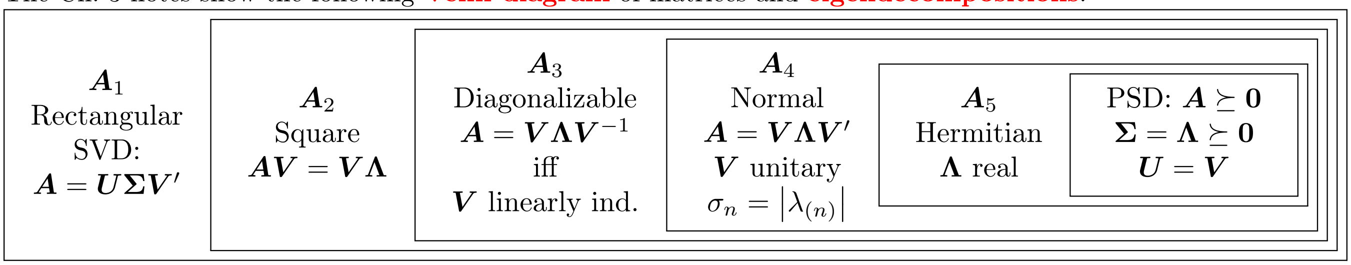
Pr. 1.

(a) Let $\boldsymbol{Q}$ denote an $M \times K$ matrix having **orthonormal** columns.
Show that $\boldsymbol{Q}'\boldsymbol{Q} = \boldsymbol{I}_K$. Thus $\|\boldsymbol{Q}\boldsymbol{x}\|_2 = \|\boldsymbol{x}\|_2$ for all $\boldsymbol{x} \in \mathbb{C}^K$.

(b) Show that the following converse is true:
if $\boldsymbol{A}$ is an $M \times K$ matrix for which $\|\boldsymbol{A}\boldsymbol{x}\|_2 = \|\boldsymbol{x}\|_2$ for all $\boldsymbol{x} \in \mathbb{C}^K$, then $\boldsymbol{A}'\boldsymbol{A} = \boldsymbol{I}_K$.
Your proof should be general enough to cover the case where $\boldsymbol{A}$ has complex elements.
Hint. Examine products with standard unit vectors $\boldsymbol{e}_j$ and combinations like $\boldsymbol{e}_j + \boldsymbol{e}_k$, or consider an eigendecomposition of $\boldsymbol{A}'\boldsymbol{A} - \boldsymbol{I}$.

Pr. 2.
Let $\boldsymbol{A}$ be an $N \times N$ Hermitian matrix with unitary eigendecomposition $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}'$, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0$ and $0 \geq \lambda_{k+1} \geq \ldots \geq \lambda_N$, for some $1 < k < N$. Determine an **SVD** of $\boldsymbol{A}$ in terms of the given components.

Pr. 3.
The Ch. 3 notes show the following **Venn diagram** of matrices and **eigendecompositions**:



Each of the categories shown above is a *strict superset* of the categories nested with in it.
Provide example matrices $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_5$ that belong to the each of the categories above but *not* the next category nested within it. Try to provide the simplest possible example in each case.
For example, the matrix $\boldsymbol{A}_1 = [0 \; 1]$ is rectangular, but not square. Now you do $\boldsymbol{A}_2, \ldots, \boldsymbol{A}_5$.
Hint. All the examples can be 1×1 or 2×2, often with simple "0" and "1" elements.

Pr. 4.
Let $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(a) Determine the **null space** of $\boldsymbol{A}$, denoted by $\mathcal{N}(\boldsymbol{A})$, and the **range space** or column space of $\boldsymbol{A}$ denoted by $\mathcal{R}(\boldsymbol{A})$.

(b) Are they equal? Does your answer hold in general? If not, provide a counterexample.

Pr. 5.
Let $\boldsymbol{A} \in \mathbb{F}^{M \times N}$.

(a) Suppose $\boldsymbol{W} \in \mathbb{F}^{M \times M}$ and $\boldsymbol{Q} \in \mathbb{F}^{N \times N}$ are each **unitary** matrices.

Show that $\boldsymbol{A}$ and $\boldsymbol{C} \triangleq \boldsymbol{W}\boldsymbol{A}\boldsymbol{Q}$ have the same singular values.
Consequently, $\boldsymbol{A}$ and $\boldsymbol{C}$ have the same **rank**, the same Frobenius norm and the same operator norm. This is why the Frobenius norm and the operator norm are called **unitarily invariant** norms. Their value does not change when the matrix is multiplied from the left and/or the right by a unitary matrix. Any norm that depends only on the singular values of $\boldsymbol{A}$ will, by definition, be unitarily invariant.

