Example: KKT Conditions and Slater's Condition

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Consider the convex constrained optimization problem

$$\begin{aligned} \arg\min_{\boldsymbol{u}\in\mathbb{R}^2} \quad & f(\boldsymbol{u}) = 2u_1^2 + u_2^2 - 8u_1 - 2u_2 \\ \text{subject to} \quad & g(\boldsymbol{u}) = u_1^2 + u_2^2 - 2 \le 0 \\ & h(\boldsymbol{u}) = u_1 + 2u_2 - 3 = 0. \end{aligned} \tag{1}$$

Equivalently,

$$f(\boldsymbol{u}) = \frac{1}{2}\boldsymbol{u}^\top \boldsymbol{Q}\boldsymbol{u} + \boldsymbol{c}^\top \boldsymbol{u}, \qquad \boldsymbol{Q} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}, \qquad \boldsymbol{c} = \begin{bmatrix} -8 \\ -2 \end{bmatrix}.$$

Geometrically, the feasible set is the portion of the line $u_1+2u_2 = 3$ contained in the disk $u_1^2+u_2^2 \le 2$. This example illustrates both equality and inequality constraints in a relatively simple setting, but not so simple that the solution is immediately obvious. So using constrained optimization methodology is compelling. The parameter choices in this example were aided by Gemini.

1. Convexity and uniqueness

The Hessian of the objective is

$$\nabla^2 f(\boldsymbol{u}) = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \succ \boldsymbol{0}.$$

Thus, f is strictly convex. The equality constraint h is affine, and the inequality function g is convex because

$$\nabla^2 g(\boldsymbol{u}) = 2\boldsymbol{I} \succ \boldsymbol{0}.$$

Therefore, (1) is a convex optimization problem. Moreover, strict convexity of the objective implies that there can be at most one optimal point.

2. Slater's condition

In the presence of an equality constraint, Slater's condition requires a point that satisfies the affine equality exactly and the inequality constraint(s) strictly.
As illustrated in Fig. 1, consider

$$\bar{\boldsymbol{u}} = \begin{bmatrix} 3/5 \\ 6/5 \end{bmatrix}.$$

It satisfies the equality constraint:

$$h(\bar{\boldsymbol{u}}) = \frac{3}{5} + 2\left(\frac{6}{5}\right) - 3 = 0.$$

It also satisfies the inequality strictly:

$$g(\bar{\boldsymbol{u}}) = \left(\frac{3}{5}\right)^2 + \left(\frac{6}{5}\right)^2 - 2 = \frac{9}{5} - 2 = -\frac{1}{5} < 0.$$

Hence, Slater's condition holds. Because this is a convex problem, it follows that strong duality holds, optimal multipliers exist, and the Karush-Kuhn-Tucker (KKT) conditions are necessary and sufficient for optimality. So we first tackle the problem via the Lagrangian and KKT conditions.

3. Lagrangian and KKT conditions

Let $\nu \in \mathbb{R}$ denote the unrestricted multiplier associated with the equality constraint, and let $\lambda \geq 0$ denote the multiplier associated with the inequality constraint. The Lagrangian is

$$\begin{aligned} L(\boldsymbol{u}, \nu, \lambda) &= f(\boldsymbol{u}) + \lambda g(\boldsymbol{u}) + \nu h(\boldsymbol{u}) \\ &= 2u_1^2 + u_2^2 - 8u_1 - 2u_2 + \lambda(u_1^2 + u_2^2 - 2) + \nu(u_1 + 2u_2 - 3). \end{aligned} \tag{2}$$

The KKT conditions are as follows.

Stationarity: (gradient w.r.t. $\boldsymbol{u}$ is zero):

$$\begin{aligned} 4u_1 - 8 + \nu + 2\lambda u_1 &= 0 \\ 2u_2 - 2 + 2\nu + 2\lambda u_2 &= 0. \end{aligned}$$

Primal feasibility:

$$\begin{aligned} g(\boldsymbol{u}) &= u_1^2 + u_2^2 - 2 \leq 0 \\ h(\boldsymbol{u}) &= u_1 + 2u_2 - 3 = 0. \end{aligned}$$

Dual feasibility:

$$\lambda \geq 0.$$

Complementary slackness:

$$\lambda\,(u_1^2 + u_2^2 - 2) = 0.$$

Note that ν is unrestricted in sign, because it is associated with an equality constraint.

