, and includes convex and nonconvex functions that have been widely used in inverse problems;

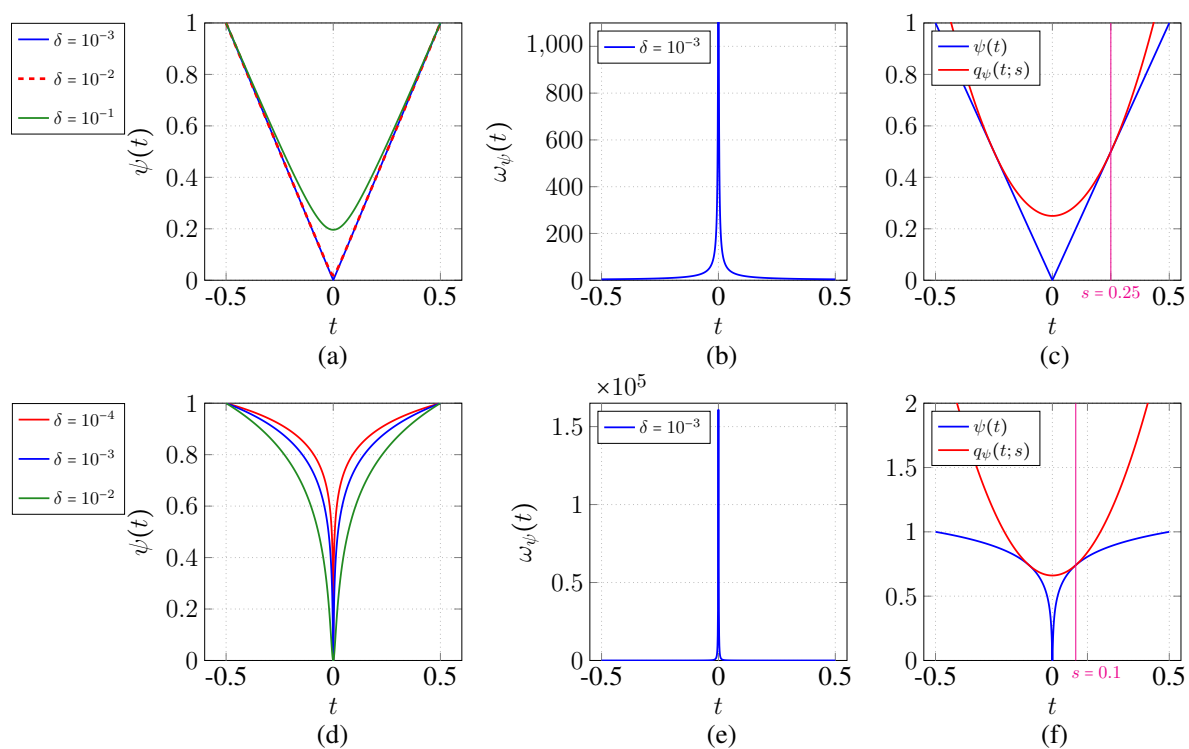


**Figure 1.** Examples of Huber potential functions. (a) Hyperbola function (convex) as defined in (2.2) for three values of  $\delta$ . Each curve has been normalized for illustration purposes. The respective potential weighting function is shown in (b) for  $\delta = 10^{-3}$ . For this case, (c) shows the quadratic majorizer as defined in (2.4) for a particular value of  $s$ . (d), (e), and (f) show analogous results for the Cauchy function (nonconvex) as defined in (2.3).

see, e.g., [5, 16, 24]. The assumption that  $\omega_\psi$  is nonnegative means that  $\dot{\psi}$  must be nonnegative for  $t > 0$ , which means that  $\psi$  must be nondecreasing for  $t > 0$ . There are examples of redescending potential functions in the imaging literature [12, 69] (although perhaps not in terms of singular values). We leave extensions to descending  $\psi$  for future work.

A representative example for  $\psi$  is the hyperbola (cf. Figure 1(a))

$$(2.2) \quad \psi(t) = \delta^2 \sqrt{1 + (t/\delta)^2},$$

which is approximately  $\delta|t|$  for  $t \gg \delta > 0$ , so  $R_\psi$  can approximate the nuclear norm very well by taking  $\delta$  to be small (cf. Figure 1(a)). Nonconvex (but still smooth) choices for  $\psi$  include the Cauchy function (cf. Figure 1(d)):

$$(2.3) \quad \psi(t) = \frac{1}{2} \delta^2 \log \left( 1 + \left( \frac{t}{\delta} \right)^2 \right).$$

A notable property of a potential function  $\psi$  is that it always has a quadratic majorizer having curvature  $\omega_\psi$ . Specifically, for any  $s \in \mathbb{R}$ , the function<sup>4</sup>

$$(2.4) \quad q_\psi(t; s) \triangleq \psi(s) + \dot{\psi}(s)(t - s) + \frac{1}{2}\omega_\psi(s)(t - s)^2$$

satisfies

$$(2.5) \quad \psi(s) = q_\psi(s; s),$$

$$(2.6) \quad \psi(t) \leq q_\psi(t; s) \quad \forall t \in \mathbb{R}.$$

Moreover,  $q_\psi$  is the quadratic majorizer for  $\psi$  satisfying (2.5) and (2.6) with optimal (i.e., smallest) curvature [26, p. 186]. This surrogate will be particularly useful when studying majorizers for  $R_\psi$  in section 3. Interestingly, this surrogate also allows one to show that  $\dot{\psi}$  is Lipschitz continuous. Given that  $\omega_\psi$  is symmetric and monotonically nonincreasing for  $t > 0$ , it follows that:

$$(2.7) \quad \begin{aligned} \psi(t) \leq q_\psi(t; s) &\leq \psi(s) + \dot{\psi}(s)(t - s) + \frac{1}{2} \left( \sup_{t \in \mathbb{R}} \omega_\psi(t) \right) (t - s)^2 \quad \forall t, s \in \mathbb{R}, \\ &= \psi(s) + \dot{\psi}(s)(t - s) + \frac{1}{2}\omega_\psi(0)(t - s)^2, \end{aligned}$$

which is equivalent to  $\dot{\psi}$  being  $\omega_\psi(0)$ -Lipschitz [1, Thm. 5.8].

Figure 1 shows two examples of potential functions with their respective weighting functions and quadratic majorizers. Figure 1(a) and Figure 1(d) illustrate the (convex) hyperbola and the (nonconvex) Cauchy potential functions, respectively.

The following subsection shows that many of the properties of the potential function  $\psi$  (e.g., convexity, differentiability, Lipschitz continuity of the derivative) are transferred to the regularizer  $R_\psi$  as a direct consequence of  $R_\psi$  being a singular value function. For this purpose, in the following derivations we consider  $\mathbb{C}^{M \times N}$  as a vector space endowed with the (Frobenius) inner product  $\langle \cdot, \cdot \rangle : \mathbb{C}^{M \times N} \times \mathbb{C}^{M \times N} \rightarrow \mathbb{C}$  defined as

$$(2.8) \quad \langle \mathbf{A}, \mathbf{B} \rangle = \text{trace}\{\mathbf{A}\mathbf{B}'\} \quad \forall \mathbf{A}, \mathbf{B} \in \mathbb{C}^{M \times N},$$

where  $(\cdot)'$  denotes the Hermitian transpose operation.

**2.2. Properties of Huber-based low-rank regularizers.** The regularizer proposed in (1.3) and (2.1) is contained in a class of functions known as unitarily invariant functions or *singular value functions* [34, 35, 36, 38, 39], which in general terms are functions that depend only on the singular values of the matrix argument. For completeness, we provide the formal definition in the following.

**Definition 2.1 (unitarily invariant function or singular value function [34, 38, 39]).** A function  $F : \mathbb{C}^{M \times N} \rightarrow [-\infty, +\infty]$  is unitarily invariant if  $F(\mathbf{U}\mathbf{X}\mathbf{V}) = F(\mathbf{X})$  for any  $\mathbf{X} \in \mathbb{C}^{M \times N}$  and any suitably sized unitary matrices  $\mathbf{U}$  and  $\mathbf{V}$ .  $F(\mathbf{X})$  is also called a singular value function, as it depends only on the singular values of  $\mathbf{X}$ .

To facilitate the analysis of the proposed regularizers we also need the following definition.

<sup>4</sup>Throughout the paper we use the notation  $f(\cdot; \cdot)$  to indicate that the function  $f$  depends on parameters that are given after the semicolon symbol.

**Definition 2.2** (absolutely symmetric function [34]). A function  $f : \mathbb{R}^r \rightarrow [-\infty, +\infty]$  is absolutely symmetric if  $f(\mathbf{x}) = f(\check{\mathbf{x}})$  for any vector  $\mathbf{x} \in \mathbb{R}^r$ , where  $\check{\mathbf{x}} \in \mathbb{R}^r$  corresponds to the vector formed by arranging the entries of the vector  $|\mathbf{x}| \triangleq [|x_1|, \dots, |x_r|]^T$  in nonincreasing order.

There is a one-to-one correspondence between unitarily invariant functions defined on  $\mathbb{C}^{M \times N}$  and absolutely symmetric functions defined on  $\mathbb{R}^r$  [34]. In fact, if  $f : \mathbb{R}^r \rightarrow [-\infty, +\infty]$  is absolutely symmetric, then  $f \circ \sigma$  is unitarily invariant. Conversely, if  $F : \mathbb{C}^{M \times N} \rightarrow [-\infty, +\infty]$  is unitarily invariant, then  $f(\mathbf{x}) = F(\text{Diag}\{\mathbf{x}\})$  is absolutely symmetric, where  $\text{Diag} : \mathbb{R}^r \rightarrow \mathbb{C}^{M \times N}$  corresponds to a linear operator that creates a rectangular diagonal matrix using the entries of its input. Specifically, the entry in the  $k$ th row and  $l$ th column of  $\text{Diag}\{\mathbf{x}\} \in \mathbb{C}^{M \times N}$  is given by  $[\text{Diag}\{\mathbf{x}\}]_{kl} = x_k$  if  $k = l$  and 0 otherwise. Establishing the correspondence between singular value functions and absolutely symmetric functions is important, as many relevant properties of  $F$  are directly related to the analogous properties of the corresponding absolutely symmetric function  $f$ . Particularly, convexity and differentiability were studied by Lewis and Sordov in [34, 35, 38, 39]. We summarize these results in the following proposition, as these properties are relevant to our optimization framework.

**Proposition 2.3** (convexity and differentiability of singular value functions [34, 35, 38, 39]). Let  $F = f \circ \sigma : \mathbb{C}^{M \times N} \rightarrow [-\infty, +\infty]$  be a singular value function with corresponding absolutely symmetric function  $f : \mathbb{R}^r \rightarrow [-\infty, +\infty]$ . Then the following hold:

- (i)  $F$  is convex if and only if  $f$  is convex.
- (ii) If  $f$  is convex, then  $F$  is differentiable at  $\mathbf{X} \in \mathbb{C}^{M \times N}$  if and only if  $f$  is differentiable at  $\sigma(\mathbf{X})$ . In this case the gradient of  $F$  is given by

$$(2.9) \quad \nabla F(\mathbf{X}) = \mathbf{U} \text{Diag}\{\nabla f(\sigma(\mathbf{X}))\} \mathbf{V}',$$

where  $\mathbf{U}$  and  $\mathbf{V}$  are suitably sized unitary matrices such that  $\mathbf{X} = \mathbf{U} \text{Diag}\{\sigma(\mathbf{X})\} \mathbf{V}'$ .

- (iii) If  $f$  is nonconvex and locally Lipschitz around  $\sigma(\mathbf{X})$ , then  $F$  is differentiable at  $\mathbf{X}$  if and only if  $f$  is differentiable at  $\sigma(\mathbf{X})$ . The gradient of  $F$  is given by (2.9).

We can study the nuclear norm as an example of the previous proposition. The nuclear norm is clearly unitarily invariant, and its corresponding absolutely symmetric function is given by  $f_*(\mathbf{x}) \triangleq \sum_{k=1}^r |x_k|$ , i.e.,  $\|\mathbf{X}\|_* = (f_* \circ \sigma)(\mathbf{X})$ . The function  $f_*$  also corresponds to a symmetric gauge function [48]. It can be shown that absolutely symmetric functions correspond to symmetric gauge functions when, in addition, they are positively homogeneous [34]. Given that  $f_*$  is convex and nondifferentiable at points where some of the entries are equal to zero, then, according to Proposition 2.3,  $\|\cdot\|_*$  is convex and nondifferentiable at matrices where some of the singular values are equal to zero.

In our setting, it is straightforward to show that the function  $f_\psi$  in (2.1) is absolutely symmetric and therefore the corresponding regularizer  $R_\psi$  is a singular value function (i.e., unitarily invariant). Then we can establish conditions for the convexity and differentiability of  $R_\psi$  by analyzing  $f_\psi$  and using Proposition 2.3. In our context, these conditions are directly related to the potential function  $\psi$  as stated in the next theorem. Appendix A gives the straightforward proof.

**Theorem 2.4 (convexity and differentiability of Huber-based low-rank regularizers).** Let  $R_\psi$  be a Huber-based low-rank regularizer defined as in (1.3) and (2.1). Then the following hold:

- (i)  $R_\psi$  is convex if and only if  $\psi$  is convex.
- (ii)  $R_\psi$  is differentiable at  $\mathbf{X} \in \mathbb{C}^{M \times N}$ , and its gradient is given by

$$(2.10) \quad \nabla R_\psi(\mathbf{X}) = \mathbf{U} \text{Diag}\{\dot{\psi}(\boldsymbol{\sigma}(\mathbf{X}))\} \mathbf{V}',$$

where  $\mathbf{U}$  and  $\mathbf{V}$  are suitably sized unitary matrices such that  $\mathbf{X} = \mathbf{U} \text{Diag}\{\boldsymbol{\sigma}(\mathbf{X})\} \mathbf{V}'$ .

In (2.10),  $\dot{\psi}(\boldsymbol{\sigma}(\mathbf{X}))$  denotes the vector formed by applying the function  $\dot{\psi}$  elementwise to the vector  $\boldsymbol{\sigma}(\mathbf{X})$ , i.e.,  $\dot{\psi}(\boldsymbol{\sigma}(\mathbf{X})) = [\dot{\psi}(\sigma_1(\mathbf{X})), \dots, \dot{\psi}(\sigma_r(\mathbf{X}))]^T$ . This notation arises in the Julia language [2], and we use it throughout.

**Remark 2.5.** The differentiability of  $R_\psi$  is independent of its convexity, so we do not have two cases, as Proposition 2.3 might suggest. This is because  $f_\psi$  is always locally Lipschitz as  $\nabla f_\psi$  is Lipschitz continuous, which happens even when  $\psi$  is nonconvex. This follows from the fact that  $\dot{\psi}$  is Lipschitz continuous (cf. (2.7)). Appendix A provides more details in the proof of Theorem 2.4.

Another important property to study is the Lipschitz continuity of  $\nabla R_\psi(\mathbf{X})$ . Specifically, we want to determine conditions under which  $\nabla R_\psi$  is Lipschitz continuous. By following the same principles in Theorem 2.4, the next proposition shows that these conditions are directly related to the Lipschitz properties of  $\dot{\psi}$ . Appendix C gives a proof for this proposition based on the results in section 3.

**Proposition 2.6 (Lipschitz continuity of the gradient of Huber-based low-rank regularizers).** Let  $R_\psi$  be a regularizer defined as in (1.3) and (2.1). Then  $\nabla R_\psi$  is  $L$ -Lipschitz continuous where  $L = \omega_\psi(0)$ , i.e.,

$$(2.11) \quad \|\nabla R_\psi(\mathbf{X}) - \nabla R_\psi(\mathbf{Y})\|_F \leq \omega_\psi(0) \|\mathbf{X} - \mathbf{Y}\|_F \quad \forall \mathbf{X}, \mathbf{Y} \in \mathbb{C}^{M \times N}.$$

The properties shown in Theorem 2.4 and Proposition 2.6 suffice to enable standard iterative gradient-based optimization methods for solving (1.4) and (1.5) as shown in section 5; however, the fast convergence of these optimization methods highly depends on the adopted step-size selection rule. The next section proposes a new class of majorizers for Huber-based low-rank regularizers. These majorizers are then used to propose efficient step-size selection strategies that enable the fast convergence of iterative gradient-based optimization methods when using Huber-based low-rank regularizers.

**3. Majorizers for Huber-based low-rank regularizers and their associated line search functions.** The previous section showed that many relevant properties of the potential function  $\psi$  are transferred to the Huber-based low-rank regularizer  $R_\psi$ . Following this principle, and given that quadratic majorizers can be defined for  $\psi$  as shown in (2.4), this section derives novel majorizers for  $R_\psi$ . Inspired by the work of Lewis and Sendov in [37], which derives a quadratic expansion to approximate spectral functions defined on real symmetric matrices, here we provide a general expression for majorizers for Huber-based low-rank regularizers. Our results are directly stated in the more general case where Huber-based low-rank regularizers are defined on complex rectangular matrices.

A majorizer for  $R_\psi$  at  $\mathbf{S} \in \mathbb{C}^{M \times N}$ , denoted by  $J(\cdot; \mathbf{S})$ , corresponds to a matrix-valued function that satisfies

$$(3.1) \quad R_\psi(\mathbf{S}) = J(\mathbf{S}; \mathbf{S}),$$

$$(3.2) \quad R_\psi(\mathbf{X}) \leq J(\mathbf{X}; \mathbf{S}) \quad \forall \mathbf{X} \in \mathbb{C}^{M \times N}.$$

We show below that (3.1) and (3.2) are satisfied for functions with the form

$$(3.3) \quad Q(\mathbf{X}; \mathbf{S}, G_\psi) = R_\psi(\mathbf{S}) + \text{real}\{\langle \nabla R_\psi(\mathbf{S}), \mathbf{X} - \mathbf{S} \rangle\} + \frac{1}{2}(\mathbf{1}'_M (G_\psi(\sigma(\mathbf{S}))) \odot |\mathbf{U}'(\mathbf{X} - \mathbf{S})\mathbf{V}|.^2 \mathbf{1}_N),$$

where  $\mathbf{1}_K$  denotes a vector of length  $K$  with each entry equal to 1;  $\odot$  denotes an entry-wise (Hadamard) product;  $\mathbf{U}$  and  $\mathbf{V}$  are suitably sized unitary matrices such that  $\mathbf{S} = \mathbf{U} \text{Diag}\{\sigma(\mathbf{S})\} \mathbf{V}'$ ; and  $G_\psi: \mathbb{R}^r \rightarrow \mathbb{C}^{M \times N}$  is an operator that depends on  $\psi$  that constructs a matrix out of the entries of its input. Finally, the notation  $|\mathbf{A}|.^2$  for  $\mathbf{A} \in \mathbb{C}^{M \times N}$  corresponds to a matrix in  $\mathbb{R}^{M \times N}$  whose entries are the entries of  $\mathbf{A}$  after taking the absolute value and square operations entrywise. If  $\mathbf{B} = |\mathbf{A}|.^2$ , then  $B_{kl} = |A_{kl}|^2$ . The derivation for the expression in (3.3) was inspired on the results found in [37], where the authors derive a quadratic expansion to approximate spectral functions. In their case, the third term in the expansion involves an entrywise multiplication using specific factors that depend on first derivatives. For our case, we made the observation that these factors could be modified such that  $Q(\mathbf{X}; \mathbf{S}, G_\psi)$  could be a majorizer for  $R_\psi$ . These factors compose the entries of the matrix given by the output of  $G_\psi$ . In the following, we show that different choices for  $G_\psi$  provide different majorizers for  $R_\psi$ , that is,  $Q(\mathbf{X}; \mathbf{S}, G_\psi)$  satisfy (3.1) and (3.2). Particularly, we show two options for the operator  $G_\psi$ . The first operator is defined in terms of the weighting function  $\omega_\psi$  and is denoted by  $G_\psi^W$ . If  $\mathbf{s} \in \mathbb{R}^r$  and  $M < N$  (wide case),  $G_\psi^W$  is defined as

$$(3.4) \quad G_\psi^W(\mathbf{s}) \triangleq (\omega_\psi(\mathbf{s})) \mathbf{1}'_N,$$

and when  $M > N$  (tall case),

$$(3.5) \quad G_\psi^W(\mathbf{s}) \triangleq \mathbf{1}_M (\omega_\psi(\mathbf{s}))'.$$

The second operator is defined in terms of the Lipschitz constant of the gradient of  $R_\psi$  (cf. Proposition 2.6) and is denoted by  $G_\psi^L$ . For both the wide and the tall cases, this operator is defined as

$$(3.6) \quad G_\psi^L(\mathbf{s}) \triangleq \omega_\psi(0) \mathbf{1}_M \mathbf{1}'_N.$$

The following theorem states that  $Q(\cdot; \mathbf{S}, G_\psi^W)$  and  $Q(\cdot; \mathbf{S}, G_\psi^L)$  are majorizers for  $R_\psi$  at  $\mathbf{S} \in \mathbb{C}^{M \times N}$ , and it also provides equivalent expressions for both functions. Appendix B provides its proof.

