Handout

EECS 600 Grizzle

Elementary Proof of the Heine-Borel Theorem

Thm (H.-B.) Let $(X, \|\cdot\|)$ be a finite dimensional normed linear space. Then, TFAE for a subset $S \subset X$.

(A) S is compact (sequence defn.)

(B) S is closed and bounded.

Lemma: Let $(X, \|\cdot\|)$ be a normed linear space and $S \subset X$ a subset. Then, TFAE

(a) For every sequence (x_n) in S, $\exists\ \bar{x} \in S$ and a subsequence (x_{n_i}) of (x_n) s.t. $x_{n_i} \to \bar{x}$.

(b) Every infinite subset of S has a ~~cluster~~ limit point in S.

Def. (i) $F \subset S$ is infinite if it contains an infinite number of elements.

(ii) ~~[illegible]~~ $\hat{x} \in X$ is a limit point of F if, $\forall \epsilon > 0$ $\exists\, y \in F$, $y \neq \bar{x}$, s.t. $\|y - \bar{x}\| < \epsilon$.

It is left to you to prove this lemma; note that part (a) is clearly of definition of compactness.

Key Lemma: A closed and bounded subset of $\mathbb{R}$ is compact.

Pf. For $a \in \mathbb{R}$ and $\delta \in \mathbb{R}$, $\delta > 0$ define

$$C_a^\delta = \{ x \in \mathbb{R} \mid a - \delta \leq x \leq a + \delta \} = [a-\delta, a+\delta],$$

the closed interval centered at a with length 2δ.

Observe that $C_a^\delta = C_{a-\delta/2}^{\delta/2} \cup C_{a+\delta/2}^{\delta/2}$.

Now, let $S \subset \mathbb{R}$ be any closed and bounded subset and let $F \subset S$ be any infinite subset.

To show: (by the Lemma) $\exists\, \bar{x} \in S$ s.t. $\forall \epsilon > 0$ $\exists\, y \in F$ s.t. $y \neq \bar{x}$ and $|\bar{x} - y| < \epsilon$.

Since S is bounded, $\exists\, m < \infty$ s.t. $S \subset C_0^{2^m} = C_{-2^{m-1}}^{2^{m-1}} \cup C_{2^{m-1}}^{2^{m-1}}$. Since F is infinite and $F \subset S$, then, either $F \cap C_{-2^{m-1}}^{2^{m-1}}$ or $F \cap C_{2^{m-1}}^{2^{m-1}}$ (or both) contains an infinite number of elements. Let $a_1 = \pm 2^{m-1}$ be such that $F \cap C_{a_1}^{2^{m-1}}$ has an infinite number of elements. Now, write

$$C_{a_1}^{2^{m-1}} = C_{a_1 - 2^{m-2}}^{2^{m-2}} \cup C_{a_1 + 2^{m-2}}^{2^{m-2}}$$

and repeat this process. Let $a_2 = \pm 2^{m-2}$ be such that $F \cap C_{a_1+a_2}^{2^{m-2}}$ has an infinite number of elements.

By induction, define a_n such that $F \cap C_{a_1+a_2+\cdots+a_n}^{2^{m-n}}$ has an infinite number of elements.

Claim: The sequence of partial sums, $\left(\sum_{i=1}^{n} a_i\right)$ converges.

Proof: $\sum_{i=1}^{\infty} |a_i| = \sum_{i=1}^{\infty} |\pm 2^{m-i}| = \sum_{i=1}^{\infty} 2^m 2^{-i}$

$= 2^m < \infty.$

$\square$

Let $\bar{x} := \sum_{i=1}^{\infty} a_i$

Claim: $\bar{x}$ is a limit point of F

Pf. Let $\epsilon > 0$ be given. Choose n s.t. $2^{m-n} < \frac{\epsilon}{2}$.

Then, $|\bar{x} - \sum_{i=1}^{n} a_i| = |\sum_{i=n+1}^{\infty} a_i \leq \sum_{i=n+1}^{\infty} 2^{m-i} =$

$= \sum_{j=1}^{\infty} 2^{m-n-j} = 2^{m-n}.$

$\therefore$ $\bar{x} \in C^{2^{m-n}}_{\sum_{i=1}^{n} a_i}$, which contains an infinite number of elements of F. Thus, $\exists$ $y \in F$, $y \neq \bar{x}$ s.t.

$|\bar{x} - y| \leq 2^{m-n+1} = 2(2^{m-n}) < \epsilon.$

$\square$

Claim $\bar{x} \in S$.

Pf. $\bar{x}$ a limit point of $F \Rightarrow \bar{x} \in \bar{F} \Rightarrow$
$\bar{x} \in \bar{S} \Rightarrow \bar{x} \in S$ because S is closed.

$\square$

This completes the proof of the key lemma.

$\square$

Proof of the H-B Theorem: Let $\{e_1, \ldots, e_k\}$ be a basis for X. Let (x_n) be a sequence in S and write $x_n = a_1^n e_1 + a_2^n e_2 + \cdots + a_k^n e_k$.

Claim $\exists\, M < \infty$ s.t., $\forall n$, $\forall i = 1, \ldots, k$, $|a_i^n| \le M$.

Pf. Let $Y_j = [e_1, \ldots, e_{j-1}, e_{j+1}, \ldots, e_k]$ and define $\delta_j := d(e_j, Y_j)$. Then, from class, $\delta_j > 0$, $j = 1, \ldots, k$ and $\|x_n\| \ge \max_j |a_j^n| \cdot \delta$, where $\delta = \min\{\delta_1, \ldots, \delta_k\}$.

$\therefore$ Since S is bounded, $\exists\, K < \infty$ s.t. $\|y\| \le K$ $\forall y \in S$, and this implies that

$\forall n, \forall j, \quad |a^n_j| \leq \frac{K}{\delta} =: M < \infty$.

$\square$

Therefore, each of the sequences (a^n_j) is contained in a compact subset of $\mathbb{R}$ and therefore one can choose convergent subsequences. We do this in the following way: Let $(a^{n_i}_1)$ be a subsequence of (a^n_1) such that $\exists\ \bar{a}_1 \in \mathbb{R}$, $a^{n_i}_1 \xrightarrow[i\to\infty]{} \bar{a}_1$.

Let $x^1_i := a^{n_i}_1 e_1 + a^{n_i}_2 e_2 + \cdots + a^{n_i}_k e_k$

and define ${}^1a^i_j := a^{n_i}_j$, $j = 1, .., k$, $i \geq 1$.

(x^1_n) is therefore a subsequence of (x_n) and its first component converges. Let $({}^1a^{n_i}_2)$ be a subsequence of $({}^1a^i_2)$ such that $\exists\ \bar{a}_2 \in \mathbb{R}$ s.t. ${}^1a^{n_i}_2 \to \bar{a}_2 \in \mathbb{R}$. This exists because $({}^1a^i_2)$, being a subsequence of a^n_2, lies in a compact subset of $\mathbb{R}$. Note that ${}^1a_1^{n_i} \to \bar{a}_1$.

Define $x_i^2 := {}^1a_1^{n_i} e_1 + {}^1a_2^{n_i} e_2 + \cdots + {}^1a_k^{n_i} e_k$,

and let ${}^2a_j^i := {}^1a_j^{n_i}$, $j = 1, \ldots, k$, $i \geq 1$. (x_n^2)

is therefore a subsequence of (x_n) and its first

2 component converge. Proceeding in

this manner, we obtain a subsequence

(x_n^k) of (x_n) s.t. (x_n^k) is convergent. Let

$\bar{x}$ be its limit point. Then $\bar{x} \in S$ because

S is closed. $\square$