Now we delve deeper into the physics and the corresponding mathematics.

5.1 Bloch equation

In a heterogeneous object, all $M$’s and $B$’s depend on time and spatial location, i.e., $\vec{M}(r, t) = \vec{M}(x, y, z, t)$ where $r = (x, y, z)$. We need a mathematical model how the input $\vec{B}$ affects the object’s magnetization $\vec{M}$.

The **Bloch equation** provides a phenomenological description of time evolution of local magnetization:

$$\frac{d}{dt} \vec{M} = \vec{M} \times \gamma \vec{B}_\text{precession} - \frac{M_x \vec{i} + M_y \vec{j}}{T_2 \text{relaxation}} - \frac{(M_x - M_y) \vec{k}}{T_1 \text{to equilibrium}}.$$

where $\vec{i}$ and $\vec{j}$ are unit vectors in $x$ and $y$ directions respectively. Note that every quantity above varies with position $r$.

This equation describes how the magnetization evolves over time in response to the (mostly) external input field $\vec{B}$ as well as due to the internal relaxation processes.

The equation captures the three key phenomena:

- precession,
- transverse and longitudinal relaxation,
- equilibrium.

Ignoring chemical shift, the external input field $\vec{B}$ is composed of

- main field $B_0 = B_{20} \vec{k}$,
- RF field $B_1(t)$,
- longitudinal field strength gradients $\left( r \cdot \vec{G}(t) \right) \vec{k} = (x G_x(t) + y G_y(t) + z G_z(t)) \vec{k}$.

The RF pulse $B_1(t)$ and gradient waveforms $\vec{G}(t)$ are user-controlled.

Recall that the cross product of two vectors $\vec{u}, \vec{v} \in \mathbb{R}^3$ is defined:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix} \vec{i} - \begin{vmatrix} u_x & u_z \\ v_x & v_z \end{vmatrix} \vec{j} + \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \vec{k} = -(\vec{v} \times \vec{u}) = \vec{e}_\perp ||\vec{u}|| ||\vec{v}|| \sin \theta_{uv}$$

where $\vec{e}_\perp$ is the unit vector perpendicular to $\vec{u}$ and $\vec{v}$, and $\theta_{uv}$ is the smaller angle between $\vec{u}$ and $\vec{v}$.

- Solutions to the Bloch equation are **time-shift invariant**, i.e., $t = 0$ is arbitrary.
- Each point in space evolves independently. (This version ignores diffusion, flow, other motion [12].)

**Why must we influence the magnetization on readout?**

The RF receiver responds (essentially) to the entire volume due the long RF wavelengths, so there is very little spatial localization on transmit or receive. It would be nice conceptually if we could somehow excite each voxel sequentially and then “listen” for the RF signal returning from that voxel, the amplitude of which would be related to the spin density $\rho$ for that voxel, and the exponential decay rate would be related to the $T_2$ for that voxel. We could thereby build a picture of $\rho(x, y, z_0)$ or of $T_2(x, y, z_0)$ for some slice $z_0$ one voxel at a time.

Because this is impractical, our principal goal now, as we focus on the readout part, is to understand how the user-controlled field gradients $\vec{G}(t)$ affect the time-evolution of the magnetization. This requires that we understand the relationship between spatial location and frequency components of the received signal. (The amplitude of each frequency component is proportional to the magnetization at some location(s).)

For now we focus on the **readout** phase of MR pulse sequence, and return to excitation later.
Solutions to the Bloch equation when applied $\text{RF}=0$

Unfortunately, there is no general closed-form solution to the Bloch equation. However, when there is no applied RF, there are explicit solutions that are easily interpreted. When the applied magnetic field $\vec{B}$ is oriented in the $z$ direction, i.e., $\vec{B}(r, t) = B_z(r, t) \hat{k}$, then by expanding the cross product in the Bloch equation we can write:

\[
\frac{d}{dt} \vec{M}(r, t) = \begin{bmatrix}
-\frac{1}{T_2(r)} & \omega(r, t) & 0 \\
-\omega(r, t) & -\frac{1}{T_2(r)} & 0 \\
0 & 0 & -\frac{1}{T_1(r)}
\end{bmatrix} \vec{M}(r, t) + \begin{bmatrix}
0 \\
0 \\
\frac{M_{20}(r)}{T_1(r)}
\end{bmatrix},
\]

where $\omega(r, t) = \gamma B_z(r, t)$ is the local Larmor frequency. Typically $B_z(r, t) = B_{z0} + r \cdot \vec{G}(t)$, so

\[
\omega(r, t) = \gamma B_z(r, t) = \gamma (B_{z0} + r \cdot \vec{G}(t)) = \omega_0 + \gamma r \cdot \vec{G}(t).
\]

Remarkably, the transverse and longitudinal components separate in this case, greatly simplifying the solution to the PDE.

The differential equation for the longitudinal component of the magnetization is

\[
\frac{d}{dt} M_z(r, t) = -\frac{1}{T_1(r)} M_z(r, t) + \frac{M_{20}(r)}{T_1(r)} \quad \text{or } \quad \frac{d}{dt} [M_x(r, t) - M_{z0}(r)] = -\frac{1}{T_1(r)} [M_x(r, t) - M_{z0}(r)].
\]

It is easy to verify that the solution to this differential equation is:

\[
M_z(r, t) = M_{z0}(r) \left(1 - e^{-t/T_1(r)}\right) + e^{-t/T_1(r)} M_z(r, 0), \quad t \geq 0.
\]

Note that (when RF=0), $\vec{B}$ has no effect on the longitudinal component; that component simply relaxes back to its equilibrium value, as illustrated below.