**Theorem 3.1 (majorizers for Huber-based low-rank regularizers).** Let  $\mathbf{S} \in \mathbb{C}^{M \times N}$ , and let  $\mathbf{U}, \mathbf{V}$  be suitably-sized unitary matrices such that  $\mathbf{S} = \mathbf{U} \text{Diag}\{\boldsymbol{\sigma}(\mathbf{S})\} \mathbf{V}'$ . Then  $Q(\cdot; \mathbf{S}, G_\psi^W)$  and  $Q(\cdot; \mathbf{S}, G_\psi^L)$  are both majorizers for  $R_\psi$  at  $\mathbf{S} \in \mathbb{C}^{M \times N}$ ; i.e., both satisfy (3.1) and (3.2). If  $M < N$  (wide case), then

$$(3.7) \quad Q(\mathbf{X}; \mathbf{S}, G_\psi^W) = R_\psi(\mathbf{S}) + \text{real}\{\langle \nabla R_\psi(\mathbf{S}), \mathbf{X} - \mathbf{S} \rangle\} + \frac{1}{2} \|\mathbf{W}'_{\psi, \mathbf{S}}(\mathbf{X} - \mathbf{S})\|_F^2,$$

where the matrix  $\mathbf{W}_{\psi, \mathbf{S}}$  is defined as

$$(3.8) \quad \mathbf{W}_{\psi, \mathbf{S}} \triangleq \mathbf{U} \text{Diag} \left\{ \sqrt{\omega_{\psi, \cdot}(\boldsymbol{\sigma}(\mathbf{s}))} \right\} \mathbf{V} \in \mathbb{C}^{M \times N},$$

where the square root is applied elementwise. If  $M > N$  (tall case), then

$$(3.9) \quad Q(\mathbf{X}; \mathbf{S}, G_\psi^W) = R_\psi(\mathbf{S}) + \text{real}\{\langle \nabla R_\psi(\mathbf{S}), \mathbf{X} - \mathbf{S} \rangle\} + \frac{1}{2} \|(\mathbf{X} - \mathbf{S}) \mathbf{T}'_{\psi, \mathbf{S}}\|_F^2,$$

where the matrix  $\mathbf{T}_{\psi, \mathbf{S}}$  is defined as

$$(3.10) \quad \mathbf{T}_{\psi, \mathbf{S}} \triangleq \mathbf{U}' \text{Diag} \left\{ \sqrt{\omega_{\psi, \cdot}(\boldsymbol{\sigma}(\mathbf{s}))} \right\} \mathbf{V}' \in \mathbb{C}^{M \times N}.$$

For the wide and tall cases, we have that

$$(3.11) \quad Q(\mathbf{X}; \mathbf{S}, G_\psi^L) = R_\psi(\mathbf{S}) + \text{real}\{\langle \nabla R_\psi(\mathbf{S}), \mathbf{X} - \mathbf{S} \rangle\} + \frac{1}{2} \omega_\psi(0) \|\mathbf{X} - \mathbf{S}\|_F^2.$$

In the following we use Theorem 3.1 to derive quadratic majorizers for the 1D line search function associated with the proposed regularizer. The construction of these quadratic majorizers is one of our main results because it provides a simple step-size selection strategy that ensures monotonic descent in the optimization framework proposed in section 5.

Using Theorem 3.1, we show in the following that quadratic majorizers can be constructed for the regularizer 1D line search function  $h_\psi: \mathbb{R} \rightarrow \mathbb{R}$  defined as

$$(3.12) \quad h_\psi(\alpha; \mathbf{X}, \boldsymbol{\Delta}) \triangleq R_\psi(\mathbf{X} + \alpha \boldsymbol{\Delta}),$$

where  $\alpha \in \mathbb{R}$  and  $\mathbf{X}, \boldsymbol{\Delta} \in \mathbb{C}^{M \times N}$  are two arbitrary (fixed) matrices. The following theorem provides a general form for these quadratic majorizers. Appendix D provides its proof using Theorem 3.1.

**Theorem 3.2.** Let  $G_\psi: \mathbb{R}^r \rightarrow \mathbb{C}^{M \times N}$  be an operator such that  $Q(\cdot; \mathbf{X} + \bar{\alpha} \boldsymbol{\Delta}, G_\psi)$ , defined as in (3.3), is a majorizer for  $R_\psi$  at  $\mathbf{X} + \bar{\alpha} \boldsymbol{\Delta}$  where  $\bar{\alpha} \in \mathbb{R}$ . Define the quadratic function

$$(3.13) \quad g_{h_\psi}(\alpha; \bar{\alpha}, \mathbf{X}, \boldsymbol{\Delta}, G_\psi) \triangleq c_{h_\psi}^{(0)}(\bar{\alpha}, \mathbf{X}, \boldsymbol{\Delta}) + c_{h_\psi}^{(1)}(\bar{\alpha}, \mathbf{X}, \boldsymbol{\Delta})(\alpha - \bar{\alpha}) + \frac{1}{2} c_{h_\psi}^{(2)}(\bar{\alpha}, \mathbf{X}, \boldsymbol{\Delta}, G_\psi)(\alpha - \bar{\alpha})^2$$

with coefficients given by

$$(3.14) \quad c_{h_\psi}^{(0)}(\bar{\alpha}, \mathbf{X}, \boldsymbol{\Delta}) = R_\psi(\mathbf{X} + \bar{\alpha} \boldsymbol{\Delta}) = h_\psi(\bar{\alpha}; \mathbf{X}, \boldsymbol{\Delta}),$$

$$(3.15) \quad c_{h_\psi}^{(1)}(\bar{\alpha}, \mathbf{X}, \boldsymbol{\Delta}) = \text{real}\{\langle \nabla R_\psi(\mathbf{X} + \bar{\alpha} \boldsymbol{\Delta}), \boldsymbol{\Delta} \rangle\},$$

$$(3.16) \quad c_{h_\psi}^{(2)}(\bar{\alpha}, \mathbf{X}, \boldsymbol{\Delta}, G_\psi) = \mathbf{1}'_M (G_\psi(\boldsymbol{\sigma}(\mathbf{X} + \bar{\alpha} \boldsymbol{\Delta})) \odot |\mathbf{U}' \boldsymbol{\Delta} \mathbf{V}|.^{\wedge 2}) \mathbf{1}_N,$$

where  $\mathbf{U}, \mathbf{V}$  are suitably sized unitary matrices such that  $\mathbf{X} + \bar{\alpha}\mathbf{\Delta} = \mathbf{U} \text{Diag}\{\boldsymbol{\sigma}(\mathbf{X} + \bar{\alpha}\mathbf{\Delta})\}\mathbf{V}'$ . Then  $g_{h_\psi}(\alpha; \bar{\alpha}, \mathbf{X}, \mathbf{\Delta}, G_\psi)$  is a quadratic majorizer for  $h_\psi(\alpha; \mathbf{X}, \mathbf{\Delta})$  at  $\bar{\alpha}$ .

Combining the results of Theorem 3.1 and Theorem 3.2 yields the following corollary, which can guide the construction of quadratic majorizers for  $h_\psi$  in practice.

**Corollary 3.3.**  $g_{h_\psi}(\alpha; \bar{\alpha}, \mathbf{X}, \mathbf{\Delta}, G_\psi^W)$  and  $g_{h_\psi}(\alpha; \bar{\alpha}, \mathbf{X}, \mathbf{\Delta}, G_\psi^L)$  are both quadratic majorizers for  $h_\psi(\alpha; \mathbf{X}, \mathbf{\Delta})$  at  $\bar{\alpha}$ , and

$$g_{h_\psi}(\alpha; \bar{\alpha}, \mathbf{X}, \mathbf{\Delta}, G_\psi^W) \leq g_{h_\psi}(\alpha; \bar{\alpha}, \mathbf{X}, \mathbf{\Delta}, G_\psi^L) \forall \alpha \in \mathbb{R}.$$

Appendix E provides the proof of this corollary.

Before showing how to use the previous results in the proposed optimization framework in section 5, the following section shows that all these previous results extend to the proposed weighted Huber-based low-rank regularizers defined in (1.6).

**4. Weighted Huber-based low-rank regularizers.** The regularizer  $R_{\psi, \mathbf{w}}$  defined in (1.6) is unitarily invariant because it can be written as

$$(4.1) \quad R_{\psi, \mathbf{w}}(\mathbf{X}) = (f_{\psi, \mathbf{w}} \circ \boldsymbol{\sigma})(\mathbf{X}),$$

where  $f_{\psi, \mathbf{w}}: \mathbb{R}^r \rightarrow \mathbb{R}$  is defined as the absolutely symmetric function

$$(4.2) \quad f_{\psi, \mathbf{w}}(\mathbf{x}) \triangleq \sum_{k=1}^r w_k \psi([\tilde{\mathbf{x}}]_k).$$

Then, according to Proposition 2.3, we can assess the convexity and differentiability of  $R_{\psi, \mathbf{w}}$  by studying  $f_{\psi, \mathbf{w}}$ . To simplify the following analysis, we assume that  $\psi$  is not a constant and that  $\mathbf{w}$  is nonzero. Both of these assumptions are reasonable in practice, as a constant potential function or having all weights equal to zero would not properly promote low-rank characteristics when constructing  $R_{\psi, \mathbf{w}}$ . Under these assumptions we begin studying the convexity of  $f_{\psi, \mathbf{w}}$ , which depends on the weights  $\mathbf{w}$  and the potential function  $\psi$  as shown in the following proposition.

**Proposition 4.1.**  $f_{\psi, \mathbf{w}}$  is convex if and only if the Huber potential  $\psi$  is convex and  $w_1 \geq w_2 \geq \dots \geq w_r \geq 0$ .

Appendix F provides the proof. The following proposition provides conditions for the differentiability of  $f_{\psi, \mathbf{w}}$  at vectors that are relevant in our analysis. Interestingly, these conditions do not depend on the vector  $\mathbf{w}$ .

**Proposition 4.2.** Let  $\mathbf{x} \in \mathbb{R}^r$  be a vector whose entries are unique, nonnegative, and arranged in decreasing order, i.e.,  $\mathbf{x} = \tilde{\mathbf{x}}$  and  $x_k > x_{k+1} \forall k \in \{1, \dots, r-1\}$ . Then  $f_{\psi, \mathbf{w}}$  is locally Lipschitz and differentiable at  $\mathbf{x}$  with its gradient given by

$$(4.3) \quad \nabla f_{\psi, \mathbf{w}}(\mathbf{x}) = \mathbf{w} \odot \dot{\psi}(\mathbf{x}).$$

Appendix G provides the proof.

The previous results allow us to establish conditions for the convexity and differentiability of  $R_{\psi, \mathbf{w}}$ . We summarize these results in the following theorem, which is provided without proof as it is a direct application of Proposition 2.3 using Proposition 4.1 and Proposition 4.2.

**Theorem 4.3** (convexity and differentiability of weighted Huber-based low-rank regularizers).

- (i)  $R_{\psi, \mathbf{w}}$  is convex if and only if  $\psi$  is convex and  $w_1 \geq w_2 \geq \dots \geq w_r \geq 0$ .
- (ii) If  $\mathbf{X} \in \mathbb{C}^{M \times N}$  has no repeated singular values, i.e.,  $\sigma_1(\mathbf{X}) > \sigma_2(\mathbf{X}) > \dots > \sigma_r(\mathbf{X}) \geq 0$ , then  $R_{\psi, \mathbf{w}}$  is differentiable at  $\mathbf{X}$  and

$$(4.4) \quad \nabla R_{\psi, \mathbf{w}}(\mathbf{X}) = \mathbf{U} \text{Diag}\{\mathbf{w} \odot \dot{\psi}(\boldsymbol{\sigma}(\mathbf{X}))\} \mathbf{V}',$$

where  $\mathbf{U}$  and  $\mathbf{V}$  are suitably sized unitary matrices such that  $\mathbf{X} = \mathbf{U} \text{Diag}\{\boldsymbol{\sigma}(\mathbf{X})\} \mathbf{V}'$ .

**Remark 4.4.** In applications we might like low-rank regularizers to penalize small singular values more than large singular values, as the former are usually associated with noise. This preference would suggest using weights  $\mathbf{w}$  whose components are in nondecreasing order, i.e.,  $0 \leq w_1 \leq w_2 \leq \dots \leq w_r$ . Under this setting, Theorem 4.3 shows that  $R_{\psi, \mathbf{w}}$  would be nonconvex, which can complicate optimization. We show empirically in the following sections that in this nonconvex setting  $R_{\psi, \mathbf{w}}$  can outperform its nonweighted version  $R_\psi$  when a good initialization is provided.

**Remark 4.5.** Theorem 4.3 provides a closed-form expression for  $\nabla R_{\psi, \mathbf{w}}(\mathbf{X})$ , whose existence is based on the assumption that  $\mathbf{X}$  does not have repeated singular values. This allows us to use Proposition 2.3 with Proposition 4.2; in the case of repeated singular values, the function  $f_{\psi, \mathbf{w}}$  would not necessarily be differentiable at  $\boldsymbol{\sigma}(\mathbf{X})$ . This condition might seem limiting, as low-rank matrices present repeated singular values. Methods based on the nuclear norm involve thresholding operations that lead to “exactly” zero singular values. In contrast, our smooth approach will shrink singular values towards zero, but never thresholds them exactly to zero, so repeated zero singular values are extremely unlikely. In addition, if the gradient  $\nabla R_{\psi, \mathbf{w}}(\mathbf{X})$  (as given in (4.4)) is calculated by computing a singular value decomposition (SVD) of  $\mathbf{X}$  using a computational solver, as is usually done in real applications, it is highly unlikely to obtain repeated singular values due to the presence of noise and numerical precision errors. Therefore, hereafter we assume that the sufficient condition of Theorem 4.3 corresponding to having no repeated singular values is satisfied when calculating the gradient of  $R_{\psi, \mathbf{w}}$ .

Under the setting of nondecreasing weights it is also possible to construct majorizers for  $R_{\psi, \mathbf{w}}$  that are analogous to the ones provided for  $R_\psi$  in (3.3). If  $\mathbf{w} \in \mathbb{R}^r$  such that  $0 \leq w_1 \leq w_2 \leq \dots \leq w_r$ , then these majorizers have the form

$$(4.5) \quad \begin{aligned} Q_{\mathbf{w}}(\mathbf{X}; \mathbf{S}, G_{\psi, \mathbf{w}}) &= R_{\psi, \mathbf{w}}(\mathbf{S}) + \text{real}\{\langle \nabla R_{\psi, \mathbf{w}}(\mathbf{S}), \mathbf{X} - \mathbf{S} \rangle\} \\ &\quad + \frac{1}{2}(\mathbf{1}'_M (G_{\psi, \mathbf{w}}(\boldsymbol{\sigma}(\mathbf{S})) \odot |\mathbf{U}'(\mathbf{X} - \mathbf{S})\mathbf{V}|.^2) \mathbf{1}_N), \end{aligned}$$

where  $\mathbf{U}, \mathbf{V}$  are suitably sized unitary matrices such that  $\mathbf{S} = \mathbf{U} \text{Diag}\{\boldsymbol{\sigma}(\mathbf{S})\} \mathbf{V}'$  and  $G_{\psi, \mathbf{w}} : \mathbb{R}^r \rightarrow \mathbb{C}^{M \times N}$  is an operator that depends on  $\psi$  and  $\mathbf{w}$ . Appendix H shows that two operators denoted by  $G_{\psi, \mathbf{w}}^W$  and  $G_{\psi, \mathbf{w}}^L$  provide valid majorizers for  $R_{\psi, \mathbf{w}}$ . Specifically, we state an extended version of Theorem 3.1. Using this theorem we can show that the gradient of  $R_{\psi, \mathbf{w}}$  is Lipschitz continuous.

**Proposition 4.6** (Lipschitz continuity of the gradient of weighted Huber-based low-rank regularizers). For nonnegative and nondecreasing  $\mathbf{w} \in \mathbb{R}^r$ ,  $\nabla R_{\psi, \mathbf{w}}$  is  $L$ -Lipschitz continuous where

$L = \|\mathbf{w}\|_\infty \omega_\psi(0)$ , i.e.,

$$(4.6) \quad \|\nabla R_{\psi, \mathbf{w}}(\mathbf{X}) - \nabla R_{\psi, \mathbf{w}}(\mathbf{Y})\|_F \leq \|\mathbf{w}\|_\infty \omega_\psi(0) \|\mathbf{X} - \mathbf{Y}\|_F \quad \forall \mathbf{X}, \mathbf{Y} \in \mathbb{C}^{M \times N}.$$

Appendix I provides the proof.

Using the previous results it is also possible to construct quadratic majorizers for the 1D function  $h_{\psi, \mathbf{w}}(\alpha; \mathbf{X}, \Delta) \triangleq R_{\psi, \mathbf{w}}(\mathbf{X} + \alpha \Delta)$ . The procedure is analogous to the one described in Theorem 3.2 and is not provided for the sake of space.