4. Is the inequality constraint active?

First suppose that the inequality constraint is inactive, i.e., $g(\boldsymbol{u}) < 0$. Complementary slackness then gives $\lambda = 0$. The stationarity equations and equality constraint simplify to three equations in three unknowns:

$$\begin{aligned} 4u_1 + \nu &= 8 \\ u_2 + \nu &= 1 \\ u_1 + 2u_2 &= 3. \end{aligned}$$

The first two equations give

$$u_1 = 2 - \nu/4, \qquad u_2 = 1 - \nu.$$

Substitution into the equality constraint gives

$$2 - \nu/4 + 2(1 - \nu) = 3, \implies \nu = 4/9.$$

Therefore, the minimizer subject only to the equality constraint is

$$\boldsymbol{u}_{\text{eq}} = \begin{bmatrix} 17/9 \\ 5/9 \end{bmatrix}.$$

However,

$$g(\boldsymbol{u}_{\text{eq}}) = \left(\frac{17}{9}\right)^2 + \left(\frac{5}{9}\right)^2 - 2 = \frac{314}{81} - 2 > 0.$$

Thus, $\boldsymbol{u}_{\text{eq}}$ violates the disk constraint. Our assumption that the inequality is inactive is therefore impossible. The inequality *must* be active at the solution.

5. Solving the active constraints

Because the inequality is active, the solution must satisfy

$$u_1 + 2u_2 = 3, \qquad u_1^2 + u_2^2 = 2.$$

This example is simple enough that we can proceed by examining the constraints. Other problems need the dual that is illustrated later.
From the equality constraint:

$$u_1 = 3 - 2u_2.$$

Substituting into the active inequality gives

$$(3 - 2u_2)^2 + u_2^2 = 2 \implies 5u_2^2 - 12u_2 + 7 = 0 \implies (5u_2 - 7)(u_2 - 1) = 0.$$

Thus, there are two candidate points:

$$\boldsymbol{u}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{u}^{(2)} = \begin{bmatrix} 1/5 \\ 7/5 \end{bmatrix}.$$

6. Testing the KKT candidates

At $\boldsymbol{u}^{(1)} = (1, 1)$, the stationarity equations become

$$-4 + \nu + 2\lambda = 0, \qquad 2\nu + 2\lambda = 0.$$

Solving gives

$$\nu = -4, \qquad \lambda = 4.$$

Since $\lambda = 4 \geq 0$, this candidate satisfies dual feasibility.
At $\boldsymbol{u}^{(2)} = (1/5, 7/5)$, stationarity becomes

$$-\frac{36}{5} + \nu + \frac{2}{5}\lambda = 0, \qquad \frac{4}{5} + 2\nu + \frac{14}{5}\lambda = 0.$$

Solving gives

$$\nu = \frac{44}{5}, \qquad \lambda = -4.$$

This candidate violates dual feasibility because $\lambda < 0$. So it cannot satisfy the KKT conditions. Consequently, the unique solution and its multipliers are

$$\boxed{\boldsymbol{u}^* = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}, \qquad \boxed{\nu^* = -4}, \qquad \boxed{\lambda^* = 4}.$$

The optimal (primal) cost value is

$$p^* = f(1,1) = 2 + 1 - 8 - 2 = \boxed{-7}.$$

Because the problem is convex and Slater's condition holds, satisfying the KKT conditions proves global optimality. Strict convexity of f establishes uniqueness.

7. Direct solution of the dual problem

Because it is needed in other applications (including SVM), we also illustrate how to solve this problem using the dual function. In general, the dual function is

$$d(\nu, \lambda) = \inf_{\boldsymbol{u} \in \mathbb{R}^2} L(\boldsymbol{u}, \nu, \lambda), \qquad \nu \in \mathbb{R}, \quad \lambda \geq 0.$$

Writing the Lagrangian as

$$L(\boldsymbol{u}, \nu, \lambda) = (2 + \lambda)u_1^2 + (\nu - 8)u_1 + (1 + \lambda)u_2^2 + (2\nu - 2)u_2 - 3\nu - 2\lambda,$$

and minimizing separately over u_1 and u_2 gives

$$u_1(\nu, \lambda) = \frac{8 - \nu}{2(2 + \lambda)}, \qquad u_2(\nu, \lambda) = \frac{1 - \nu}{1 + \lambda}.$$

Therefore, our dual function is

$$d(\nu, \lambda) = -\frac{(\nu - 8)^2}{4(2 + \lambda)} - \frac{(\nu - 1)^2}{1 + \lambda} - 3\nu - 2\lambda. \tag{3}$$

The dual problem is

$$\max_{\nu \in \mathbb{R},\, \lambda \geq 0} d(\nu, \lambda).$$

We now solve this maximization directly by calculus.