(b) Suppose that $\boldsymbol{W}$ and $\boldsymbol{Q}$ are **nonsingular** but not necessarily unitary matrices. Do $\boldsymbol{A}$ and $\boldsymbol{D} \triangleq \boldsymbol{W}\boldsymbol{A}\boldsymbol{Q}$ have the same **rank**? Prove or give a counterexample.

(c) Continuing (b), do $\boldsymbol{A}$ and $\boldsymbol{D} = \boldsymbol{W}\boldsymbol{A}\boldsymbol{Q}$ have the same singular values? Prove or give a counterexample.

Pr. 6.
Prove that the **orthogonal complement** of the **range** of a matrix $\boldsymbol{A}$ equals the **null space** of $\boldsymbol{A}'$: $\mathcal{R}^{\perp}(\boldsymbol{A}) = \mathcal{N}(\boldsymbol{A}')$.

Pr. 7.

Let $\boldsymbol{A} = \boldsymbol{x}\boldsymbol{y}'$, where neither $\boldsymbol{x}$ nor $\boldsymbol{y}$ is $\boldsymbol{0}$.

(a) How many **linearly independent** columns does $\boldsymbol{A}$ have? I.e., of all the sets of columns from this matrix where the set is linearly independent, what is the maximum cardinality?

(b) What is the **rank** of $\boldsymbol{A}$?

(c) Enter (cut and paste) this code into `Julia`:

```
using LinearAlgebra: svdvals
using Plots
using LaTeXStrings
n = 100
x = randn(n); y = randn(n); A = x*y'
s = svdvals(A)
scatter(s, yscale = :log10, label= "",
    title = "singular values: log scale",
    xlabel="i", ylabel=L"\sigma_i") # to make σ_i
```

(Ideally the ylabel should look like σ_i, but if it does not, it is OK. Comment out the `ylabel` part if needed.)

- How many numerically computed singular values of $\boldsymbol{A}$ are nonzero?
- What answer do you get when you type `rank(A)` ?
- Turn in your plot of the singular values and the answers.

You will have just seen that a theoretically rank-1 matrix will have multiple nonzero singular values when expressed in machine precision arithmetic; this property is due to roundoff errors (finite numerical precision). (Round-off errors should not be thought of as "wrong answers," but just a fact of real-life computing to be understood and not feared.)

(d) To help locate the code for the `rank` function, type this into `Julia`:

`@less rank(ones(3,2))`

Examine the code for the `rank` function and write down the formula used to determine the threshold (aka tolerance) below which, due to roundoff errors, the code considers the singular value to be "zero."

(e) Determine the numerical value of the threshold when $\boldsymbol{A} = [9\ 9]$.

(f) Optional. Check your answer to the previous part by using the `Julia` debugger to step into the `rank` function and examine the tolerance `tol` .

Pr. 8.

(a) Prove the **vec trick** , *i.e.*, for $\boldsymbol{A} \in \mathbb{F}^{P\times M}$, $\boldsymbol{X} \in \mathbb{F}^{M\times N}$, $\boldsymbol{B} \in \mathbb{F}^{N\times Q}$. :

$$\operatorname{vec}(\boldsymbol{A}\boldsymbol{X}\boldsymbol{B}) = (\boldsymbol{B}^\top \otimes \boldsymbol{A})\operatorname{vec}(\boldsymbol{X}), \tag{1}$$

where **vec(.)** is the operator that stacks the columns of the input matrix into a vector, and $\otimes$ denotes the **Kronecker product**. Here we really mean transpose $\boldsymbol{B}^\top$ even if $\boldsymbol{B}$ is complex valued!

(b) Suppose $\boldsymbol{A}$, $\boldsymbol{X}$, and $\boldsymbol{B}$ are all $N \times N$ dense matrices. Determine how many scalar-scalar multiplications are needed for the LHS $\operatorname{vec}(\boldsymbol{A}\boldsymbol{X}\boldsymbol{B})$ and compare to the number for the RHS $(\boldsymbol{B}^\top \otimes \boldsymbol{A})\operatorname{vec}(\boldsymbol{X})$. Which version uses fewer?

One use of (1) is analyzing the 2D **discrete Fourier transform** (DFT), which may be explored in later HW.

Pr. 9.