One important case is when the vector of weights is defined such that the first  $K$  entries are equal to zero and the rest are equal to one. Denoting this vector by  $\mathbf{w}_K$ , it follows that

$$(4.7) \quad [\mathbf{w}_K]_k \triangleq \begin{cases} 0 & \text{if } k \leq K, \\ 1 & \text{otherwise,} \end{cases}$$

where  $K \in \{0, \dots, r-1\}$ . If  $K$  is a good estimate of the true rank of  $\mathbf{X}$ , then  $R_{\psi, \mathbf{w}_K}$  will not penalize the first  $K$  singular values, as desired. It would only penalize the tail singular values. We call this version of the weighted regularizer the *tail Huber-based low-rank regularizer*:

$$(4.8) \quad R_{\psi, K}(\mathbf{X}) \triangleq R_{\psi, \mathbf{w}_K}(\mathbf{X}) = \sum_{k=K+1}^r \psi(\sigma_k(\mathbf{X})).$$

This regularizer is nonconvex, given that the entries of  $\mathbf{w}_K$  are in nondecreasing order (see Theorem 4.3 part (i)). Nevertheless, we can calculate its gradient using Theorem 4.3 part (ii) and also construct majorizers for it using Theorem H.1. Therefore, it is possible to construct quadratic majorizers for the associated 1D line search function using the procedure in Theorem 3.2.

At this point we have derived a closed-form expression for the gradients of the proposed regularizers and also provided quadratic majorizers for the associated 1D line search functions. Section 5 leverages these results to solve optimization problems (1.4) and (1.5) using standard iterative gradient-based optimization methods.

**5. Smooth optimization framework using Huber-based low-rank regularizers.** Instead of promoting low-rank models using the (nonsmooth) nuclear norm in (1.1) and (1.2), we propose to use smooth Huber-based low-rank regularizers,<sup>5</sup> i.e.,  $R_\psi$  as defined in (1.3), to minimize (1.4) and (1.5).

Since (1.5) corresponds to a more general model than<sup>6</sup> (1.4), we focus on the local low-rank optimization problem. To use iterative gradient-based optimization algorithms to minimize  $\Psi_{\text{Local}}$ , we start by calculating its gradient by using Theorem 2.4. This gradient is simply given by

$$(5.1) \quad \nabla \Psi_{\text{Local}}(\mathbf{X}) = \nabla f(\mathbf{X}) + \beta \sum_{s \in \Lambda} \sum_{p \in \Gamma} \mathcal{S}_s^*(\mathcal{P}_p^*(\nabla R_\psi(\mathcal{P}_p(\mathcal{S}_s(\mathbf{X}))))),$$

<sup>5</sup>The following analysis can be easily extended to weighted Huber-based low-rank regularizers. We omit those derivations for the sake of notation simplicity.

<sup>6</sup>We can recover (1.4) from (1.5) by selecting  $\mathcal{P}_p$  and  $\mathcal{S}_s$  as the identity operator and scaling  $\beta$  according to the cardinality of  $\Lambda$  and  $\Gamma$ .

where  $\nabla R_\psi(\mathcal{P}_p(\mathcal{S}_s(\mathbf{X})))$  is calculated using (2.10), and  $\mathcal{S}_s^*$  and  $\mathcal{P}_p^*$  correspond to the adjoint operators of  $\mathcal{S}_s$  and  $\mathcal{P}_p$ , respectively. This expression enables the minimization of  $\Psi_{\text{Local}}$  using gradient-based optimization algorithms like NCG or quasi-Newton methods; however, the fast convergence of these iterative algorithms highly depends on a step-size selection strategy that guarantees a monotonic decrease of  $\Psi_{\text{Local}}$  in each iteration. The ideal step size could be determined by minimizing the line search function:

$$(5.2) \quad h(\alpha; \mathbf{X}, \mathbf{\Delta}) \triangleq \Psi_{\text{Local}}(\mathbf{X} + \alpha \mathbf{\Delta}),$$

where  $\mathbf{X}, \mathbf{\Delta} \in \mathbb{C}^{M \times N}$  are two arbitrary (fixed) matrices.

For notation simplicity, we define the matrices  $\mathbf{X}^{(p,s)} \triangleq \mathcal{P}_p(\mathcal{S}_s(\mathbf{X}))$  and  $\mathbf{\Delta}^{(p,s)} \triangleq \mathcal{P}_p(\mathcal{S}_s(\mathbf{\Delta}))$ , where  $(p, s) \in \Gamma \times \Lambda$ , and we also define the 1D functions

$$(5.3) \quad h_f(\alpha; \mathbf{X}, \mathbf{\Delta}) \triangleq f(\mathbf{X} + \alpha \mathbf{\Delta}),$$

$$(5.4) \quad h_{\psi, \text{Local}}(\alpha; \mathbf{X}, \mathbf{\Delta}) \triangleq \sum_{s \in \Lambda} \sum_{p \in \Gamma} h_\psi(\alpha; \mathbf{X}^{(p,s)}, \mathbf{\Delta}^{(p,s)}),$$

where  $h_\psi$  was defined in (3.12). Then we can express the line search function  $h$  as follows:

$$\begin{aligned} h(\alpha; \mathbf{X}, \mathbf{\Delta}) &= \Psi_{\text{Local}}(\mathbf{X} + \alpha \mathbf{\Delta}) \\ &= f(\mathbf{X} + \alpha \mathbf{\Delta}) + \beta \sum_{s \in \Lambda} \sum_{p \in \Gamma} R_\psi(\mathcal{P}_p(\mathcal{S}_s(\mathbf{X} + \alpha \mathbf{\Delta}))) \\ &= f(\mathbf{X} + \alpha \mathbf{\Delta}) + \beta \sum_{s \in \Lambda} \sum_{p \in \Gamma} R_\psi(\mathcal{P}_p(\mathcal{S}_s(\mathbf{X})) + \alpha \mathcal{P}_p(\mathcal{S}_s(\mathbf{\Delta}))) \\ &= h_f(\alpha; \mathbf{X}, \mathbf{\Delta}) + \beta \sum_{s \in \Lambda} \sum_{p \in \Gamma} h_\psi(\alpha; \mathbf{X}^{(p,s)}, \mathbf{\Delta}^{(p,s)}) \\ (5.5) \quad &= h_f(\alpha; \mathbf{X}, \mathbf{\Delta}) + \beta h_{\psi, \text{Local}}(\alpha; \mathbf{X}, \mathbf{\Delta}), \end{aligned}$$

where the third equality follows from the linearity of  $\mathcal{P}_p$  and  $\mathcal{S}_s$ . Thus, (5.5) reveals that  $h(\alpha; \mathbf{X}, \mathbf{\Delta})$  can be minimized by addressing the minimization of the functions  $h_f(\alpha; \mathbf{X}, \mathbf{\Delta})$  and  $h_{\psi, \text{Local}}(\alpha; \mathbf{X}, \mathbf{\Delta})$ . Our strategy for this purpose is based on using quadratic majorizers for each of these 1D functions. In many inverse problems (e.g., dynamic MRI reconstruction),  $f$  has a quadratic structure and therefore it is straightforward to derive a quadratic majorizer for  $h_f(\alpha; \mathbf{X}, \mathbf{\Delta})$ . Then for the following derivations we assume that there exists a quadratic function defined as

$$(5.6) \quad g_f(\alpha; \bar{\alpha}, \mathbf{X}, \mathbf{\Delta}) \triangleq c_f^{(0)}(\bar{\alpha}, \mathbf{X}, \mathbf{\Delta}) + c_f^{(1)}(\bar{\alpha}, \mathbf{X}, \mathbf{\Delta})(\alpha - \bar{\alpha}) + \frac{1}{2} c_f^{(2)}(\bar{\alpha}, \mathbf{X}, \mathbf{\Delta})(\alpha - \bar{\alpha})^2,$$

that is, a majorizer for  $h_f(\alpha; \mathbf{X}, \mathbf{\Delta})$  at  $\bar{\alpha} \in \mathbb{R}$ , where  $\{c_f^{(l)}(\bar{\alpha}, \mathbf{X}, \mathbf{\Delta})\}_{l=0}^2$  are three scalar coefficients that depend on  $\bar{\alpha}$ ,  $\mathbf{X}$ , and  $\mathbf{\Delta}$ . We use Theorem 3.2 and Corollary 3.3 to construct a quadratic majorizer for  $h_{\psi, \text{Local}}(\alpha; \mathbf{X}, \mathbf{\Delta})$ . The following theorem provides this construction.

**Theorem 5.1.** Let  $G_\psi$  be an operator<sup>7</sup> such that  $Q(\cdot; \mathbf{X}^{(\mathbf{p}, \mathbf{s})} + \bar{\alpha} \Delta^{(\mathbf{p}, \mathbf{s})}, G_\psi)$ , defined as in (3.3), is a majorizer for  $R_\psi$  at  $\mathbf{X}^{(\mathbf{p}, \mathbf{s})} + \bar{\alpha} \Delta^{(\mathbf{p}, \mathbf{s})} \forall (\mathbf{p}, \mathbf{s}) \in \Gamma \times \Lambda$ . Then the quadratic function

$$(5.7) \quad \begin{aligned} g_{h_\psi, \text{Local}}(\alpha; \bar{\alpha}, \mathbf{X}, \Delta, G_\psi) &\triangleq c_{h_\psi, \text{Local}}^{(0)}(\bar{\alpha}, \mathbf{X}, \Delta) + c_{h_\psi, \text{Local}}^{(1)}(\bar{\alpha}, \mathbf{X}, \Delta)(\alpha - \bar{\alpha}) \\ &\quad + \frac{1}{2} c_{h_\psi, \text{Local}}^{(2)}(\bar{\alpha}, \mathbf{X}, \Delta, G_\psi)(\alpha - \bar{\alpha})^2 \end{aligned}$$

with coefficients given by

$$(5.8) \quad \begin{aligned} c_{h_\psi, \text{Local}}^{(0)}(\bar{\alpha}, \mathbf{X}, \Delta) &= \sum_{\mathbf{s} \in \Lambda} \sum_{\mathbf{p} \in \Gamma} c_{h_\psi}^{(0)}(\bar{\alpha}, \mathbf{X}^{(\mathbf{p}, \mathbf{s})}, \Delta^{(\mathbf{p}, \mathbf{s})}), \\ c_{h_\psi, \text{Local}}^{(1)}(\bar{\alpha}, \mathbf{X}, \Delta) &= \sum_{\mathbf{s} \in \Lambda} \sum_{\mathbf{p} \in \Gamma} c_{h_\psi}^{(1)}(\bar{\alpha}, \mathbf{X}^{(\mathbf{p}, \mathbf{s})}, \Delta^{(\mathbf{p}, \mathbf{s})}), \end{aligned}$$

$$(5.9) \quad c_{h_\psi, \text{Local}}^{(2)}(\bar{\alpha}, \mathbf{X}, \Delta, G_\psi) = \sum_{\mathbf{s} \in \Lambda} \sum_{\mathbf{p} \in \Gamma} c_{h_\psi}^{(2)}(\bar{\alpha}, \mathbf{X}^{(\mathbf{p}, \mathbf{s})}, \Delta^{(\mathbf{p}, \mathbf{s})}, G_\psi)$$

is a majorizer for  $h_{\psi, \text{Local}}(\alpha; \mathbf{X}, \Delta)$  at  $\bar{\alpha}$ .

*Proof.* It follows directly from using (3.13) to provide a quadratic majorizer for each term in the sum on the right-hand side of (5.4). ■

Given that we have constructed quadratic majorizers for  $h_f$  and  $h_{\psi, \text{Local}}$ , the following corollary of Theorem 5.1 derives a quadratic majorizer for the line search function  $h$ .

**Corollary 5.2.** Let  $G_\psi$  be an operator defined as in Theorem 5.1. Then the quadratic function

$$(5.10) \quad \begin{aligned} g_h(\alpha; \bar{\alpha}, \mathbf{X}, \Delta, G_\psi) &\triangleq g_{h_f}(\alpha; \bar{\alpha}, \mathbf{X}, \Delta) + \beta g_{h_\psi, \text{Local}}(\alpha; \bar{\alpha}, \mathbf{X}, \Delta, G_\psi) \\ &= c_h^{(0)}(\bar{\alpha}, \mathbf{X}, \Delta) + c_h^{(1)}(\bar{\alpha}, \mathbf{X}, \Delta)(\alpha - \bar{\alpha}) \end{aligned}$$

$$(5.11) \quad + \frac{1}{2} c_h^{(2)}(\bar{\alpha}, \mathbf{X}, \Delta, G_\psi)(\alpha - \bar{\alpha})^2$$

with coefficients given by

$$(5.12) \quad \begin{aligned} c_h^{(0)}(\bar{\alpha}, \mathbf{X}, \Delta) &= c_f^{(0)}(\bar{\alpha}, \mathbf{X}, \Delta) + \beta c_{h_\psi, \text{Local}}^{(0)}(\bar{\alpha}, \mathbf{X}, \Delta), \\ c_h^{(1)}(\bar{\alpha}, \mathbf{X}, \Delta) &= c_f^{(1)}(\bar{\alpha}, \mathbf{X}, \Delta) + \beta c_{h_\psi, \text{Local}}^{(1)}(\bar{\alpha}, \mathbf{X}, \Delta), \end{aligned}$$

$$(5.13) \quad c_h^{(2)}(\bar{\alpha}, \mathbf{X}, \Delta, G_\psi) = c_f^{(2)}(\bar{\alpha}, \mathbf{X}, \Delta) + \beta c_{h_\psi, \text{Local}}^{(2)}(\bar{\alpha}, \mathbf{X}, \Delta, G_\psi)$$

is a majorizer for  $h(\alpha; \mathbf{X}, \Delta)$  at  $\bar{\alpha}$ .

**Remark 5.3.** The existence of a majorizer for the line search function  $h$  given by Corollary 5.2 enables the minimization of  $h$  using a simple iterative majorize-minimize (MM) approach [16, § IV]. Given an initial guess  $\alpha_0$ , it is straightforward to show that the iterative step

<sup>7</sup>The input and output dimensions of  $G_\psi$  are different in (3.13) and (5.7). We have adopted the same notation in both cases for simplicity, as the dimensions should be clear from the context.

$$(5.14) \quad \alpha_{k+1} \triangleq \arg \min_{\alpha \in \mathbb{R}} g_h(\alpha; \alpha_k, \mathbf{X}, \Delta, G_\psi) = \alpha_k - \frac{c_h^{(1)}(\alpha_k, \mathbf{X}, \Delta)}{c_h^{(2)}(\alpha_k, \mathbf{X}, \Delta, G_\psi)}$$

decreases  $h(\alpha; \mathbf{X}, \Delta)$  monotonically, i.e.,  $h(\alpha_{k+1}; \mathbf{X}, \Delta) \leq h(\alpha_k; \mathbf{X}, \Delta)$ . Let  $n_\alpha$  denote the number of iterations of (5.14) that we use to descend  $h(\alpha; \mathbf{X}, \Delta)$ .

To apply the MM step-size selection method in (5.14), one must calculate  $c_{h_\psi, \text{Local}}^{(1)}$  and  $c_{h_\psi, \text{Local}}^{(2)}$  as given in (5.8) and (5.9), respectively, which can be computationally demanding because one SVD should be performed for each pair  $(\mathbf{s}, \mathbf{p}) \in \Lambda \times \Gamma$ . In the following, we provide a heuristic fast strategy for this calculation.

Instead of considering every pair  $(\mathbf{s}, \mathbf{p}) \in \Lambda \times \Gamma$  for calculating  $c_{h_\psi, \text{Local}}^{(1)}$  and  $c_{h_\psi, \text{Local}}^{(2)}$ , we have empirically observed that it suffices to use only the pairs associated with a single shift index, denoted by  $\bar{\mathbf{s}} \in \Lambda$ , that we typically set to  $\mathbf{s} = \mathbf{0}$  for simplicity. For our fast variant, we approximate the majorizer  $g_{h_\psi, \text{Local}}(\alpha; \bar{\alpha}, \mathbf{X}, \Delta, G_\psi)$  by the quadratic function

$$(5.15) \quad \bar{g}_{h_\psi, \text{Local}}(\alpha; \bar{\alpha}, \mathbf{X}, \Delta, G_\psi) \triangleq \bar{c}_{h_\psi, \text{Local}}^{(0)}(\bar{\alpha}, \mathbf{X}, \Delta) + \bar{c}_{h_\psi, \text{Local}}^{(1)}(\bar{\alpha}, \mathbf{X}, \Delta)(\alpha - \bar{\alpha}) + \frac{1}{2} \bar{c}_{h_\psi, \text{Local}}^{(2)}(\bar{\alpha}, \mathbf{X}, \Delta, G_\psi)(\alpha - \bar{\alpha})^2$$

with coefficients given by

$$\begin{aligned} \bar{c}_{h_\psi, \text{Local}}^{(0)}(\bar{\alpha}, \mathbf{X}, \Delta) &= |\Lambda| \sum_{\mathbf{p} \in \Gamma} c_{h_\psi}^{(0)}(\bar{\alpha}, \mathbf{X}^{(\mathbf{p}, \bar{\mathbf{s}})}, \Delta^{(\mathbf{p}, \bar{\mathbf{s}})}), \\ \bar{c}_{h_\psi, \text{Local}}^{(1)}(\bar{\alpha}, \mathbf{X}, \Delta) &= |\Lambda| \sum_{\mathbf{p} \in \Gamma} c_{h_\psi}^{(1)}(\bar{\alpha}, \mathbf{X}^{(\mathbf{p}, \bar{\mathbf{s}})}, \Delta^{(\mathbf{p}, \bar{\mathbf{s}})}), \\ \bar{c}_{h_\psi, \text{Local}}^{(2)}(\bar{\alpha}, \mathbf{X}, \Delta, G_\psi) &= |\Lambda| \sum_{\mathbf{p} \in \Gamma} c_{h_\psi}^{(2)}(\bar{\alpha}, \mathbf{X}^{(\mathbf{p}, \bar{\mathbf{s}})}, \Delta^{(\mathbf{p}, \bar{\mathbf{s}})}, G_\psi). \end{aligned}$$

This subsequently provides an approximation for the majorizer  $g_h(\alpha; \bar{\alpha}, \mathbf{X}, \Delta, G_\psi)$  given by the quadratic function

$$\begin{aligned} \bar{g}_h(\alpha; \bar{\alpha}, \mathbf{X}, \Delta, G_\psi) &\triangleq g_{h_f}(\alpha; \bar{\alpha}, \mathbf{X}, \Delta) + \beta \bar{g}_{h_\psi, \text{Local}}(\alpha; \bar{\alpha}, \mathbf{X}, \Delta, G_\psi) \\ &= \bar{c}_h^{(0)}(\bar{\alpha}, \mathbf{X}, \Delta) + \bar{c}_h^{(1)}(\bar{\alpha}, \mathbf{X}, \Delta)(\alpha - \bar{\alpha}) \\ &\quad + \frac{1}{2} \bar{c}_h^{(2)}(\bar{\alpha}, \mathbf{X}, \Delta, G_\psi)(\alpha - \bar{\alpha})^2 \end{aligned}$$

with coefficients given by

$$\begin{aligned} \bar{c}_h^{(0)}(\bar{\alpha}, \mathbf{X}, \Delta) &= c_f^{(0)}(\bar{\alpha}, \mathbf{X}, \Delta) + \beta \bar{c}_{h_\psi, \text{Local}}^{(0)}(\bar{\alpha}, \mathbf{X}, \Delta), \\ \bar{c}_h^{(1)}(\bar{\alpha}, \mathbf{X}, \Delta) &= c_f^{(1)}(\bar{\alpha}, \mathbf{X}, \Delta) + \beta \bar{c}_{h_\psi, \text{Local}}^{(1)}(\bar{\alpha}, \mathbf{X}, \Delta), \\ \bar{c}_h^{(2)}(\bar{\alpha}, \mathbf{X}, \Delta, G_\psi) &= c_f^{(2)}(\bar{\alpha}, \mathbf{X}, \Delta) + \beta \bar{c}_{h_\psi, \text{Local}}^{(2)}(\bar{\alpha}, \mathbf{X}, \Delta, G_\psi). \end{aligned}$$