Maximization over ν

For every fixed $\lambda \geq 0$, the function $d(\nu, \lambda)$ is a strictly concave quadratic function of ν. Its maximizer w.r.t. ν therefore satisfies

$$0 = \frac{\partial d}{\partial \nu} = -\frac{\nu - 8}{2(2 + \lambda)} - \frac{2(\nu - 1)}{1 + \lambda} - 3.$$

Multiplication by $2(2 + \lambda)(1 + \lambda)$ gives

$$-(\nu - 8)(1 + \lambda) - 4(\nu - 1)(2 + \lambda) - 6(2 + \lambda)(1 + \lambda) = 0.$$

After collecting terms, we can solve for the optimal ν in terms of λ:

$$(9 + 5\lambda)\nu = -6\lambda^2 - 6\lambda + 4 \implies \boxed{\nu(\lambda) = \frac{4 - 6\lambda - 6\lambda^2}{9 + 5\lambda}.}$$

Define the reduced dual objective

$$\phi(\lambda) = d\big(\nu(\lambda), \lambda\big), \qquad \lambda \geq 0.$$

Because $\nu(\lambda)$ maximizes $d(\nu, \lambda)$ over ν, the original dual problem reduces to the 1D problem:

$$\max_{\lambda \geq 0} \phi(\lambda).$$

Maximization over λ

We approach this 1D maximization using the total derivative:

$$\phi'(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\phi(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\lambda} d\big(\nu(\lambda), \lambda\big) = \underbrace{\frac{\partial d(\nu, \lambda)}{\partial \nu}\bigg|_{\nu=\nu(\lambda)}}_{=0} \nu'(\lambda) + \frac{\partial d(\nu, \lambda)}{\partial \lambda}\bigg|_{\nu=\nu(\lambda)} = \frac{\partial d(\nu, \lambda)}{\partial \lambda}\bigg|_{\nu=\nu(\lambda)}.$$

The first term is zero because $\nu(\lambda)$ is the maximizer of d w.r.t. ν for any given λ. This simplification is called the envelope theorem. Differentiating:

$$\frac{\partial}{\partial \lambda} d(\nu, \lambda) = \frac{(\nu - 8)^2}{4(2 + \lambda)^2} + \frac{(\nu - 1)^2}{(1 + \lambda)^2} - 2.$$

Substituting the expression for $\nu(\lambda)$ and simplifying substantially gives

$$\boxed{\phi'(\lambda) = -\frac{(\lambda - 4)(5\lambda + 38)}{(5\lambda + 9)^2}.}$$

For $\lambda \geq 0$, the factors $5\lambda + 38$ and $(5\lambda + 9)^2$ are positive. Consequently,

$$\phi'(\lambda) > 0 \quad \text{for } 0 \leq \lambda < 4,$$

and

$$\phi'(\lambda) < 0 \quad \text{for } \lambda > 4.$$

Thus, the unique maximizer of the reduced dual objective is

$$\boxed{\lambda^* = 4}.$$

Substitution into $\nu(\lambda)$ gives

$$\nu^* = \nu(4) = \frac{4 - 6(4) - 6(4)^2}{9 + 5(4)} = \frac{-116}{29} = \boxed{-4.}$$

Thus, the dual optimal multipliers have been obtained directly from the dual maximization:

$$\boxed{(\nu^*, \lambda^*) = (-4, 4).}$$

The associated minimizer of the Lagrangian is

$$u_1(\nu^*, \lambda^*) = \frac{8 - \nu^*}{2(2 + \lambda^*)} = \frac{12}{12} = 1$$
$$u_2(\nu^*, \lambda^*) = \frac{1 - \nu^*}{1 + \lambda^*} = \frac{5}{5} = 1.$$

Therefore,

$$\boldsymbol{u}^* = \boldsymbol{u}(\nu^*, \lambda^*) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Finally, evaluating the dual objective gives

$$d^* = d(-4, 4) - \frac{(-4-8)^2}{4(2+4)} - \frac{(-4-1)^2}{1+4} - 3(-4) - 2(4) = -7.$$

The primal calculation gave $p^* = -7$, so

$$\boxed{d^* = p^* = -7.}$$

Thus, the dual problem independently recovers both the optimal value and the primal optimizer, and the zero duality gap confirms strong duality.

8. Geometric interpretation

Completing the square gives

$$f(\boldsymbol{u}) = 2(u_1 - 2)^2 + (u_2 - 1)^2 - 9.$$

Therefore, the unconstrained minimizer is

$$\boldsymbol{u}_{\text{unc}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

and the contours of the objective are axis-aligned ellipses centered at $(2, 1)$. See Fig. 1.
The optimal contour $f(\boldsymbol{u}) = -7$ is

$$2(u_1 - 2)^2 + (u_2 - 1)^2 = 2.$$

This contour passes through the optimal point $\boldsymbol{u}^* = (1, 1)$. At that point, the stationarity condition can be written geometrically as

$$\nabla f(\boldsymbol{u}^*) + \nu^* \nabla h(\boldsymbol{u}^*) + \lambda^* \nabla g(\boldsymbol{u}^*) = \boldsymbol{0}.$$