(a) Find an orthonormal basis for the range of the $M \times N$ matrix that is all ones.

(b) Find an orthonormal basis for the range of $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$.

(c) Find an orthonormal basis for the range of any 2×2 rotation matrix.

(d) Use the fact that $\mathcal{R}(\boldsymbol{A}) = \mathcal{R}(\boldsymbol{U}_r)$ to modify the **compact SVD** code in §4.5 to write a function that returns an orthonormal basis for the range of a given matrix, akin to how `nullspace` returns a null space basis.

In `Julia`, your file should be named `matrixrange.jl` and should contain the following function:

```
"""
    matrixrange(A::AbstractMatrix)
Return orthonormal basis matrix for the range of matrix `A`.
"""
function matrixrange(A::AbstractMatrix)
```

Email your solution as an attachment to `eecs551@autograder.eecs.umich.edu`.

Check your code using your answers to the preceding parts.

(e) Optional. Find an orthonormal basis for the range of any 2×2 rotation matrix, such that the basis matrix has the smallest possible trace.

Non-graded problem(s) below
(Solutions will be provided for self check; do not submit to gradescope.)

Pr. 10.
Let $\boldsymbol{A}$ be a **normal** matrix of the (unnamed) type where each eigenvalue either has a different magnitude than all other eigenvalues, or has the same value as all other eigenvalues with its magnitude. In other words, having both $|\lambda_j| = |\lambda_i|$ and $\lambda_j \neq \lambda_i$ is not allowed. For example, $\boldsymbol{A}$ might have eigenvalues $(3, 4\imath, 4\imath, -5)$ but cannot have eigenvalues $(3, 4, 4\imath, -5)$.
Prove that every right singular vector of $\boldsymbol{A}$ is also an eigenvector of $\boldsymbol{A}$.
This problem finishes the story on relating **SVD** and eigendecomposition.

Pr. 11.
We write $\boldsymbol{A} \preceq \boldsymbol{B}$ iff $\boldsymbol{B} - \boldsymbol{A} \succeq \boldsymbol{0}$, *i.e.*, iff $\boldsymbol{B} - \boldsymbol{A}$ is **positive semidefinite**.
For $a > 0$, determine $\left\{ x \in \mathbb{R} : x \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \preceq \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \right\}$.

Pr. 12.
Suppose that $\boldsymbol{X} = \boldsymbol{U}_x \boldsymbol{\Sigma}_x \boldsymbol{V}_x'$ and $\boldsymbol{Y} = \boldsymbol{U}_y \boldsymbol{\Sigma}_y \boldsymbol{V}_y'$ denote **SVD** s of the matrices $\boldsymbol{X} \in \mathbb{F}^{m \times n}$ and $\boldsymbol{Y} \in \mathbb{F}^{p \times q}$.

(a) Show that

$$\boldsymbol{X} \otimes \boldsymbol{Y} = (\boldsymbol{U}_x \otimes \boldsymbol{U}_y)(\boldsymbol{\Sigma}_x \otimes \boldsymbol{\Sigma}_y)(\boldsymbol{V}_x \otimes \boldsymbol{V}_y)'$$

is an SVD of $\boldsymbol{X} \otimes \boldsymbol{Y}$ (up to a permutation of the singular values). The block diagonal matrix $\boldsymbol{\Sigma}_x \otimes \boldsymbol{\Sigma}_y$ contains the pairwise *products* of the singular values of $\boldsymbol{X}$ and $\boldsymbol{Y}$.

Hint. First show that the formula is correct, then argue that it is an SVD by showing that $\boldsymbol{U}_x \otimes \boldsymbol{U}_y$ and $\boldsymbol{V}_x \otimes \boldsymbol{V}_y$ are unitary matrices.

Hint: You may find the following properties of **Kronecker product** useful:

- $(\boldsymbol{X} \otimes \boldsymbol{Y})' = \boldsymbol{X}' \otimes \boldsymbol{Y}'$
- $(\boldsymbol{X} \otimes \boldsymbol{Y})(\boldsymbol{W} \otimes \boldsymbol{Z}) = (\boldsymbol{X}\boldsymbol{W}) \otimes (\boldsymbol{Y}\boldsymbol{Z})$

(b) Now suppose that $\boldsymbol{A} = \boldsymbol{Q}_a \boldsymbol{\Lambda}_a \boldsymbol{Q}_a'$ and $\boldsymbol{B} = \boldsymbol{Q}_b \boldsymbol{\Lambda}_b \boldsymbol{Q}_b'$ are unitary **eigendecompositions** of the (normal) matrices $\boldsymbol{A} \in \mathbb{F}^{N \times N}$ and $\boldsymbol{B} \in \mathbb{F}^{M \times M}$.