Then, instead of using (5.14), we use the iterative step

$$(5.16) \quad \alpha_{k+1} \triangleq \arg \min_{\alpha \in \mathbb{R}} \bar{g}_h(\alpha; \alpha_k, \mathbf{X}, \Delta, G_\psi) = \alpha_k - \frac{\bar{c}_h^{(1)}(\alpha_k, \mathbf{X}, \Delta)}{\bar{c}_h^{(2)}(\alpha_k, \mathbf{X}, \Delta, G_\psi)}.$$

---

**Algorithm 5.1.** Nonlinear conjugate gradient algorithm for solving (1.5).

---

**Require:**  $\mathbf{X}_0 \in \mathbb{C}^{M \times N}$  (initial estimation),  $\alpha_0$  (initial step size),  $n_\alpha \in \mathbb{N}$  (number of iterations for the MM step-size selection strategy),  $M_{\text{iter}} \in \mathbb{N}$  (number of iterations),  $\eta \in \mathbb{R}$  (convergence threshold)

- 1:  $\mathbf{G}_0 = \nabla \Psi_{\text{Local}}(\mathbf{X}_0)$  (initial gradient calculation using (5.1) and (2.10))
- 2:  $\Delta_0 = -\mathbf{G}_0$  (initial search direction)
- 3: **for**  $k = 0, 1, \dots, M_{\text{iter}} - 1$  **do**
- 4:    $\alpha_k \leftarrow \alpha_0$
- 5:   **for**  $l = 1, \dots, n_\alpha$  **do** (MM step-size selection)
- 6:      $\alpha_k \leftarrow \alpha_k - \frac{c_h^{(1)}(\alpha_k, \mathbf{X}_k, \Delta_k)}{c_h^{(2)}(\alpha_k, \mathbf{X}_k, \Delta_k, G_\psi)}$  (using (5.12) and (5.13))
- 7:   **end for**
- 8:    $\mathbf{X}_{k+1} \leftarrow \mathbf{X}_k + \alpha_k \Delta_k$  (estimation update)
- 9:    $\mathbf{G}_{k+1} \leftarrow \nabla \Psi_{\text{Local}}(\mathbf{X}_{k+1})$  (gradient calculation using (5.1) and (2.10))
- 10:    $\beta_k \leftarrow \|\mathbf{G}_{k+1}\|_{\text{F}}^2 / \|\mathbf{G}_k\|_{\text{F}}^2$  (conjugate gradient parameter update using Fletcher–Reeves rule)
- 11:    $\Delta_{k+1} \leftarrow -\mathbf{G}_{k+1} + \beta_k \Delta_k$  (search direction update)
- 12:   **if**  $\|\mathbf{G}_{k+1}\|_{\text{F}} < \eta$  **then**
- 13:     **break**
- 14:   **end if**
- 15: **end for**

---

This heuristic approach is substantially faster than the one in (5.14), as only one SVD is performed instead of  $|\Lambda||\Gamma|$ ; however, a step size produced with this method is not guaranteed to decrease  $h(\alpha; \mathbf{X}, \Delta)$  monotonically, as  $\bar{g}_h(\alpha; \bar{\alpha}, \mathbf{X}, \Delta, G_\psi)$  is not necessarily a majorizer for  $h(\alpha; \mathbf{X}, \Delta)$ . We have empirically seen that  $\bar{g}_h(\alpha; \bar{\alpha}, \mathbf{X}, \Delta, G_\psi)$  resembles  $g_h(\alpha; \bar{\alpha}, \mathbf{X}, \Delta, G_\psi)$  quite well and does not considerably affect performance when estimating  $\mathbf{X}$ . Intuitively, the good performance of this method could be attributed to some isotropy of the local low-rank regularizer, in the sense that every shift  $\mathbf{s} \in \Lambda$  contributes similarly to the cost function.

The gradient formula in (5.1) and the step-size iterative selection strategy in (5.14) allow the minimization of  $\Psi_{\text{Local}}$  using standard iterative gradient-based optimization algorithms. In the following section, we empirically illustrate this procedure in the context of dynamic MRI reconstruction using the NCG algorithm; corresponding pseudocode is provided in Algorithm 5.1.

**6. Application in dynamic MRI reconstruction using local low-rank models.** In dynamic MRI applications it is typical to acquire a limited amount of samples to accelerate the acquisition, as data collection is inherently slow in MRI. Overall, the main goal is to reconstruct a series of images from a partial set of samples that have also been corrupted by noise. In the following, we assume that these images are 2D, although our analysis can be easily extended to higher dimensions. Mathematically, the data-collection process is usually modeled as

$$(6.1) \quad \mathbf{Y} = \mathcal{A}(\mathbf{X}) + \mathbf{E} \in \mathbb{C}^{S \times N},$$

where  $\mathbf{X} = [\mathbf{x}_1 \ \mathbf{x}_2 \ \cdots \ \mathbf{x}_N] \in \mathbb{C}^{M \times N}$  is a matrix whose columns  $\{\mathbf{x}_n\}_{n=1}^N$  are the (vectorized) images to be recovered;  $\mathbf{Y} = [\mathbf{y}_1 \ \mathbf{y}_2 \ \cdots \ \mathbf{y}_N] \in \mathbb{C}^{S \times N}$  is a matrix whose columns  $\{\mathbf{y}_n\}_{n=1}^N$  are the observed data samples in the Fourier domain (a.k.a.  $k$ -space) for each image;  $\mathbf{E}$  is a matrix that represents the noise in the observations;<sup>8</sup> and  $\mathcal{A}: \mathbb{C}^{M \times N} \rightarrow \mathbb{C}^{S \times N}$  is the linear forward operator that represents the MRI acquisition process.<sup>9</sup> The acquisition of a limited amount of samples is modeled by including an undersampling matrix in  $\mathcal{A}$ . This makes recovering  $\mathbf{X}$  from  $\mathbf{Y}$  an ill-posed inverse problem. One option to recover  $\mathbf{X}$  is by minimizing a data-consistency cost function of the form

$$(6.2) \quad f(\mathbf{X}) \triangleq \frac{1}{2} \|\mathcal{A}(\mathbf{X}) - \mathbf{Y}\|_{\text{F}}^2,$$

which can be done using well-known least-squares solvers; however, additional regularizers are needed in cases when data are heavily undersampled. In the following, we study how to estimate  $\mathbf{X}$  when local low-rank regularizers are added by using the proposed smooth optimization framework from section 5. We start by describing how the local low-rank model introduced in (1.2) and (1.5) is applied in the context of dynamic MRI reconstruction.

The underlying assumption for the use of local low-rank models is that patches extracted from the same spatial locations across images should be similar. More specifically, if  $\mathbf{p} \in \mathbb{Z}^2$  corresponds to the spatial location of one specific voxel, it is then possible to extract from each image the values in a neighborhood around  $\mathbf{p}$  corresponding to a patch of dimensions  $n_x \times n_y$ . In our examples, we used rectangular neighborhoods and we denote by  $P = n_x n_y$  the number of voxels in each of them. Another possibility is to use ellipsoidal neighborhoods [43]. Then low-rank characteristics are expected for the matrix constructed by concatenating the vectorized patches corresponding to  $\mathbf{p}$  from the different images. This matrix has a Casorati structure, and we denote it by  $\mathcal{P}_{\mathbf{p}}(\mathbf{X})$ , where  $\mathcal{P}_{\mathbf{p}}: \mathbb{C}^{M \times N} \rightarrow \mathbb{C}^{P \times N}$  is the linear operator that performs the Casorati matrix construction for the patch corresponding to  $\mathbf{p}$ . The adjoint operator  $\mathcal{P}_{\mathbf{p}}^*: \mathbb{C}^{P \times N} \rightarrow \mathbb{C}^{M \times N}$  constructs a matrix that is zero everywhere, except for the entries corresponding to the patch  $\mathbf{p}$ , which are filled according to the entries of the input. Figure 2 illustrates how  $\mathcal{P}_{\mathbf{p}}$  and  $\mathcal{P}_{\mathbf{p}}^*$  perform their respective matrix constructions for a series of three images. Local low-rank models aim to leverage on the low-rank characteristics of the Casorati matrices constructed from multiple patches at different spatial locations. As in (1.2) and (1.5), we denote the set of spatial locations by  $\Gamma$ , and in our experiments we design it such that the patches corresponding to different spatial locations jointly cover the whole voxel grid while not overlapping with each other. However, one important consideration for local low-rank models to properly work is that the patches should overlap to avoid blocky artifacts in the reconstructed images [57]. One way to impose this is by considering the patch extraction process for circularly shifted versions of the image. If  $\mathbf{s} \in \mathbb{Z}^2$  denotes a shift index, then it is expected that the matrix  $\mathcal{P}_{\mathbf{p}}(\mathcal{S}_{\mathbf{s}}(\mathbf{X}))$  would possess low-rank characteristics, where  $\mathcal{S}_{\mathbf{s}}: \mathbb{C}^{M \times N} \rightarrow \mathbb{C}^{M \times N}$  is the linear operator that performs the circular shift by  $\mathbf{s}$ . The adjoint operator  $\mathcal{S}_{\mathbf{s}}^*: \mathbb{C}^{M \times N} \rightarrow \mathbb{C}^{M \times N}$  performs a circular shift by  $-\mathbf{s}$ . The number of shifts that we

<sup>8</sup>The entries in  $\mathbf{E}$  are usually modeled as independent and identically distributed variables following a white Gaussian complex distribution [40].

<sup>9</sup>See [11] for a detailed explanation of  $\mathcal{A}$ . In general terms,  $\mathcal{A}$  involves multiplication with coil sensitivities, Fourier transformation, and an undersampling process.

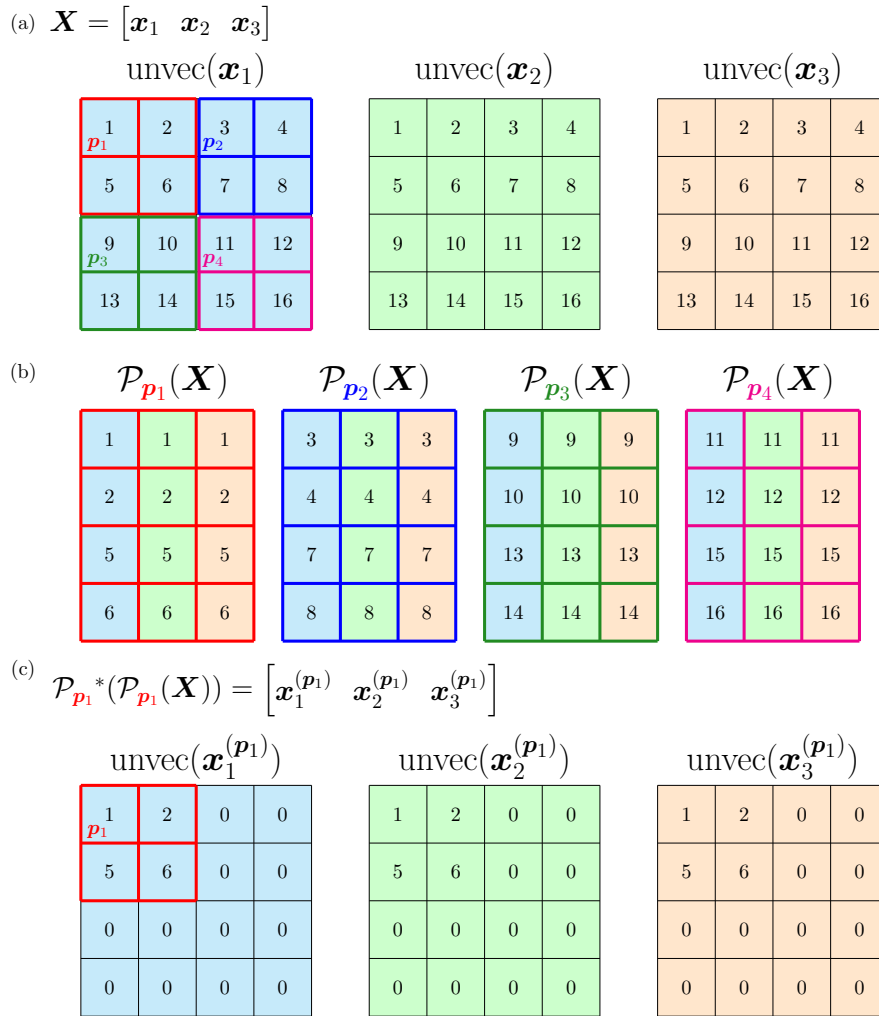


**Figure 2.** Illustration of the operators  $\mathcal{P}_{\mathbf{p}}$  and  $\mathcal{P}_{\mathbf{p}}^*$  for a series of  $N = 3$  2D-images with  $M = 16$  voxels each. (a) shows each image (i.e., the columns of  $\mathbf{X}$ ) as a 2D grid after unvectorizing the corresponding column in  $\mathbf{X}$ ; the value of each voxel is indicated in black at the center of each square of the grid. Each image has been filled out with a different color (light blue, light green, and light orange) to identify how the voxels are taken from each image once the Casorati matrices are constructed. The set of locations  $\Gamma = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\}$  has been indicated in the first image using a color code; the same locations are used in the other two images. (b) shows the Casorati matrices created after extracting the patches corresponding to the locations in  $\Gamma$ . A grid with the same color of the location index has been used for each Casorati matrix. This should help the reader to identify which voxels were taken from each image to construct the Casorati matrices, as the same grid color has been used in (a) to indicate the neighborhood of voxels that forms the patch corresponding to each location. In this example the dimensions of each patch were  $n_x \times n_y = 2 \times 2$  (i.e.,  $P = 4$ ). (c) shows an example of how  $\mathcal{P}_{\mathbf{p}}^*$  performs its matrix construction. The columns of  $\mathcal{P}_{\mathbf{p}_1}^*(\mathcal{P}_{\mathbf{p}_1}(\mathbf{X}))$ , denoted by  $\{\mathbf{x}_n^{(\mathbf{p}_1)}\}_{n=1}^3$ , are shown after unvectorization.

considered depended on the number of voxels in each patch. If each patch is considered as a 2D grid, then each location in the grid provides a shift index. Let  $\Lambda$  denote the set of shifts that also includes  $\mathbf{s} = [0, 0]^T$  (i.e., no shift), so that  $|\Lambda| = n_x n_y$  if  $n_x$  and  $n_y$  are both even, which is the case considered in our experiments. More specifically,

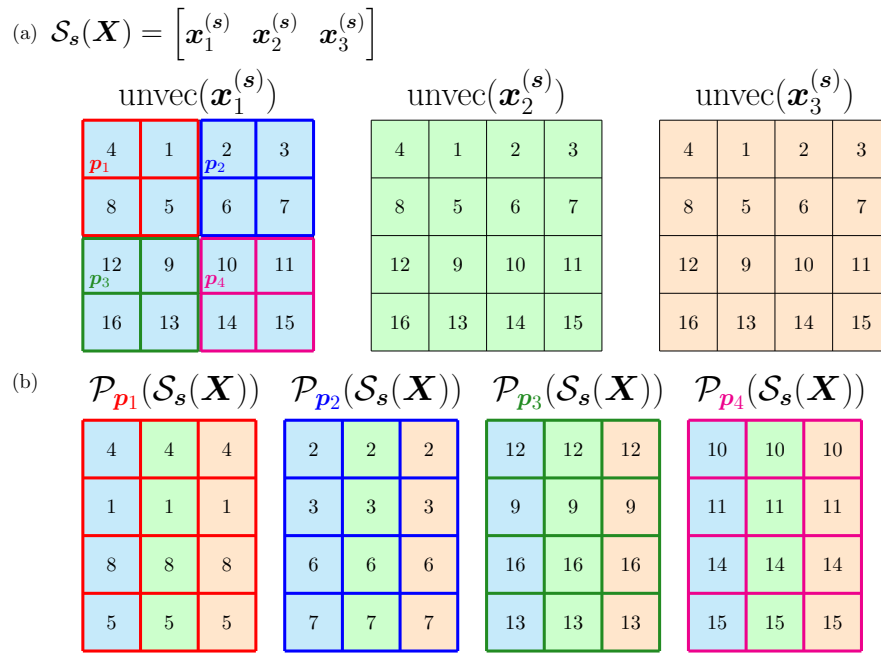


**Figure 3.** Illustration of the shifting operator using the same images and color code used in Figure 2. (a) shows how each column of  $\mathbf{X}$  is modified after applying the operator  $\mathcal{S}_s$  when  $\mathbf{s} = [1, 0]^T$ . The modified columns are denoted by  $\{\mathbf{x}_n^{(s)}\}_{n=1}^3$  and correspond to the vectorized images after circularly shifting the images according to  $\mathbf{s}$ . The first and second entries of  $\mathbf{s}$  indicate how many positions each voxel should be shifted horizontally and vertically, respectively; in this example,  $\mathbf{s}$  indicates one position to the right and no vertical shifts. The total number of shift vectors depends on the number of voxels in each patch. In this case,  $\Lambda = \{[0, 0]^T, [0, 1]^T, [1, 0]^T, [1, 1]^T\}$  according to (6.3). (b) shows the Casorati matrices created after extracting the patches corresponding to the locations in  $\Gamma$  as done in Figure 2(b), after circularly shifting the images according to  $\mathbf{s}$ .

$$(6.3) \quad \Lambda \triangleq \left\{ \mathbf{s} = [s_x, s_y]^T; s_x \in \left[-\frac{n_x}{2} + 1, \frac{n_x}{2}\right], s_y \in \left[-\frac{n_y}{2} + 1, \frac{n_y}{2}\right] \right\}.$$

Figure 3 illustrates how the shifting operation is combined with patch extraction using the same toy example as in Figure 2. Under this setting, we propose to estimate  $\mathbf{X}$  by solving (1.5) using standard iterative gradient-based optimization algorithms by using the results in (5.1) and (5.14). As indicated in section 5, we first must calculate  $\nabla f(\mathbf{X})$  and a quadratic majorizer for  $h_f(\alpha; \mathbf{X}, \Delta)$ . Given the quadratic structure of  $f$  in (6.2), it follows that

$$(6.4) \quad \nabla f(\mathbf{X}) = \mathcal{A}^*(\mathcal{A}(\mathbf{X}) - \mathbf{Y})$$

and that the coefficients of the quadratic majorizer in (5.6) are given by

$$\begin{aligned} c_f^{(0)}(\bar{\alpha}, \mathbf{X}, \Delta) &= \frac{1}{2} \|\mathcal{A}(\mathbf{X} + \bar{\alpha}\Delta) - \mathbf{Y}\|_{\mathbb{F}}^2, \\ c_f^{(1)}(\bar{\alpha}, \mathbf{X}, \Delta) &= \text{real}\{\langle \mathcal{A}(\mathbf{X} + \bar{\alpha}\Delta) - \mathbf{Y}, \mathcal{A}(\Delta) \rangle\}, \\ c_f^{(2)}(\Delta) &= \|\mathcal{A}(\Delta)\|_{\mathbb{F}}^2. \end{aligned}$$

Using the gradient calculation results in (2.10), (5.1), and (6.4), in addition to the MM step-size selection strategy in (5.14) or its fast (heuristic) version in (5.16), we use a method like

NCG to solve (1.5). In the following, we show experiments using this optimization approach and dynamic MRI data.