Indeed,

$$\nabla f(1, 1) = \begin{bmatrix} -4 \\ 0 \end{bmatrix}, \qquad \nabla h(1, 1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad \nabla g(1, 1) = \begin{bmatrix} 2 \\ 2 \end{bmatrix},$$

and

$$\begin{bmatrix} -4 \\ 0 \end{bmatrix} - 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

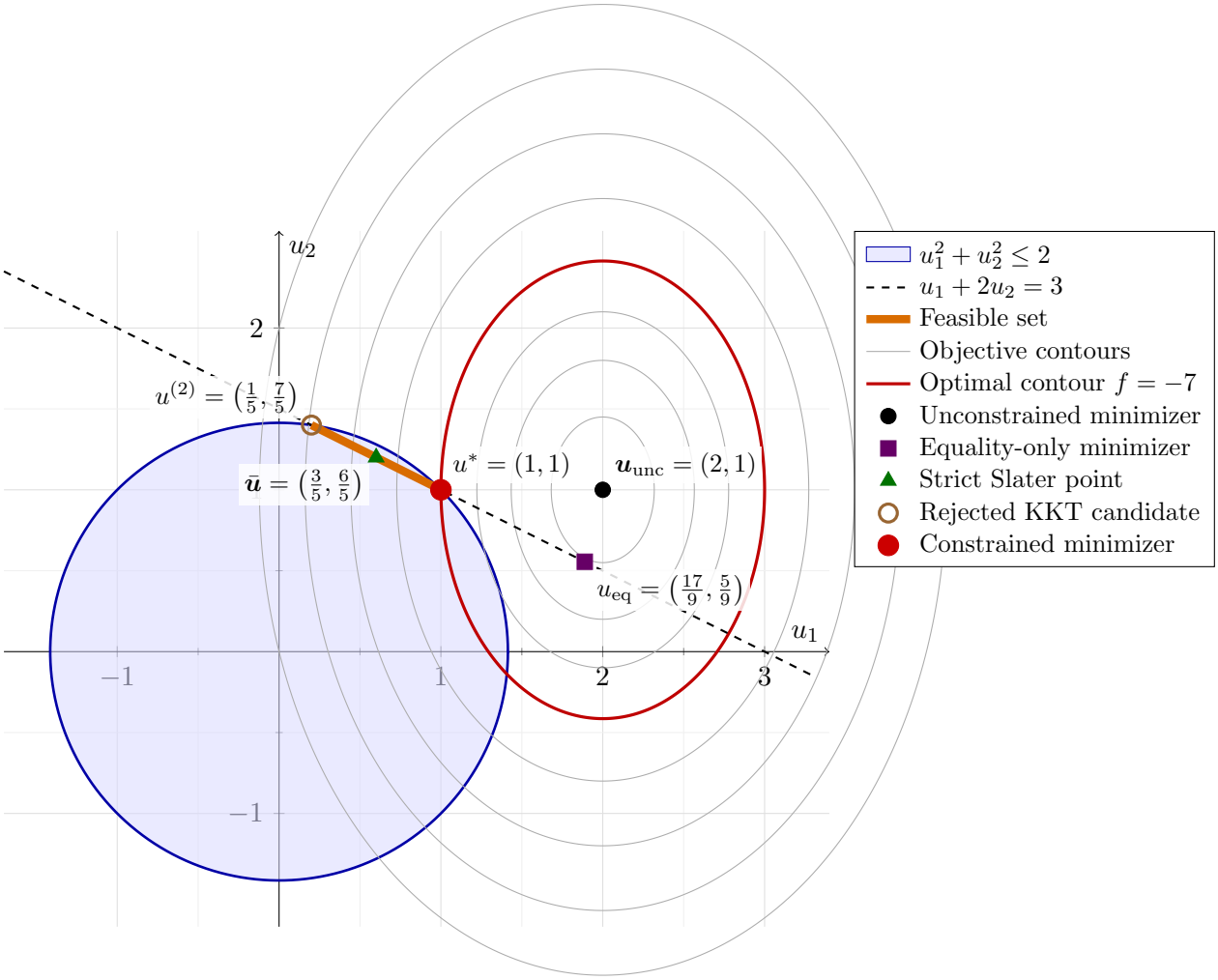


Figure 1: The feasible set is the thick orange segment of the equality line $u_1 + 2u_2 = 3$ contained in the disk $u_1^2 + u_2^2 \leq 2$. The equality-only minimizer lies outside the disk, so the inequality constraint must be active. Of the two endpoints of the feasible segment, only $\boldsymbol{u}^* = (1, 1)$ satisfies the KKT conditions with $\lambda \geq 0$. The red optimal contour $f(\boldsymbol{u}) = -7$ passes through $\boldsymbol{u}^*$.