Show that

$$\boldsymbol{A} \otimes \boldsymbol{B} = (\boldsymbol{Q}_a \otimes \boldsymbol{Q}_b)(\boldsymbol{\Lambda}_a \otimes \boldsymbol{\Lambda}_b)(\boldsymbol{Q}_a \otimes \boldsymbol{Q}_b)'$$

is a unitary eigendecomposition of $\boldsymbol{A} \otimes \boldsymbol{B}$. The diagonal matrix $\boldsymbol{\Lambda}_a \otimes \boldsymbol{\Lambda}_b$ contains the pairwise *products* of the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{B}$.

Hint. First show that the formula is correct, then argue that it is an eigendecomposition by showing that $\boldsymbol{Q}_a \otimes \boldsymbol{Q}_b$ is a unitary matrix.

(c) Continuing (b), show that

$$\boldsymbol{A} \oplus \boldsymbol{B} = (\boldsymbol{Q}_a \otimes \boldsymbol{Q}_b)(\boldsymbol{\Lambda}_a \oplus \boldsymbol{\Lambda}_b)(\boldsymbol{Q}_a \otimes \boldsymbol{Q}_b)'$$

is an eigendecomposition of the **Kronecker sum** $\boldsymbol{A} \oplus \boldsymbol{B} \triangleq (\boldsymbol{A} \otimes \boldsymbol{I}) + (\boldsymbol{I} \otimes \boldsymbol{B})$, where you must determine the size(s) of the "$\boldsymbol{I}$" in that expression. The diagonal matrix $\boldsymbol{\Lambda}_a \oplus \boldsymbol{\Lambda}_b$ contains the pairwise *sums* of the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{B}$.

Hint. Start by writing $\boldsymbol{I}_N = \boldsymbol{Q}_a \boldsymbol{I}_N \boldsymbol{Q}_a'$ and $\boldsymbol{I}_M = \boldsymbol{Q}_b \boldsymbol{I}_M \boldsymbol{Q}_b'$ in the definition of $\boldsymbol{A} \oplus \boldsymbol{B}$, and then apply (b).

Pr. 13.

Complete the "Introduction to Matrix Math" tutorial at `https://pathbird.com/courses/register`. (Use the registration code shown on the ECE 551 Canvas home page.) This tutorial should be helpful for anyone who wants deeper understanding of `Julia` for operations with matrices and vectors. Such operations are a major part of ECE 551.

Pr. 14.

For this problem you will show that projecting onto a subspace $\mathcal{S} \subseteq \mathbb{F}^N$ is a linear operation, i.e., there exists an $N \times N$ matrix $\boldsymbol{P}$ such that $\mathcal{P}_{\mathcal{S}}(\boldsymbol{v}) = \boldsymbol{P}\boldsymbol{v},\ \forall \boldsymbol{v} \in \mathbb{F}^N$.

(a) Using the subspace decomposition theorem in Ch. 4, any $\boldsymbol{v} \in \mathcal{V}$ has a unique decomposition $\boldsymbol{v} = \boldsymbol{x} + \boldsymbol{z}$, where $\boldsymbol{x} \in \mathcal{S}$ and $\boldsymbol{z} \in \mathcal{S}^\perp$. Show that $\mathcal{P}_{\mathcal{S}}(\boldsymbol{v}) = \mathcal{P}_{\mathcal{S}}(\boldsymbol{x} + \boldsymbol{z}) = \boldsymbol{x}$.

(b) Show that $\mathcal{P}_{\mathcal{S}}(\boldsymbol{v} + \boldsymbol{u}) = \mathcal{P}_{\mathcal{S}}(\boldsymbol{v}) + \mathcal{P}_{\mathcal{S}}(\boldsymbol{u})$.

(c) Show that $\mathcal{P}_{\mathcal{S}}(\alpha \boldsymbol{v}) = \alpha \mathcal{P}_{\mathcal{S}}(\boldsymbol{v})$.

(d) Describe how to construct the matrix $\boldsymbol{P}$ from the projection operator $\mathcal{P}_{\mathcal{S}}(\cdot)$.

(e) Find $\boldsymbol{P}$ when $\mathcal{V} = \mathbb{F}^3$ and $\mathcal{S} = \operatorname{span}\left(\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}\right)$.