**6.1. Experiments.** In this section we test the proposed optimization framework by reconstructing realistic retrospectively undersampled dynamic MRI data. We studied imposing local low-rank models using the Huber-based low-rank regularizer  $R_\psi$  and its tail version  $R_{\psi,K}$ . The standard iterative gradient-based algorithm used in our experiments was NCG, which was compared with ad-hoc heuristic versions of two proximal gradients methods: POGM and FISTA. For these two proximal gradient methods, the cost function was the one in (1.2). Since the local low-rank regularizer is not prox friendly, for both POGM and FISTA we *approximated* the proximal mapping for the regularizer using the average of the proximal mappings of each term inside the sum (a.k.a. a proximal average [19]).

**6.1.1. Data description and forward operator.** The data used in our experiments corresponded to a 2D cardiac perfusion dataset of size  $128 \times 128$  (i.e.,  $M = 16384$ ) with  $N = 40$  time frames [51]. In this case we have data from 12 coils; therefore, the forward operator  $\mathcal{A}$  includes multiplication with coil sensitivities, Fourier transformation, and undersampling in the  $k$ -space corresponding to each time frame. The multicoil  $k$ -space data corresponding to each time frame was undersampled to obtain a  $\times 7$ -accelerated dataset. The undersampling scheme guaranteed that each  $k$ -space sample was present at least once when considering all time frames.

**6.1.2. Optimization framework parameters.** To use the proposed optimization framework, we first needed a Huber potential function  $\psi$  such that  $R_\psi$  and  $R_{\psi,K}$  were able to impose low-rank models. We focused our analysis on the (convex) hyperbola function defined in (2.2) for  $\delta = 10^{-3}$ , as  $R_\psi$  and  $R_{\psi,K}$  in this case can be seen as smooth versions of the nuclear norm (cf. Figure 1(a)) and the regularizer corresponding to the sum of the tail singular values [49], respectively; however, we also provide some results using the (nonconvex) Cauchy potential function defined in (2.3). In all of our experiments we set  $K = 1$  for  $R_{\psi,K}$ . This value was chosen assuming that, for this type of data, the spatial region covered by each patch uniformly changes its intensity over time. Therefore, each column in the corresponding Casorati matrix of each patch is considered as a scaled version of any other column, which produces a rank equal to one. To use the MM step-size selection strategy in (5.14), we studied  $G_\psi = G_\psi^W$  and  $G_\psi = G_\psi^L$  when using  $R_\psi$ , as both provide valid quadratic majorizers according to Corollary 3.3. For  $R_{\psi,K}$ , we used the analogous operators  $G_{\psi,w} = G_{\psi,w_K}^W$  and  $G_{\psi,w} = G_{\psi,w_K}^L$  defined in Appendix H. We set the number of iterations for the MM step-size selection strategy to  $n_\alpha = 1$  and the initial step size as  $\alpha_0 = 0$ . In cases where the (heuristic) fast step-size selection strategy was used, we set  $\bar{s} = [0, 0]^T$ . The operator  $\mathcal{P}_p$  was designed to extract  $8 \times 8$  patches (i.e.,  $n_x = n_y = 8$ ), and we selected  $\Gamma$  such that the  $128 \times 128$  grid of voxels could be completely covered using nonoverlapping patches before applying the shifting operator. Thus,  $|\Gamma| = (128/n_x)(128/n_y) = 256$  in this example, and  $|\Lambda| = n_x n_y = 64$ . Each method was initialized using a data-sharing reconstruction [29] where each time frame was reconstructed by filling missing  $k$ -space samples using available  $k$ -space samples in neighboring time frames. Each method was run for 500 iterations and implemented in-house using MATLAB R2023a. All computations were performed on a MacBook Pro laptop computer with

an Apple M2 Pro chip, 12 cores, and 32 GB RAM. The authors have made an open-source implementation available at [https://github.com/ralobos/smooth\\_LLR](https://github.com/ralobos/smooth_LLR).

**6.1.3. Performance metrics.** To study the convergence rate of each method, we calculated the normalized distance to the limit

$$(6.5) \quad \frac{\|\mathbf{X}_k - \hat{\mathbf{X}}_\infty\|_F}{\|\hat{\mathbf{X}}_\infty\|_F}$$

at each iteration, where  $\mathbf{X}_k$  was the estimate at the  $k$ th iteration and  $\hat{\mathbf{X}}_\infty$  was the estimate after 500 iterations. To measure reconstruction quality, we calculated the normalized-root-mean-square-error (NRMSE) at each iteration defined as

$$(6.6) \quad \frac{\|\mathbf{X}_k - \mathbf{X}_{\text{GS}}\|_F}{\|\mathbf{X}_{\text{GS}}\|_F},$$

where  $\mathbf{X}_{\text{GS}}$  corresponded to the gold-standard reconstruction obtained after coil combination of the fully sampled data using a SENSE-based approach [52].

**6.2. Results.** Figure 4 shows the measures in (6.5) and (6.6) for each method. In Figure 4(a), NCG converged faster than the other two methods, using either  $R_\psi$  or  $R_{\psi,K}$ , and POGM converged faster than FISTA. The faster convergence of POGM compared to FISTA was also observed for global low-rank models [41]. NCG using  $R_\psi$  yielded NRMSE values similar to those in POGM, while taking fewer iterations for NCG to converge. NCG using  $R_{\psi,K}$  obtained a better NRMSE than POGM at the point of convergence, and converged the fastest. The convergence of NCG is theoretically ensured [20], whereas the convergence of POGM and FISTA and their points of convergence are unclear in this context because of the heuristic approximation of the proximal mapping in both cases. NCG using the tail regularizer

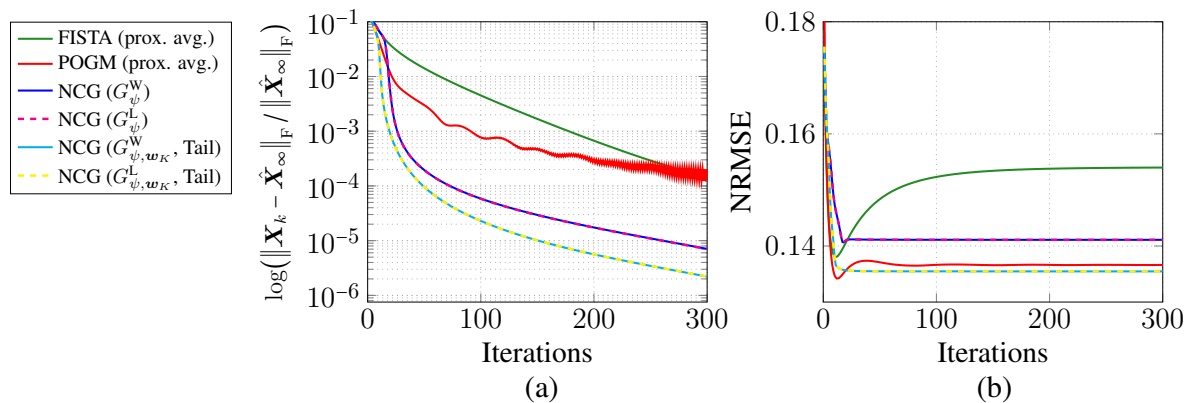


**Figure 4.** Convergence and NRMSE metrics for each method as functions of the number of iterations (only 300 of the 500 iterations are shown). In both plots, NCG using  $R_\psi$  corresponds to  $\text{NCG}(G_\psi^W)$  and  $\text{NCG}(G_\psi^L)$ , and NCG using  $R_{\psi,K}$  corresponds to  $\text{NCG}(G_{\psi,w_K}^W, \text{Tail})$  and  $\text{NCG}(G_{\psi,w_K}^L, \text{Tail})$ . The operator in parentheses is the one used for the MM step-size selection strategy in (5.14). For POGM and FISTA, we have indicated in parentheses that proximal average has been used as an approximation of the proximal mapping.

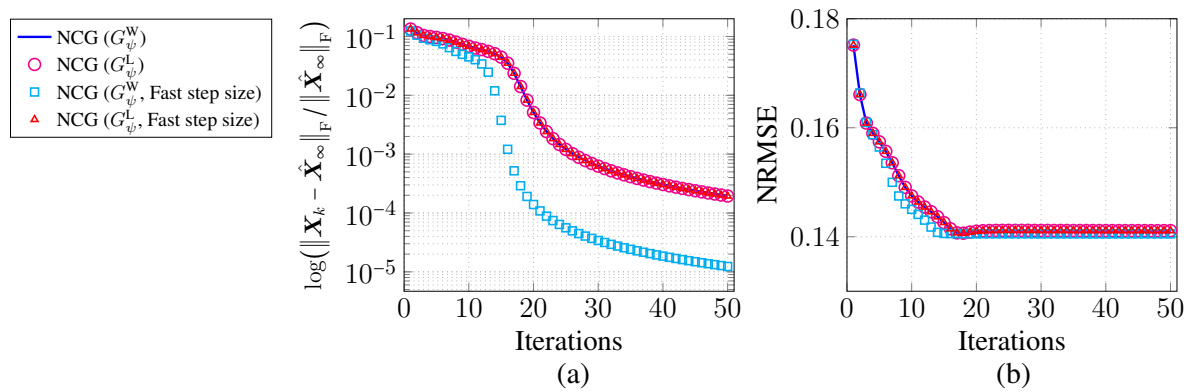


**Figure 5.** Convergence and NRMSE metrics for NCG using  $R_\psi$  with different MM step-size selection strategies as functions of the number of iterations (only 50 of the 500 iterations are shown). We indicate using the heuristic step-size selection strategy by writing “Fast step size”.

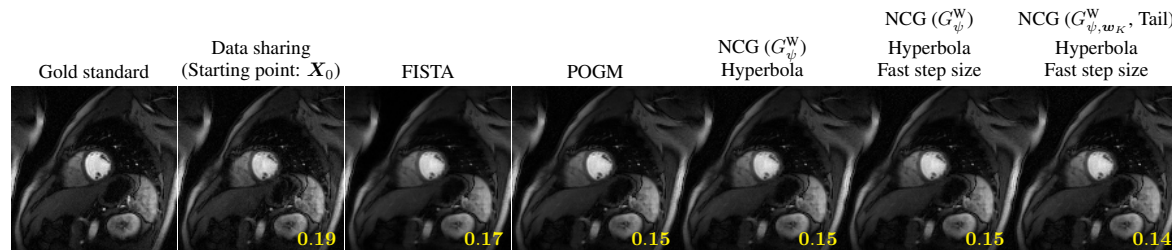


**Figure 6.** Reconstruction results for different methods using only 25 iterations. Only one image corresponding to one specific time frame is shown for each method, and the NRMSE for that particular image is indicated in yellow in the bottom right corner. For NCG using  $R_\psi$ , results are shown using both step-size selection strategies for a visual comparison between them. For NCG using  $R_{\psi,K}$ , only the reconstruction using the heuristic (fast) step-size selection strategy is shown, as no substantial differences were observed compared to using the nonheuristic version. We indicated use of the heuristic step-size selection strategy by writing “Fast step size”. Similar results were obtained for NCG when using  $G_\psi^L$  instead of  $G_\psi^W$  and  $G_{\psi,w_K}^L$  instead of  $G_{\psi,w_K}^W$  (not shown).

$R_{\psi,K}$  converged faster and obtained better image quality than NCG using the nonweighted regularizer  $R_\psi$ , as expected, because the tail regularizer exploits an a priori estimate of the true rank and penalizes only tail singular values.

Although NCG converged faster than POGM and FISTA, it was the most computationally expensive, as many SVD calculations were needed for the MM step-size selection strategy in (5.14). For this reason, we also explored the heuristic step-size selection strategy proposed in (5.16). Figure 5 shows NCG results using  $R_\psi$  and both step-size selection strategies. Neither convergence nor NRMSE were considerably affected when adopting the fast (heuristic) step-size selection strategy (similar results were obtained for NCG using  $R_{\psi,K}$ ). The heuristic step-size selection did not visibly affect the quality of the reconstructed images either. Figure 6 shows examples of the reconstructed images obtained with each method using only 25 iterations in each case. Using the heuristic (fast) step-size selection strategy did not cause visible changes. However, the fast step-size strategy significantly reduced NCG compute time, as shown in Table 1, which reports the median times in all cases after repeating

Table 1

Reconstruction time and NRMSE after 25 iterations.

| Method                                                    | Time [secs] | NRMSE        |
|-----------------------------------------------------------|-------------|--------------|
| POGM                                                      | <b>35.1</b> | 0.138        |
| NCG ( $G_{\psi}^W$ )                                      | 75.9        | 0.141        |
| NCG ( $G_{\psi, \mathbf{w}_K}^W$ , Tail)                  | 76          | <b>0.136</b> |
| NCG ( $G_{\psi}^W$ ) - Fast step size                     | 39          | 0.141        |
| NCG ( $G_{\psi, \mathbf{w}_K}^W$ , Tail) - Fast step size | 38.9        | <b>0.136</b> |

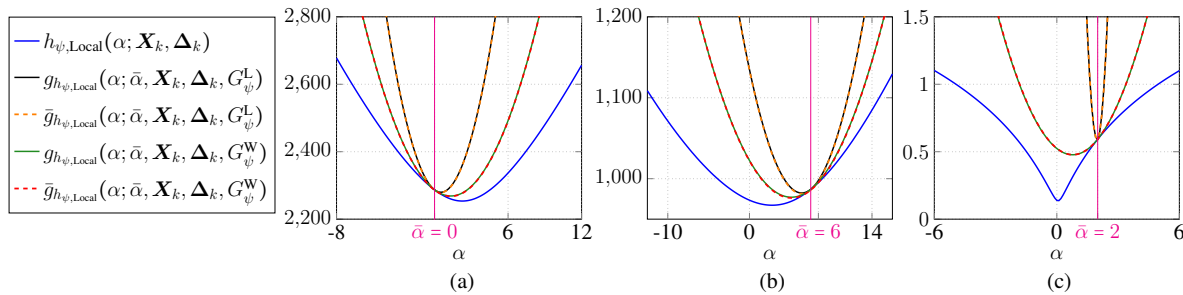


**Figure 7.** Illustration of the proposed quadratic majorizers for  $h_{\psi, \text{Local}}$  given in Theorem 5.1 when using (a)  $R_{\psi}$  with  $\psi$  as the hyperbola function, (b)  $R_{\psi, K}$  with  $\psi$  as the hyperbola function, and (c)  $R_{\psi}$  with  $\psi$  as the Cauchy function. We also show the approximation of each quadratic majorizer (dashed lines) as defined in (5.15). Different values of  $\bar{\alpha}$  are used in each case to illustrate different examples of our theoretical results. When using the tail regularizer, the function  $h_{\psi, \text{Local}}$  is defined in terms of  $R_{\psi, K}$ , and the quadratic majorizers and their approximations are calculated considering the operators  $G_{\psi, \mathbf{w}_K}^W$  and  $G_{\psi, \mathbf{w}_K}^L$  (cf. Appendix H).

each experiment five times. Best results are marked in bold. Only the results for  $G_{\psi}^W$  and  $G_{\psi, \mathbf{w}_K}^W$  are shown, as no substantial differences were observed when using  $G_{\psi}^L$  and  $G_{\psi, \mathbf{w}_K}^L$  instead. To further study the similarities between the two step-size selection strategies in terms of convergence and image quality, we calculated the quadratic functions  $g_{h_{\psi, \text{Local}}}$  and  $\bar{g}_{h_{\psi, \text{Local}}}$  for one particular NCG iteration using (5.7) and (5.15), respectively, as they represent the main difference between the two strategies. Denoting the estimation and search direction for this iteration as  $\mathbf{X}_k$  and  $\Delta_k$ , respectively, Figure 7(a) shows  $h_{\psi, \text{Local}}(\alpha; \mathbf{X}_k, \Delta_k)$  calculated for different values of  $\alpha$ :  $g_{h_{\psi, \text{Local}}}(\alpha; \bar{\alpha}, \mathbf{X}_k, \Delta_k, G_{\psi})$  and  $\bar{g}_{h_{\psi, \text{Local}}}(\alpha; \bar{\alpha}, \mathbf{X}_k, \Delta_k, G_{\psi})$  when  $\bar{\alpha} = 0$  and  $G_{\psi} = G_{\psi}^W$  or  $G_{\psi} = G_{\psi}^L$ . As indicated by our theoretical results in Theorem 5.1,  $g_{h_{\psi, \text{Local}}}(\alpha; \bar{\alpha}, \mathbf{X}_k, \Delta_k, G_{\psi})$  majorizes  $h_{\text{Local}}(\alpha; \mathbf{X}_k, \Delta_k)$  at every  $\alpha$  and they are equal at  $\bar{\alpha}$ , both for  $G_{\psi} = G_{\psi}^W$  or  $G_{\psi} = G_{\psi}^L$ . Notably,  $\bar{g}_{h_{\psi, \text{Local}}}(\alpha; \bar{\alpha}, \mathbf{X}_k, \Delta_k, G_{\psi})$  resembled  $g_{h_{\psi, \text{Local}}}(\alpha; \bar{\alpha}, \mathbf{X}_k, \Delta_k, G_{\psi})$  quite well for each choice of  $G_{\psi}$ , providing further explanation for why choosing the heuristic step-size selection strategy did not affect convergence or estimation quality. Analogous results are observed in Figure 7(b) when using the tail regularizer, although in this case the quadratic functions were calculated based on  $G_{\psi, \mathbf{w}_K}^W$  and  $G_{\psi, \mathbf{w}_K}^L$ . For completeness, Figure 7(c) shows an analogous analysis using the Cauchy function as the potential function for  $R_{\psi}$ . In this case,  $h_{\psi, \text{Local}}(\alpha; \mathbf{X}_k, \Delta_k)$  has a challenging non-convex behavior; however, it can still be minimized using the proposed strategy based on

quadratic majorizers. Figure 7(a) shows that  $G_\psi^W$  provided better majorizing characteristics than  $G_\psi^L$ , as the global minimum of  $h_{\psi, \text{Local}}(\alpha; \mathbf{X}_k, \mathbf{\Delta}_k)$  was closer to the global minimum of  $g_{h_{\psi, \text{Local}}}(\alpha; \bar{\alpha}, \mathbf{X}_k, \mathbf{\Delta}_k, G_\psi^W)$  than the global minimum of  $g_{h_{\psi, \text{Local}}}(\alpha; \bar{\alpha}, \mathbf{X}_k, \mathbf{\Delta}_k, G_\psi^L)$ . This is consistent with the theoretical results in Corollary 3.3. However, the results in Figure 4 and Figure 5 illustrate that the differences between the majorizers associated with  $G_\psi^W$  and  $G_\psi^L$  did not have a big impact when minimizing the cost function in (1.5). This can be attributed to the fact that the data-consistency term given by  $f$  had a bigger contribution in our experiments than the regularizer. As a consequence, the function  $g_f$  dominated over  $g_{h_{\psi, \text{Local}}}$  when calculating  $g_h$  in (5.10) to majorize the line search function  $h$ , which mitigated the effects of using  $G_\psi^L$  instead of  $G_\psi^W$ .

**7. Discussion and conclusion.** In this work we have proposed new classes of smooth regularizers to impose either global or local low-rank models in inverse problems. We have studied several properties of these regularizers using singular value function theory, establishing conditions for their convexity, differentiability, and Lipschitz continuity of the gradient. We have also provided a novel theory that enables the construction of majorizers for the proposed regularizers, which corresponds to one of our main theoretical contributions. Moreover, we have shown that these regularizers enable the use of standard iterative gradient-based optimization methods to solve inverse problems using low-rank models. For this purpose, we have shown how to calculate the regularizer gradient in closed form and also how to efficiently select the step size in each iteration using an MM strategy based on quadratic majorizers of the regularizer line search function.

Another point to highlight is the flexibility offered by the proposed theoretical framework, in the sense that a big family of low-rank regularizers can be analyzed using the same theoretical results. Many Huber-like potential functions have been proposed, and each of them would provide a regularizer according to our theory. Interestingly, some of these Huber-based low-rank regularizers can be considered smooth versions of commonly used nonsmooth low-rank regularizers. For example, the nuclear norm can be approximated by selecting a potential function like the hyperbola function that approximates the absolute value function, and non-convex Schatten- $p$  quasi norms with  $0 < p < 1$  can be approximated by selecting the potential function to be  $\psi(t) = (|t|^2 + \delta^2)^{p/2}$  for a small value of  $\delta$ . This same principle holds for the proposed smooth weighted Huber-based low-rank regularizers (cf. section 4). Remarkably, the proposed optimization framework enables the minimization of each Huber-based low-rank regularizer using the same iterative gradient-based optimization algorithm.

As an application, we showed that the proposed optimization framework based on smooth low-rank regularizers can be used to address the dynamic MRI reconstruction problem when using local low-rank models with overlapping patches. Particularly, we showed that NCG can be used to minimize the associated cost function. NCG showed a faster convergence and similar or better image quality than other heuristic ad-hoc proximal methods that use the nuclear norm as a regularizer. Moreover, methods like NCG provide well-known convergence guarantees [20] that are not available for proximal methods with approximate proximal maps in this context. In addition, we have observed (not shown) that quasi-Newton methods like L-BFGS can also be used instead of NCG, providing similar reconstruction quality.

As future work we would like to explore other applications for the proposed tail Huber-based low-rank regularizers studied in section 4. Interestingly, by selecting the Huber potential function as the hyperbola function, the proposed regularizer can be seen as a smooth version of the regularizer corresponding to the sum of the tail singular values, which has been used in robust PCA applications [56, 49]. Similarly, by selecting the Huber potential function as the parabola, this type of regularizer coincides with the regularizer proposed in [21], which has been extensively used in MRI reconstruction applications using structured low-rank models [3, 23, 32, 44]. The theory and optimization framework proposed in this work could allow a complementary analysis for these applications.

From a theoretical point of view, a relevant question is whether “better” majorizers can be constructed for the proposed regularizers. The majorizer  $Q(\mathbf{X}; \mathbf{S}, G_\psi)$  proposed in (3.3) depends on the operator  $G_\psi$ , and Theorem 3.1 shows that two choices for this operator provide valid majorizers. However, we have not studied if these majorizers satisfy any optimality criterion. Inspired by the criterion used to study the optimality of quadratic majorizers for Huber potential functions [26, p. 186], we would like to study the existence and construction of an operator  $G_\psi^*$  such that

- (i)  $Q(\mathbf{X}; \mathbf{S}, G_\psi^*)$  is a valid majorizer for  $R_\psi$ , and
- (ii)  $Q(\mathbf{X}; \mathbf{S}, G_\psi^*) \leq Q(\mathbf{X}; \mathbf{S}, G_\psi) \forall \mathbf{X} \in \mathbb{C}^{M \times N}$ , for any operator  $G_\psi$  that makes  $Q(\mathbf{X}; \mathbf{S}, G_\psi)$  a valid majorizer for  $R_\psi$ .

Under this criterion the operator  $G_\psi^W$  defined in (3.4) and (3.5) would be better than the operator  $G_\psi^L$  defined in (3.6); however, it is an open question if there is a better operator than  $G_\psi^W$ . Answering this is important, as it would allow us to define better quadratic majorizers for the regularizer line search function, and therefore it would improve our proposed majorize-minimize step-size selection strategy.

**Appendix A. Proof of Theorem 2.4.** We first prove (i). Suppose  $\psi$  is convex. Then, for  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^r$  and  $\alpha \in (0, 1)$ ,

$$\begin{aligned} f_\psi(\alpha \mathbf{x} + (1 - \alpha) \mathbf{y}) &= \sum_{k=1}^r \psi(\alpha x_k + (1 - \alpha) y_k) \\ &\leq \sum_{k=1}^r \alpha \psi(x_k) + (1 - \alpha) \psi(y_k) = \alpha f_\psi(\mathbf{x}) + (1 - \alpha) f_\psi(\mathbf{y}), \end{aligned}$$

using the convexity of  $\psi$  in the inequality. Thus,  $f_\psi$  is convex and hence  $R_\psi$  is convex by Proposition 2.3 part (i) using the equivalent formulation for  $R_\psi$  provided in (2.1).

Conversely, if  $R_\psi$  is convex, then Proposition 2.3 part (i) shows that  $f_\psi$  is convex. Because  $\psi(t) = \frac{1}{r} f_\psi(t \mathbf{1}_r)$ , where  $\mathbf{1}_r \in \mathbb{R}^r$  is the vector with each entry equal to 1, it follows that for  $\alpha \in (0, 1)$  and  $t, s \in \mathbb{R}$

$$\begin{aligned} \psi(\alpha t + (1 - \alpha)s) &= \frac{1}{r} f_\psi(\alpha t \mathbf{1}_r + (1 - \alpha)s \mathbf{1}_r) \\ &\leq \alpha \frac{1}{r} f_\psi(t \mathbf{1}_r) + (1 - \alpha) \frac{1}{r} f_\psi(s \mathbf{1}_r) = \alpha \psi(t) + (1 - \alpha) \psi(s), \end{aligned}$$

using the convexity of  $f_\psi$  in the inequality.

To prove (ii), we first calculate  $\nabla f_\psi(\sigma(\mathbf{X}))$ . By the definition of  $f_\psi$ , for any  $\mathbf{x} \in \mathbb{R}^r$ ,

$$\nabla f_\psi(\mathbf{x}) = [\dot{\psi}(x_1), \dots, \dot{\psi}(x_r)]^T = \dot{\psi}(\mathbf{x}),$$

as  $\psi$  is differentiable. Now we show that  $f_\psi$  is locally Lipschitz, which follows from showing that  $\nabla f_\psi$  is Lipschitz continuous. If  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^r$ , then

$$\begin{aligned} \|\nabla f_\psi(\mathbf{x}) - \nabla f_\psi(\mathbf{y})\|^2 &= \sum_{k=1}^r |\dot{\psi}(x_k) - \dot{\psi}(y_k)|^2 \\ &\leq \sum_{k=1}^r \omega_\psi(0)^2 |x_k - y_k|^2 = \omega_\psi^2(0) \|\mathbf{x} - \mathbf{y}\|^2, \end{aligned}$$

using the fact that  $\dot{\psi}$  is  $\omega_\psi(0)$ -Lipschitz continuous (cf. (2.7)) in the inequality. Therefore,  $\nabla f_\psi$  is  $\omega_\psi(0)$ -Lipschitz continuous, which implies that  $f_\psi$  is locally Lipschitz. The result (2.10) follows from Proposition 2.3 parts (ii) and (iii) using the equivalent formulation for  $R_\psi$  provided in (2.1).

**Appendix B. Proof of Theorem 3.1.** It is clear from their definitions that  $Q(\cdot; \mathbf{S}, G_\psi^W)$  and  $Q(\cdot; \mathbf{S}, G_\psi^L)$  both satisfy the equality (3.1). Next, we prove that  $Q(\cdot; \mathbf{S}, G_\psi^W)$  satisfies (3.2), which is the main part of the proof. We then prove that  $Q(\cdot; \mathbf{S}, G_\psi^L)$  satisfies (3.2) by showing that  $Q(\mathbf{X}; \mathbf{S}, G_\psi^W) \leq Q(\mathbf{X}; \mathbf{S}, G_\psi^L) \forall \mathbf{X} \in \mathbb{C}^{M \times N}$ . Without loss of generality, we assume that  $M < N$  (wide case).

A key step in our proof uses the following inequality to lower bound the trace of a product of two Hermitian matrices [25, Thm. 4.3.53].

**Proposition B.1 (lower and upper bounds for the trace of the product of two Hermitian matrices).** Let  $\mathbf{A} \in \mathbb{C}^{M \times M}$ ,  $\mathbf{B} \in \mathbb{C}^{M \times M}$  be two Hermitian matrices with eigenvalues  $\{\lambda_k(\mathbf{A})\}_{k=1}^M$  and  $\{\lambda_k(\mathbf{B})\}_{k=1}^M$ , respectively. Assuming both sets of eigenvalues are arranged in nonincreasing order, i.e.,  $\lambda_1(\mathbf{A}) \geq \lambda_2(\mathbf{A}) \geq \dots \geq \lambda_M(\mathbf{A})$  and  $\lambda_1(\mathbf{B}) \geq \lambda_2(\mathbf{B}) \geq \dots \geq \lambda_M(\mathbf{B})$ , it follows that

$$(B.1) \quad \sum_{k=1}^M \lambda_{M-k+1}(\mathbf{A}) \lambda_k(\mathbf{B}) \leq \text{trace}\{\mathbf{A}\mathbf{B}\} \leq \sum_{k=1}^M \lambda_k(\mathbf{A}) \lambda_k(\mathbf{B}).$$

Equipped with this inequality, we now continue our proof. We start by showing the inequality (3.2) when  $\mathbf{S} \in \mathbb{C}^{M \times N}$  is a rectangular diagonal matrix of the form  $\mathbf{S} = \text{Diag}(\mathbf{s})$ , where  $\mathbf{s} \in \mathbb{R}^M$  has nonnegative entries arranged in nonincreasing order (i.e.,  $\mathbf{s} = \tilde{\mathbf{s}}$ ). We then generalize the result to generic rectangular matrices by leveraging the unitary invariance of  $R_\psi$ .

An important result used in our proof is given in the following lemma.

**Lemma B.2.** Let  $t, s \in \mathbb{R}$ . Then

$$(B.2) \quad \psi(s) - \psi(t) + \frac{1}{2}\omega_\psi(s)(t^2 - s^2) \geq 0.$$

*Proof.* Because  $q_\psi$  is a majorizer for  $\psi$ ,

$$\begin{aligned} 0 \leq q_\psi(t; s) - \psi(t) &= \psi(s) - \psi(t) + \dot{\psi}(s)(t - s) + \frac{1}{2}\omega_\psi(s)(t - s)^2 \\ &= \psi(s) - \psi(t) + s\omega_\psi(s)(t - s) + \frac{1}{2}\omega_\psi(s)(t^2 - 2ts + s^2) \\ &= \psi(s) - \psi(t) + \frac{1}{2}\omega_\psi(s)(t^2 - s^2). \end{aligned}$$

Next, we extend Lemma B.2 to prove that  $Q(\mathbf{X}; \mathbf{S}, G_\psi^W) - R_\psi(\mathbf{X}) \geq 0 \forall \mathbf{X} \in \mathbb{C}^{M \times N}$  when  $\mathbf{S} = \text{Diag}\{\mathbf{s}\}$ . In this case we have that  $\mathbf{S} = \mathbf{U} \text{Diag}\{\boldsymbol{\sigma}(\mathbf{S})\} \mathbf{V}'$ , where  $\boldsymbol{\sigma}(\mathbf{S}) = \mathbf{s}$ ,  $\mathbf{U} = \mathbf{I}_M$ ,  $\mathbf{V} = \mathbf{I}_N$ , and  $\mathbf{I}_K$  denotes the  $K \times K$  identity matrix. This implies that  $\nabla R_\psi(\mathbf{S}) = \text{Diag}\{\dot{\psi}(\mathbf{s})\}$  by Theorem 2.4. It follows that

$$(B.3) \quad Q(\mathbf{X}; \mathbf{S}, G_\psi^W) = R_\psi(\text{Diag}\{\mathbf{s}\}) + \text{real}\left\{\left\langle \text{Diag}\{\dot{\psi}(\mathbf{s})\}, \mathbf{X} - \text{Diag}\{\mathbf{s}\} \right\rangle\right\}$$

$$(B.4) \quad + \frac{1}{2}(\mathbf{1}'_M (G_\psi^W(\mathbf{s}) \odot |\mathbf{X} - \text{Diag}\{\mathbf{s}\}|.^2) \mathbf{1}_N)$$

$$(B.5) \quad = \sum_{k=1}^M \psi(s_k) + \text{real}\left\{\sum_{k=1}^M \dot{\psi}(s_k)(X_{kk} - s_k)\right\}$$

$$\begin{aligned} &+ \frac{1}{2} \sum_{k=1}^M \omega_\psi(s_k) |X_{kk} - s_k|^2 + \frac{1}{2} \sum_{k=1}^M \omega_\psi(s_k) \sum_{\substack{l=1 \\ l \neq k}}^N |X_{kl}|^2 \\ &= \sum_{k=1}^M \psi(s_k) + s_k \omega_\psi(s_k) (\text{real}\{X_{kk}\} - s_k) \\ &\quad + \frac{1}{2} \omega_\psi(s_k) (|X_{kk}|^2 - 2s_k \text{real}\{X_{kk}\} + s_k^2) + \frac{1}{2} \sum_{\substack{l=1 \\ l \neq k}}^N \omega_\psi(s_k) |X_{kl}|^2 \\ &= \sum_{k=1}^M \psi(s_k) + \frac{1}{2} \omega_\psi(s_k) (|X_{kk}|^2 - s_k^2) + \frac{1}{2} \sum_{\substack{l=1 \\ l \neq k}}^N \omega_\psi(s_k) |X_{kl}|^2 \\ &= \left( \sum_{k=1}^M \psi(s_k) - \frac{1}{2} \omega_\psi(s_k) s_k^2 \right) + \frac{1}{2} \sum_{k=1}^M \omega_\psi(s_k) \sum_{l=1}^N |X_{kl}|^2. \end{aligned}$$

The next step is to derive a lower bound for the last sum of the previous equation in terms of the singular values of  $\mathbf{X}$ . For this we first write this term as the trace of the product of two Hermitian matrices and use (B.1) as follows:

$$\begin{aligned}
\sum_{k=1}^M \omega_{\psi}(s_k) \sum_{l=1}^N |X_{kl}|^2 &= \text{trace} \left\{ \left( \text{Diag}\{\sqrt{\omega_{\psi}(\cdot)(\mathbf{s})}\} \right)' \mathbf{X} \left( \left( \text{Diag}\{\sqrt{\omega_{\psi}(\cdot)(\mathbf{s})}\} \right)' \mathbf{X} \right)' \right\} \\
&= \text{trace} \left\{ \underbrace{\text{Diag}\{\sqrt{\omega_{\psi}(\cdot)(\mathbf{s})}\} \left( \text{Diag}\{\sqrt{\omega_{\psi}(\cdot)(\mathbf{s})}\} \right)'}_{\mathbf{A} \in \mathbb{C}^{M \times M}} \underbrace{\mathbf{X} \mathbf{X}'}_{\mathbf{B} \in \mathbb{C}^{M \times M}} \right\} \\
&= \text{trace} \{ \mathbf{A} \mathbf{B} \} \geq \sum_{k=1}^M \lambda_{M-k+1}(\mathbf{A}) \lambda_k(\mathbf{B}).
\end{aligned}$$

We have that  $\lambda_{M-k+1}(\mathbf{A}) = \omega_{\psi}(s_k)$ , given that  $\omega_{\psi}$  is monotone nonincreasing and that  $\lambda_k(\mathbf{B}) = \sigma_k^2(\mathbf{X})$ . It follows that

$$(B.6) \quad \sum_{k=1}^M \omega_{\psi}(s_k) \sum_{l=1}^N |X_{kl}|^2 \geq \sum_{k=1}^M \omega_{\psi}(s_k) \sigma_k^2(\mathbf{X}).$$

Using this inequality to lower bound the right-hand side in (B.5), Lemma B.2 shows that

$$(B.7) \quad Q(\mathbf{X}; \mathbf{S}, G_{\psi}^{\mathbf{W}}) - R_{\psi}(\mathbf{X}) \geq \sum_{k=1}^M \psi(s_k) - \psi(\sigma_k(\mathbf{X})) + \frac{1}{2} \omega_{\psi}(s_k) (\sigma_k^2(\mathbf{X}) - s_k^2) \geq 0.$$

Now we prove the general case. Let  $\mathbf{Y}, \mathbf{S} \in \mathbb{C}^{M \times N}$  and  $\mathbf{U} \in \mathbb{C}^{M \times M}, \mathbf{V} \in \mathbb{C}^{N \times N}$  denote two unitary matrices such that  $\mathbf{S} = \mathbf{U} \text{Diag}\{\boldsymbol{\sigma}(\mathbf{S})\} \mathbf{V}'$ . Because  $R_{\psi}$  is unitarily invariant, it follows that

$$\begin{aligned}
R_{\psi}(\mathbf{Y} + \mathbf{S}) &= R_{\psi}(\mathbf{U}'(\mathbf{Y} + \mathbf{S})\mathbf{V}) = R_{\psi}(\mathbf{U}'\mathbf{Y}\mathbf{V} + \text{Diag}\{\boldsymbol{\sigma}(\mathbf{S})\}) \\
&\leq Q(\mathbf{U}'\mathbf{Y}\mathbf{V} + \text{Diag}\{\boldsymbol{\sigma}(\mathbf{S})\}; \text{Diag}\{\boldsymbol{\sigma}(\mathbf{S})\}, G_{\psi}^{\mathbf{W}}) \quad \text{by (B.7)} \\
&= R_{\psi}(\text{Diag}\{\boldsymbol{\sigma}(\mathbf{S})\}) + \text{real}\{\langle \nabla R_{\psi}(\text{Diag}\{\boldsymbol{\sigma}(\mathbf{S})\}), \mathbf{U}'\mathbf{Y}\mathbf{V} \rangle\} \\
&\quad + \frac{1}{2}(\mathbf{1}'_M(G_{\psi}^{\mathbf{W}}(\boldsymbol{\sigma}(\mathbf{S})) \odot |\mathbf{U}'\mathbf{Y}\mathbf{V}|.^2) \mathbf{1}_N) \quad \text{by (3.3)} \\
&= R_{\psi}(\mathbf{S}) + \text{real}\left\{ \left\langle \text{Diag}\{\dot{\psi}(\boldsymbol{\sigma}(\mathbf{S}))\}, \mathbf{U}'\mathbf{Y}\mathbf{V} \right\rangle \right\} \quad \text{by (2.10)} \\
&\quad + \frac{1}{2}(\mathbf{1}'_M(G_{\psi}^{\mathbf{W}}(\boldsymbol{\sigma}(\mathbf{S})) \odot |\mathbf{U}'\mathbf{Y}\mathbf{V}|.^2) \mathbf{1}_N) \\
&= R_{\psi}(\mathbf{S}) + \text{real}\{\text{trace}\{\text{Diag}\{\dot{\psi}(\boldsymbol{\sigma}(\mathbf{S}))\} \mathbf{V}'\mathbf{Y}'\mathbf{U}\}\} \\
&\quad + \frac{1}{2}(\mathbf{1}'_M(G_{\psi}^{\mathbf{W}}(\boldsymbol{\sigma}(\mathbf{S})) \odot |\mathbf{U}'\mathbf{Y}\mathbf{V}|.^2) \mathbf{1}_N) \\
&= R_{\psi}(\mathbf{S}) + \text{real}\left\{ \text{trace}\left\{ \mathbf{U} \text{Diag}\left\{ \dot{\psi}(\boldsymbol{\sigma}(\mathbf{S})) \right\} \mathbf{V}'\mathbf{Y}' \right\} \right\} \\
&\quad + \frac{1}{2}(\mathbf{1}'_M(G_{\psi}^{\mathbf{W}}(\boldsymbol{\sigma}(\mathbf{S})) \odot |\mathbf{U}'\mathbf{Y}\mathbf{V}|.^2) \mathbf{1}_N) \\
(B.8) \quad &= R_{\psi}(\mathbf{S}) + \text{real}\{\langle \nabla R_{\psi}(\mathbf{S}), \mathbf{Y} \rangle\} + \frac{1}{2}(\mathbf{1}'_M(G_{\psi}^{\mathbf{W}}(\boldsymbol{\sigma}(\mathbf{S})) \odot |\mathbf{U}'\mathbf{Y}\mathbf{V}|.^2) \mathbf{1}_N),
\end{aligned}$$

using the cyclic property of the trace in the next to last equality. The result follows from setting  $\mathbf{Y}$  to  $\mathbf{X} - \mathbf{S}$  in (B.8).

To show that  $Q(\mathbf{X}; \mathbf{S}, G_\psi^W) \leq Q(\mathbf{X}; \mathbf{S}, G_\psi^L) \forall \mathbf{X} \in \mathbb{C}^{M \times N}$ , it is enough to show that  $\mathbf{1}'_M (G_\psi^W(\boldsymbol{\sigma}(\mathbf{S})) \odot |\mathbf{U}'(\mathbf{X} - \mathbf{S})\mathbf{V}|.^{\wedge}2) \mathbf{1}_N \leq \mathbf{1}'_M (G_\psi^L(\boldsymbol{\sigma}(\mathbf{S})) \odot |\mathbf{U}'(\mathbf{X} - \mathbf{S})\mathbf{V}|.^{\wedge}2) \mathbf{1}_N$ , which is equivalent to showing that

$$\begin{aligned} \sum_{k=1}^M \omega_\psi(\sigma_k(\mathbf{S})) \sum_{l=1}^N |[\mathbf{U}'(\mathbf{X} - \mathbf{S})\mathbf{V}]_{kl}|^2 &\leq \sum_{k=1}^M \left( \sup_{t \in \mathbb{R}} \omega_\psi(t) \right) \sum_{l=1}^N |[\mathbf{U}'(\mathbf{X} - \mathbf{S})\mathbf{V}]_{kl}|^2 \\ &= \sum_{k=1}^M \sum_{l=1}^N \omega_\psi(0) |[\mathbf{U}'(\mathbf{X} - \mathbf{S})\mathbf{V}]_{kl}|^2, \end{aligned}$$

where we have used the property that  $\sup_{t \in \mathbb{R}} \omega_\psi(t) = \omega_\psi(0)$  as  $\psi$  is even and  $\omega_\psi(t)$  is monotonic nonincreasing for  $t > 0$ .

The proof for the tall case, i.e., when  $M > N$ , is analogous and is omitted for the sake of space. Finally, we prove (3.7), (3.11), and (3.9). To prove (3.7) it is enough to show that

$$\|\mathbf{W}'_{\psi, \mathbf{S}}(\mathbf{X} - \mathbf{S})\|_{\text{F}}^2 = \mathbf{1}'_M (G_\psi^W(\boldsymbol{\sigma}(\mathbf{S})) \odot |\mathbf{U}'(\mathbf{X} - \mathbf{S})\mathbf{V}|.^{\wedge}2) \mathbf{1}_N.$$

Because the Frobenius norm is unitarily invariant, we have that

$$\begin{aligned} \|\mathbf{W}'_{\psi, \mathbf{S}}(\mathbf{X} - \mathbf{S})\|_{\text{F}}^2 &= \left\| \mathbf{V}' \left( \text{Diag}\{\sqrt{\omega_\psi(\boldsymbol{\sigma}(\mathbf{S}))}\} \right)' \mathbf{U}'(\mathbf{X} - \mathbf{S}) \right\|_{\text{F}}^2 \\ &= \left\| \left( \text{Diag}\{\sqrt{\omega_\psi(\boldsymbol{\sigma}(\mathbf{S}))}\} \right)' \mathbf{U}'(\mathbf{X} - \mathbf{S})\mathbf{V} \right\|_{\text{F}}^2 \\ &= \sum_{k=1}^M \omega_\psi(\sigma_k(\mathbf{S})) \sum_{l=1}^N |[\mathbf{U}'(\mathbf{X} - \mathbf{S})\mathbf{V}]_{kl}|^2 \\ &= \mathbf{1}'_M (G_\psi^W(\boldsymbol{\sigma}(\mathbf{S})) \odot |\mathbf{U}'(\mathbf{X} - \mathbf{S})\mathbf{V}|.^{\wedge}2) \mathbf{1}_N. \end{aligned}$$

The proofs for (3.9) and (3.11) are analogous to the previous proof and are omitted.

**Appendix C. Proof of Proposition 2.6.** Let  $\mathbf{X}, \mathbf{S} \in \mathbb{C}^{M \times N}$  and  $\mathbf{U}, \mathbf{V}$  denote suitably sized unitary matrices such that  $\mathbf{S} = \mathbf{U} \text{Diag}\{\boldsymbol{\sigma}(\mathbf{S})\} \mathbf{V}'$ . Using (3.11) in Theorem 3.1,

$$(C.1) \quad R_\psi(\mathbf{X}) \leq R_\psi(\mathbf{S}) + \text{real}\{\langle \nabla R_\psi(\mathbf{S}), \mathbf{X} - \mathbf{S} \rangle\} + \frac{1}{2} \omega_\psi(0) \|\mathbf{X} - \mathbf{S}\|_{\text{F}}^2,$$

which is equivalent to  $\nabla R_\psi$  being  $\omega_\psi(0)$ -Lipschitz [1, Thm. 5.8] for the inner product in (2.8).

**Appendix D. Proof of Theorem 3.2.** We need to verify that

$$(D.1) \quad h_\psi(\bar{\alpha}; \mathbf{X}, \boldsymbol{\Delta}) = g_{h_\psi}(\bar{\alpha}; \bar{\alpha}, \mathbf{X}, \boldsymbol{\Delta}, G_\psi),$$

$$(D.2) \quad h_\psi(\alpha; \mathbf{X}, \boldsymbol{\Delta}) \leq g_{h_\psi}(\alpha; \bar{\alpha}, \mathbf{X}, \boldsymbol{\Delta}, G_\psi) \quad \forall \alpha \in \mathbb{R}.$$

Equation (D.1) is easy to verify from the definition of  $g_{h_\psi}(\cdot; \bar{\alpha}, \mathbf{X}, \Delta, G_\psi)$ . To prove (D.2), we use the fact that  $Q(\cdot; \mathbf{X} + \bar{\alpha}\Delta, G_\psi)$  is a majorizer for  $R_\psi$  at  $\mathbf{X} + \bar{\alpha}\Delta$ . We have that

$$\begin{aligned}
 h_\psi(\alpha; \mathbf{X}, \Delta) &= R_\psi(\mathbf{X} + \alpha\Delta) \\
 &\leq Q(\mathbf{X} + \alpha\Delta; \mathbf{X} + \bar{\alpha}\Delta, G_\psi) \\
 &= R_\psi(\mathbf{X} + \bar{\alpha}\Delta) + \text{real}\{\langle \nabla R_\psi(\mathbf{X} + \bar{\alpha}\Delta), \mathbf{X} + \alpha\Delta - (\mathbf{X} + \bar{\alpha}\Delta) \rangle\} \\
 &\quad + \frac{1}{2} \mathbf{1}'_M (G_\psi(\sigma(\mathbf{X} + \bar{\alpha}\Delta)) \odot (\mathbf{U}'(\mathbf{X} + \alpha\Delta - (\mathbf{X} + \bar{\alpha}\Delta))\mathbf{V}).^{\wedge 2}) \mathbf{1}_N \\
 &= R_\psi(\mathbf{X} + \bar{\alpha}\Delta) + \text{real}\{\langle \nabla R_\psi(\mathbf{X} + \bar{\alpha}\Delta), \Delta \rangle\}(\alpha - \bar{\alpha}) \\
 &\quad + \frac{1}{2} \mathbf{1}'_M (G_\psi(\sigma(\mathbf{X} + \bar{\alpha}\Delta)) \odot (\mathbf{U}'\Delta\mathbf{V}).^{\wedge 2}) \mathbf{1}_N (\alpha - \bar{\alpha})^2 \\
 &= g_{h_\psi}(\alpha; \bar{\alpha}, \mathbf{X}, \Delta, G_\psi) \quad \forall \alpha, \bar{\alpha} \in \mathbb{R}.
 \end{aligned}$$

**Appendix E. Proof of Corollary 3.3.** Without loss of generality, we assume that  $M < N$  (wide case). The first part of the corollary follows directly from Theorem 3.1 and Theorem 3.2. To show the second part, it is enough to show that  $\mathbf{1}'_M (G_\psi^W(\sigma(\mathbf{X} + \bar{\alpha}\Delta)) \odot |\mathbf{U}'\Delta\mathbf{V}|.^{\wedge 2}) \mathbf{1}_N \leq \mathbf{1}'_M (G_\psi^L(\sigma(\mathbf{X} + \bar{\alpha}\Delta)) \odot |\mathbf{U}'\Delta\mathbf{V}|.^{\wedge 2}) \mathbf{1}_N$ . This is equivalent to showing that

$$\begin{aligned}
 \sum_{k=1}^M \omega_\psi(\sigma_k(\mathbf{X} + \bar{\alpha}\Delta)) \sum_{l=1}^N |[\mathbf{U}'\Delta\mathbf{V}]_{kl}|^2 &\leq \left( \sup_{t \in \mathbb{R}} \omega_\psi(t) \right) \sum_{k=1}^M \sum_{l=1}^N |[\mathbf{U}'\Delta\mathbf{V}]_{kl}|^2 \\
 &= \omega_\psi(0) \sum_{k=1}^M \sum_{l=1}^N |[\mathbf{U}'\Delta\mathbf{V}]_{kl}|^2.
 \end{aligned}$$

**Appendix F. Proof of Proposition 4.1.** We first prove the sufficient condition by providing an equivalent expression for  $f_{\psi, \mathbf{w}}$  in (4.2) inspired by the  $c$ -spectral norm in [34]. For this purpose, we review *generalized permutation matrices*.

**Definition F.1 (generalized permutation matrix [34]).** An  $r \times r$  matrix  $\mathbf{P}$  is a generalized permutation matrix if it has exactly one nonzero entry in each row and each column, that entry being equal to 1 or  $-1$ . We denote the set of  $r \times r$  generalized permutation matrices by  $\Theta_r$ .

Using the previous definition we first show that

$$(F.1) \quad f_{\psi, \mathbf{w}}(\mathbf{x}) = \max_{\mathbf{P} \in \Theta_r} \langle \mathbf{w}, \psi(\mathbf{P}\mathbf{x}) \rangle,$$

where here  $\langle \cdot, \cdot \rangle$  denotes the Euclidean inner product in  $\mathbb{R}^r$ . Given that  $\tilde{\mathbf{x}}$  is constructed by sorting the entries of  $|\mathbf{x}|$ , there exists  $\tilde{\mathbf{P}} \in \Theta_r$  such that  $\tilde{\mathbf{x}} = \tilde{\mathbf{P}}\mathbf{x}$ . Then, from (4.2),

$$f_{\psi, \mathbf{w}}(\mathbf{x}) = \sum_{k=1}^r w_k \psi([\tilde{\mathbf{P}}\mathbf{x}]_k) = \langle \mathbf{w}, \psi(\tilde{\mathbf{P}}\mathbf{x}) \rangle \leq \max_{\mathbf{P} \in \Theta_r} \langle \mathbf{w}, \psi(\mathbf{P}\mathbf{x}) \rangle.$$

We prove the reverse inequality by contradiction. If the inequality is not true, then there are an  $\mathbf{x}$  and a generalized permutation matrix  $\mathbf{P}^* \in \Theta_r$  such that  $\mathbf{P}^*\mathbf{x} \neq \tilde{\mathbf{x}}$  and

$$0 \leq f_{\psi, \mathbf{w}}(\mathbf{x}) < \langle \mathbf{w}, \psi(\mathbf{P}^*\mathbf{x}) \rangle = \sum_{k=1}^r w_k \psi([\mathbf{P}^*\mathbf{x}]_k).$$

Because  $\mathbf{P}^* \mathbf{x} \neq \check{\mathbf{x}}$ , there are entries  $k_1, k_2 \in \{1, \dots, r\}$ , where  $k_1 < k_2$ , such that  $||[\mathbf{P}^* \mathbf{x}]_{k_1}| < |[\mathbf{P}^* \mathbf{x}]_{k_2}|$ . Because the weights are nonincreasing, and  $\psi(t)$  is even and nondecreasing for  $t > 0$ , it follows that

$$w_{k_1} \psi(|[\mathbf{P}^* \mathbf{x}]_{k_2}|) \geq w_{k_2} \psi(|[\mathbf{P}^* \mathbf{x}]_{k_1}|).$$

Let  $\mathbf{P}_{k_1, k_2}$  denote the generalized permutation matrix that swaps the  $k_1$ th and  $k_2$ th entries and multiplies each entry either by 1 or  $-1$  to make both entries nonnegative. Then we have that

$$\langle \mathbf{w}, \psi(\mathbf{P}^* \mathbf{x}) \rangle = \sum_{k=1}^r w_k \psi(|[\mathbf{P}^* \mathbf{x}]_k|) \leq \sum_{k=1}^r w_k \psi(|[\mathbf{P}_{k_1, k_2} \mathbf{P}^* \mathbf{x}]_k|) = \langle \mathbf{w}, \psi(\mathbf{P}_{k_1, k_2} \mathbf{P}^* \mathbf{x}) \rangle.$$

At this point, either  $\mathbf{P}_{k_1, k_2} \mathbf{P}^* \mathbf{x} = \check{\mathbf{x}}$  or we can find another pair of entries in  $\mathbf{P}_{k_1, k_2} \mathbf{P}^* \mathbf{x}$  that can be swapped and made nonnegative to majorize  $\langle \mathbf{w}, \psi(\mathbf{P}_{k_1, k_2} \mathbf{P}^* \mathbf{x}) \rangle$ . Continuing this process, we finally obtain that  $f_{\psi, \mathbf{w}}(\mathbf{x}) < \langle \mathbf{w}, \psi(\mathbf{P}^* \mathbf{x}) \rangle \leq f_{\psi, \mathbf{w}}(\mathbf{x})$ , which is a contradiction. Therefore,

$$f_{\psi, \mathbf{w}}(\mathbf{x}) \geq \max_{\mathbf{P} \in \Theta_r} \langle \mathbf{w}, \psi(\mathbf{P} \mathbf{x}) \rangle.$$

It is easy to show that each function  $\gamma_{\psi}(\mathbf{x}; \mathbf{w}, \mathbf{P}) \triangleq \langle \mathbf{w}, \psi(\mathbf{P} \mathbf{x}) \rangle$ ,  $\mathbf{P} \in \Theta_r$  is convex when  $\psi$  is convex. Then  $f_{\psi, \mathbf{w}}$  is convex, as it can be written as the pointwise supremum of a collection of convex functions [55, Thm. 5.5].

We now prove the necessary conditions. Noting that  $\psi(t) = f_{\psi, \mathbf{w}}(\mathbf{1}_r t) / \sum_{k=1}^r w_k$ , the convexity of  $\psi$  follows from the convexity of  $f_{\psi, \mathbf{w}}$  because convexity is preserved under affine maps and nonnegative scaling.

Now we show that the entries of  $\mathbf{w}$  must be nonincreasing, using proof by contradiction.

Consider first the case where  $r = 2$  with  $0 \leq w_1 < w_2$  and define  $\mathbf{x} = \begin{bmatrix} u \\ v \end{bmatrix}$  and  $\mathbf{y} = \begin{bmatrix} v \\ u \end{bmatrix}$  for  $0 < u < v < \epsilon u$ , where  $\epsilon > 1$ . In addition, we pick  $u$  and  $v$  such that  $\omega_{\psi}(\alpha v + (1 - \alpha)u) > 0 \forall \alpha \in (0, 1)$ . This is possible, as  $\omega_{\psi}$  is not always equal to 0 unless  $\psi$  is a constant, which is precluded by our ongoing assumptions. Define

$$\begin{aligned} g(\alpha) &\triangleq f_{\psi, \mathbf{w}}(\alpha \mathbf{x} + (1 - \alpha) \mathbf{y}) = f_{\psi, \mathbf{w}} \left( \begin{bmatrix} \alpha u + (1 - \alpha)v \\ \alpha v + (1 - \alpha)u \end{bmatrix} \right) \\ &= \begin{cases} w_1 \psi(\alpha u + (1 - \alpha)v) + w_2 \psi(\alpha v + (1 - \alpha)u), & 0 \leq \alpha \leq 1/2, \\ w_1 \psi(\alpha v + (1 - \alpha)u) + w_2 \psi(\alpha u + (1 - \alpha)v), & 1/2 \leq \alpha \leq 1. \end{cases} \end{aligned}$$

We will show that  $g$  is nonconvex, which means that  $f_{\psi, \mathbf{w}}$  must be nonconvex, by showing that  $g$  increases on  $(0, 1/2)$  and then decreases on  $(1/2, 1)$ .

If  $\alpha \in (0, \frac{1}{2})$ , then  $\dot{g}(\alpha)$ , i.e., the derivative of  $g$  with respect to  $\alpha$ , satisfies

$$\begin{aligned} \dot{g}(\alpha) &= w_1 \dot{\psi}(\alpha u + (1 - \alpha)v) (u - v) + w_2 \dot{\psi}(\alpha v + (1 - \alpha)u) (v - u) \\ &= (v - u) (w_2 \dot{\psi}(\alpha v + (1 - \alpha)u) - w_1 \dot{\psi}(\alpha u + (1 - \alpha)v)) \\ &= (v - u) (w_2 (\alpha v + (1 - \alpha)u) \omega_{\psi}(\alpha v + (1 - \alpha)u) \\ &\quad - w_1 (\alpha u + (1 - \alpha)v) \omega_{\psi}(\alpha u + (1 - \alpha)v)), \end{aligned}$$

where we used that  $\dot{\psi}(t) = t\omega_\psi(t)$  in the last equality. Next, because  $\omega_\psi(t)$  is monotonic nonincreasing for  $t > 0$ ,

$$\omega_\psi(\alpha v + (1 - \alpha)u) \geq \omega_\psi(\alpha u + (1 - \alpha)v).$$

Using this in the previous equation implies that

$$\begin{aligned} \dot{g}(\alpha) &\geq C(w_2(\alpha v + (1 - \alpha)u) - w_1(\alpha u + (1 - \alpha)v)) \\ &= C(u(w_2 - \alpha(w_2 + w_1)) - v(w_1 - \alpha(w_2 + w_1))), \end{aligned}$$

where  $C \triangleq (v - u)\omega_\psi(\alpha v + (1 - \alpha)u) > 0$ . Because  $\alpha < 1/2$ , the term multiplying  $v$  is positive. Using  $v < \epsilon u$  in the previous inequality yields

$$\begin{aligned} \dot{g}(\alpha) &> C(u(w_2 - \alpha(w_2 + w_1)) - \epsilon u(w_1 - \alpha(w_2 + w_1))) \\ &= C u(w_2(1 + \alpha(\epsilon - 1)) - w_1(\alpha + \epsilon(1 - \alpha))). \end{aligned}$$

If  $w_1 = 0$ , then we have that  $\dot{g}(\alpha) > 0$  and we can conclude that  $g$  is increasing for  $\alpha \in (0, \frac{1}{2})$ . If  $w_1 > 0$ , then we can choose  $\epsilon = \frac{w_2}{w_1}$  to simplify the lower bound and obtain that

$$\dot{g}(\alpha) > C u \alpha (w_2^2 - w_1^2) / w_1 > 0 \quad (\text{given that } w_2 > w_1).$$

Therefore,  $g$  is increasing for  $\alpha \in (0, \frac{1}{2})$ . Proceeding analogously for the case  $\alpha \in (\frac{1}{2}, 1)$ , we obtain the upper bound

$$\dot{g}(\alpha) < D u (1 - \alpha)(w_1^2 - w_2^2) / w_1 < 0 \quad (\text{given that } w_2 > w_1),$$

where  $D \triangleq (v - u)\omega_\psi(\alpha u + (1 - \alpha)v) > 0$ . Thus,  $g$  is decreasing for  $\alpha \in (\frac{1}{2}, 1)$ . We omit some of the algebraic derivations for this case, as they are analogous to the ones in the first case. Therefore,  $g$  is nonconvex as it is increasing before  $\alpha = \frac{1}{2}$  and decreasing after  $\alpha = \frac{1}{2}$ . This implies that  $f_{\psi, w}$  is nonconvex, which is a contradiction. This concludes the proof for the case  $r = 2$ .

We now prove the general case  $r > 2$  also by contradiction. Suppose there is  $K \in \{1, \dots, r - 1\}$  such that  $w_K < w_{K+1}$ , and pick  $\mathbf{x} \in \mathbb{R}^r$  such that  $0 < x_1 < x_2 < \dots < x_r$ . Let  $\mathbf{y} \in \mathbb{R}^r$  have entries that are the same entries as  $\mathbf{x}$ , except that the  $(r - K)$ th and  $(r - K + 1)$ th entries are swapped, i.e.,  $y_{r-K+1} = x_{r-K}$  and  $y_{r-K} = x_{r-K+1}$ . We choose  $x_{r-K+1} < \epsilon x_{r-K}$  for some  $\epsilon > 1$ . Under this setting we have that

$$g(\alpha) = w_K \psi(M_\alpha) + w_{K+1} \psi(m_\alpha) + \sum_{\substack{k=1 \\ k \notin \{K, K+1\}}}^r w_k \psi(x_{r-k+1}),$$

where

$$\begin{aligned} M_\alpha &\triangleq \max(\alpha x_{r-K} + (1 - \alpha)x_{r-K+1}, \alpha x_{r-K+1} + (1 - \alpha)x_{r-K}), \\ m_\alpha &\triangleq \min(\alpha x_{r-K} + (1 - \alpha)x_{r-K+1}, \alpha x_{r-K+1} + (1 - \alpha)x_{r-K}), \end{aligned}$$

which implies that

$$\dot{g}(\alpha) = w_K \dot{M}_\alpha \dot{\psi}(M_\alpha) + w_{K+1} \dot{m}_\alpha \dot{\psi}(m_\alpha),$$

which can be analyzed following the same procedure shown for the case  $r = 2$ . Therefore,  $g$  is nonconvex, which implies that  $f_{\psi, w}$  is nonconvex, which is a contradiction. This concludes our proof.

**Appendix G. Proof of Proposition 4.2.** Let  $0 < \delta < \min_{k \in \{1, \dots, r\}} |x_k - x_{k-1}|$ , where we have defined  $x_0 = 0$ . Then any vector within the ball with center at  $\mathbf{x}$  and radius  $\delta$  has entries that are nonnegative, nonrepeated, and arranged in decreasing order. More formally,

$$(G.1) \quad \mathbf{y} \in B_\delta(\mathbf{x}) \iff \mathbf{y} = \tilde{\mathbf{y}},$$

where  $B_\delta(\mathbf{x}) \triangleq \{\mathbf{y} \in \mathbb{R}^r \mid \|\mathbf{x} - \mathbf{y}\| \leq \delta\}$ , with  $\|\cdot\|$  denoting the Euclidean norm. It follows that

$$(G.2) \quad f_{\psi, \mathbf{w}}(\mathbf{y}) = \sum_{k=1}^r w_k \psi([\tilde{\mathbf{y}}]_k) = \sum_{k=1}^r w_k \psi([\mathbf{y}]_k) = \langle \mathbf{w}, \psi(\mathbf{y}) \rangle \quad \forall \mathbf{y} \in B_\delta(\mathbf{x}).$$

The result follows from showing that  $\langle \mathbf{w}, \psi(\mathbf{y}) \rangle$  is locally Lipschitz when  $\mathbf{y} \in B_\delta(\mathbf{x})$  and differentiable at  $\mathbf{x}$  with the gradient given by  $\mathbf{w} \odot \dot{\psi}(\mathbf{x})$ . Showing the differentiability and gradient formula for  $\langle \mathbf{w}, \psi(\mathbf{y}) \rangle$  is straightforward and follows from the differentiability of  $\psi$ . We now show that the gradient of  $f_{\psi, \mathbf{w}}$  is Lipschitz continuous, which implies that  $f_{\psi, \mathbf{w}}$  is locally Lipschitz at  $\mathbf{x}$ . Let  $\mathbf{u}, \mathbf{v} \in B_\delta(\mathbf{x})$ ; then

$$(G.3) \quad \begin{aligned} \|\mathbf{w} \odot \dot{\psi}(\mathbf{u}) - \mathbf{w} \odot \dot{\psi}(\mathbf{v})\|^2 &= \sum_{k=1}^r w_k^2 |\dot{\psi}(u_k) - \dot{\psi}(v_k)|^2 \\ &\leq \|\mathbf{w}\|_\infty^2 \sum_{k=1}^r |\dot{\psi}(u_k) - \dot{\psi}(v_k)|^2 \\ &\leq \|\mathbf{w}\|_\infty^2 \omega_\psi(0)^2 \|\mathbf{u} - \mathbf{v}\|^2 \end{aligned}$$

as  $\dot{\psi}$  is  $\omega_\psi(0)$ -Lipschitz continuous. Therefore, the gradient of  $\langle \mathbf{w}, \psi(\mathbf{y}) \rangle$  is  $L$ -Lipschitz continuous with  $L = \|\mathbf{w}\|_\infty \omega_\psi(0)$ .

**Appendix H. Majorizers for weighted Huber-based low-rank regularizers.** A majorizer  $Q_{\mathbf{w}}(\cdot; \mathbf{S}, G_{\psi, \mathbf{w}})$  as defined in (4.5) should satisfy

$$(H.1) \quad R_{\psi, \mathbf{w}}(\mathbf{S}) = Q_{\mathbf{w}}(\mathbf{S}; \mathbf{S}, G_{\psi, \mathbf{w}}),$$

$$(H.2) \quad R_{\psi, \mathbf{w}}(\mathbf{X}) \leq Q_{\mathbf{w}}(\mathbf{X}; \mathbf{S}, G_{\psi, \mathbf{w}}) \quad \forall \mathbf{X} \in \mathbb{C}^{M \times N}.$$

Particularly, we show that two operators, denoted by  $G_{\psi, \mathbf{w}}^W$  and  $G_{\psi, \mathbf{w}}^L$ , when used in (4.5), lead to valid majorizers for  $R_{\psi, \mathbf{w}}$ , i.e., in both cases the resulting  $Q_{\mathbf{w}}$  satisfies (H.1) and (H.2), when  $\mathbf{w}$  has nondecreasing entries. When  $M < N$  (wide case), we define  $G_{\psi, \mathbf{w}}^W$  and  $G_{\psi, \mathbf{w}}^L$  as the following generalizations of (3.4)–(3.6):

$$(H.3) \quad G_{\psi, \mathbf{w}}^W(\mathbf{s}) \triangleq (\mathbf{w} \odot \omega_\psi(\mathbf{s})) \mathbf{1}'_N,$$

$$(H.4) \quad G_{\psi, \mathbf{w}}^L(\mathbf{s}) \triangleq \omega_\psi(0) \mathbf{w} \mathbf{1}'_N,$$

and when  $M > N$  (tall case),

$$(H.5) \quad G_{\psi, \mathbf{w}}^W(\mathbf{s}) \triangleq \mathbf{1}_M (\mathbf{w} \odot \omega_\psi(\mathbf{s}))',$$

$$(H.6) \quad G_{\psi, \mathbf{w}}^L(\mathbf{s}) \triangleq \omega_\psi(0) \mathbf{1}_M \mathbf{w}'.$$

The following theorem states that  $Q_{\mathbf{w}}(\cdot; \mathbf{S}, G_{\psi, \mathbf{w}}^W)$  and  $Q_{\mathbf{w}}(\cdot; \mathbf{S}, G_{\psi, \mathbf{w}}^L)$  are majorizers for  $R_{\psi, \mathbf{w}}$  at  $\mathbf{S} \in \mathbb{C}^{M \times N}$ , and it also provides equivalent expressions for both functions.

**Theorem H.1 (majorizers for weighted Huber-based low-rank regularizers).** Let  $\mathbf{S} \in \mathbb{C}^{M \times N}$  and  $\mathbf{U}, \mathbf{V}$  denote suitably sized unitary matrices such that  $\mathbf{S} = \mathbf{U} \text{Diag}\{\boldsymbol{\sigma}(\mathbf{S})\} \mathbf{V}'$ . Let  $R_{\psi, \mathbf{w}}$  be defined as in (1.6) with nondecreasing weights, i.e.,  $0 \leq w_1 \leq w_2 \leq \dots \leq w_r$ . Then  $Q_{\mathbf{w}}(\cdot; \mathbf{S}, G_{\psi, \mathbf{w}}^{\mathbf{W}})$  and  $Q_{\mathbf{w}}(\cdot; \mathbf{S}, G_{\psi, \mathbf{w}}^{\mathbf{L}})$  are both majorizers for  $R_{\psi, \mathbf{w}}$  at  $\mathbf{S} \in \mathbb{C}^{M \times N}$ ; i.e., both satisfy (H.1) and (H.2). If  $M < N$  (wide case), then

(H.7)

$$Q_{\mathbf{w}}(\mathbf{X}; \mathbf{S}, G_{\psi, \mathbf{w}}^{\mathbf{W}}) = R_{\psi, \mathbf{w}}(\mathbf{S}) + \text{real}\{\langle \nabla R_{\psi, \mathbf{w}}(\mathbf{S}), \mathbf{X} - \mathbf{S} \rangle\} + \frac{1}{2} \|\mathbf{W}'_{\psi, \mathbf{S}, \mathbf{w}}(\mathbf{X} - \mathbf{S})\|_{\mathbf{F}}^2,$$

(H.8)

$$Q_{\mathbf{w}}(\mathbf{X}; \mathbf{S}, G_{\psi, \mathbf{w}}^{\mathbf{L}}) = R_{\psi, \mathbf{w}}(\mathbf{S}) + \text{real}\{\langle \nabla R_{\psi, \mathbf{w}}(\mathbf{S}), \mathbf{X} - \mathbf{S} \rangle\} + \frac{1}{2} \omega_{\psi}(0) \|\mathbf{L}'_{\psi, \mathbf{S}, \mathbf{w}}(\mathbf{X} - \mathbf{S})\|_{\mathbf{F}}^2,$$

where the matrices  $\mathbf{W}_{\psi, \mathbf{S}, \mathbf{w}}$  and  $\mathbf{L}_{\psi, \mathbf{S}, \mathbf{w}}$  are defined as

$$(H.9) \quad \mathbf{W}_{\psi, \mathbf{S}, \mathbf{w}} \triangleq \mathbf{U} \text{Diag}\{\sqrt{\mathbf{w} \odot (\omega_{\psi}(\boldsymbol{\sigma}(\mathbf{S})))}\} \mathbf{V} \in \mathbb{C}^{M \times N},$$

$$(H.10) \quad \mathbf{L}_{\psi, \mathbf{S}, \mathbf{w}} \triangleq \mathbf{U} \text{Diag}\{\sqrt{\mathbf{w}}\} \mathbf{V} \in \mathbb{C}^{M \times N}.$$

If  $M > N$  (tall case), then

(H.11)

$$Q_{\mathbf{w}}(\mathbf{X}; \mathbf{S}, G_{\psi, \mathbf{w}}^{\mathbf{W}}) = R_{\psi, \mathbf{w}}(\mathbf{S}) + \text{real}\{\langle \nabla R_{\psi, \mathbf{w}}(\mathbf{S}), \mathbf{X} - \mathbf{S} \rangle\} + \frac{1}{2} \|(\mathbf{X} - \mathbf{S}) \mathbf{T}'_{\psi, \mathbf{S}, \mathbf{w}}\|_{\mathbf{F}}^2,$$

(H.12)

$$Q_{\mathbf{w}}(\mathbf{X}; \mathbf{S}, G_{\psi, \mathbf{w}}^{\mathbf{L}}) = R_{\psi, \mathbf{w}}(\mathbf{S}) + \text{real}\{\langle \nabla R_{\psi, \mathbf{w}}(\mathbf{S}), \mathbf{X} - \mathbf{S} \rangle\} + \frac{1}{2} \omega_{\psi}(0) \|(\mathbf{X} - \mathbf{S}) \mathbf{H}'_{\psi, \mathbf{S}, \mathbf{w}}\|_{\mathbf{F}}^2,$$

where the matrices  $\mathbf{T}_{\psi, \mathbf{S}, \mathbf{w}}$  and  $\mathbf{H}_{\psi, \mathbf{S}, \mathbf{w}}$  are defined as

$$(H.13) \quad \mathbf{T}_{\psi, \mathbf{S}, \mathbf{w}} \triangleq \mathbf{U}' \text{Diag}\{\sqrt{\mathbf{w} \odot (\omega_{\psi}(\boldsymbol{\sigma}(\mathbf{S})))}\} \mathbf{V}' \in \mathbb{C}^{M \times N},$$

$$(H.14) \quad \mathbf{H}_{\psi, \mathbf{S}, \mathbf{w}} \triangleq \mathbf{U}' \text{Diag}\{\sqrt{\mathbf{w}}\} \mathbf{V}' \in \mathbb{C}^{M \times N}.$$

The proof for this theorem is not provided, as it is analogous to the proof of Theorem 3.1 in Appendix B.

**Appendix I. Proof of Proposition 4.6.** Let  $\mathbf{X}, \mathbf{S} \in \mathbb{C}^{M \times N}$  and  $\mathbf{U}, \mathbf{V}$  denote suitably sized unitary matrices such that  $\mathbf{S} = \mathbf{U} \text{Diag}\{\boldsymbol{\sigma}(\mathbf{S})\} \mathbf{V}'$ . Without loss of generality, we assume that  $M < N$  (wide case). By Theorem H.1 we have that

$$(I.1) \quad \begin{aligned} R_{\psi, \mathbf{w}}(\mathbf{X}) &\leq Q_{\mathbf{w}}(\mathbf{X}; \mathbf{S}, G_{\psi, \mathbf{w}}^{\mathbf{L}}) \\ &= R_{\psi, \mathbf{w}}(\mathbf{S}) + \text{real}\{\langle \nabla R_{\psi, \mathbf{w}}(\mathbf{S}), \mathbf{X} - \mathbf{S} \rangle\} \\ &\quad + \frac{1}{2} \mathbf{1}'_M (G_{\psi, \mathbf{w}}^{\mathbf{L}}(\boldsymbol{\sigma}(\mathbf{S})) \odot |\mathbf{U}'(\mathbf{X} - \mathbf{S}) \mathbf{V}|.^2) \mathbf{1}_N. \end{aligned}$$

We write the last term on the right-hand side of the previous inequality as follows:

$$(I.2) \quad \frac{1}{2} \mathbf{1}'_M (G_{\psi, \mathbf{w}}^{\mathbf{L}}(\boldsymbol{\sigma}(\mathbf{S})) \odot |\mathbf{U}'(\mathbf{X} - \mathbf{S}) \mathbf{V}|.^2) \mathbf{1}_N = \frac{1}{2} \omega_{\psi}(0) \sum_{k=1}^M w_k \sum_{l=1}^N |[\mathbf{U}'(\mathbf{X} - \mathbf{S}) \mathbf{V}]_{kl}|^2.$$

By replacing this in (I.1) it follows that

$$\begin{aligned}
 R_{\psi, \mathbf{w}}(\mathbf{X}) &\leq R_{\psi, \mathbf{w}}(\mathbf{S}) + \text{real}\{\langle \nabla R_{\psi, \mathbf{w}}(\mathbf{S}), \mathbf{X} - \mathbf{S} \rangle\} + \frac{1}{2} \omega_{\psi}(0) \sum_{k=1}^M w_k \sum_{l=1}^N |[U'(\mathbf{X} - \mathbf{S})\mathbf{V}]_{kl}|^2 \\
 &\leq R_{\psi, \mathbf{w}}(\mathbf{S}) + \text{real}\{\langle \nabla R_{\psi, \mathbf{w}}(\mathbf{S}), \mathbf{X} - \mathbf{S} \rangle\} + \frac{1}{2} \|\mathbf{w}\|_{\infty} \omega_{\psi}(0) \sum_{k=1}^M \sum_{l=1}^N |[U'(\mathbf{X} - \mathbf{S})\mathbf{V}]_{kl}|^2 \\
 &= R_{\psi, \mathbf{w}}(\mathbf{S}) + \text{real}\{\langle \nabla R_{\psi, \mathbf{w}}(\mathbf{S}), \mathbf{X} - \mathbf{S} \rangle\} + \frac{1}{2} \|\mathbf{w}\|_{\infty} \omega_{\psi}(0) \|\mathbf{U}'(\mathbf{X} - \mathbf{S})\mathbf{V}\|_{\text{F}}^2 \\
 \text{(I.3)} \quad &= R_{\psi, \mathbf{w}}(\mathbf{S}) + \text{real}\{\langle \nabla R_{\psi, \mathbf{w}}(\mathbf{S}), \mathbf{X} - \mathbf{S} \rangle\} + \frac{1}{2} \|\mathbf{w}\|_{\infty} \omega_{\psi}(0) \|\mathbf{X} - \mathbf{S}\|_{\text{F}}^2,
 \end{aligned}$$

using unitary invariance of the Frobenius norm. Then (I.3) is equivalent to  $\nabla R_{\psi, \mathbf{w}}$  being  $L$ -Lipschitz where  $L = \|\mathbf{w}\|_{\infty} \omega_{\psi}(0)$  [1, Thm. 5.8] when considering the inner product in (2.8).

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