Our primary interest is in the transverse components of the magnetization, because these induce signals in RF receive coils. The differential equation for the transverse component of the magnetization is

\[
\frac{d}{dt} \begin{bmatrix} M_x(r, t) \\ M_y(r, t) \end{bmatrix} = \begin{bmatrix}
-\frac{1}{T_2(r)} & \omega(r, t) \\
-\omega(r, t) & -\frac{1}{T_2(r)}
\end{bmatrix} \begin{bmatrix} M_x(r, t) \\ M_y(r, t) \end{bmatrix}.
\]

This differential equation is easier to solve if we combine the two transverse components using complex notation. Specifically, we define

\[
M(r, t) \triangleq M_{xy}(r, t) = M_x(r, t) + i M_y(r, t)
\]

where $i = \sqrt{-1}$. (Note that this $M$ is subscript free. It is the same as $M_{xy}(r, t)$ defined earlier.) Note that the physical quantities involved, i.e., $M_x$ and $M_y$, are both real. We make the choice to define a complex quantity $M(r, t)$ by combining these two real quantities because that choice simplifies the analysis. Using this representation, we can write the differential equation as simply:

\[
\frac{d}{dt} M(r, t) = \left(-\frac{1}{T_2(r)} - i \omega(r, t)\right) M(r, t).
\]
This differential equation is solved very easily in the case where the applied field $\vec{B}$ is static, because then $\omega(\vec{r})$ is independent of time $t$. In this case one can see directly that the solution is

$$M(\vec{r}, t) = M(\vec{r}, 0) e^{-i\omega(\vec{r}) t} e^{-t/T_2(\vec{r})}, \quad t \geq 0.$$ 

More generally, note that a differential equation of the form $\frac{d}{dt} f(t) = a(t) f(t)$ has the solution for $t \geq 0$:

$$f(t) = f(0) e^{\int_0^t a(\tau) d\tau}.$$ 

Here the “coefficient” of the differential equation is $a(t) = -\frac{1}{T_2(\vec{r})} - i\omega(\vec{r}, t)$.

Thus the general solution is

$$M(\vec{r}, t) = M(\vec{r}, 0) \exp\left(\int_0^t -\frac{1}{T_2(\vec{r})} - i\omega(\vec{r}, \tau) d\tau\right) = M(\vec{r}, 0) e^{-t/T_2(\vec{r})} e^{-\int_0^t \gamma B_0(\vec{r}, \tau) d\tau}.$$ 

**Case 1: Homogeneous object, static and uniform field**
- $\vec{M}(\vec{r}, t) = \vec{M}(t)$ (independent of spatial position $\vec{r}$)
- static: $\vec{B}(\vec{r}, t) = \vec{B}(\vec{r})$ independent of time
- uniform: $\vec{B}(\vec{r}) = B_0$ has uniform field strength over volume, $\vec{B}(\vec{r}, t) = B_0 = B_{Z0}\vec{k}$

Solution:

$$\vec{M}(t) = \begin{bmatrix} e^{-t/T_2} \\
-\sin \omega_0 t & \cos \omega_0 t & 0 \\
\cos \omega_0 t & \sin \omega_0 t & 0 \\
0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\
M(0) \\
M_{Z0} \left(1 - e^{-t/T_1}\right) \end{bmatrix},$$

where $\omega_0 = \gamma B_{Z0}$ is the resonant frequency corresponding to the main field.

In the complex representation, for this case the magnetization is a decaying complex exponential:

$$M(t) = M(0) e^{-i\omega_0 t} e^{-t/T_2}, \quad t \geq 0.$$ 

- Initial condition (determined by RF excitation and local spin density)
- Precession at Larmor frequency (all spins at same rate because uniform field)
- Relaxation (decay of transverse component)

You might think of the relaxation functions described previously as “consequences” of this differential equation, but really it is the other way around: the differential equation was developed to match the required physical properties.
We use the resonant frequency as a nominal reference (due to baseband operation) and express other frequencies as a time-varying difference relative to an on-resonance spin at the scanner center.

To review again:
- $M(r, t)$: local magnetization evolving over time (in complex notation to combine $x$ and $y$ components)
- $M(r, 0)$: initial local magnetization (following excitation)
- Precession: $e^{-\omega_0 t}$
- Transverse relaxation $M(r, t) \to 0$ as $t \to \infty$
- Phase, due to the nonuniform field strength, means that spins precess at different rates (due to gradients).

We use the resonant frequency as a nominal reference (due to baseband operation) and express other frequencies as a time-varying phase.

**Summary**

The above equation describes how the magnetization (ideally) evolves over time after RF turned off (which usually defines $t = 0$).

Key design parameters are $\vec{G}(t)$.

How can we manipulate $\vec{G}(t)$ to control signal to make image?

To answer that, we must first answer the question: What is the observed signal?
5.2

Signal equation

The time-varying magnetization $M(r, t)$, that is evolving according to the fundamental equation of NMR, induces an electromotive force (EMF) in one or more neighboring receiver coils by Faraday’s law of induction.

For simplicity, here we assume an ideal RF receiver coil, i.e.:

- uniformly sensitive over volume of interest, with sensitivity $B_{1xy}$,
- detects flux changes in transverse direction.
- noiseless.

Then the received signal is

$$s_r(t) \triangleq \iiint B_{1xy} \frac{\partial}{\partial t} M(r, t) \, dr = \iiint B_{1xy} \frac{\partial}{\partial t} M(x, y, z, t) \, dx \, dy \, dz$$

$$= \iiint B_{1xy} \frac{\partial}{\partial t} \left[ M(r, 0) e^{-i\omega_0 t} e^{-i\phi(r, t)} e^{-t/T_2(r)} \right] \, dr \approx -i\omega_0 B_{1xy} e^{-i\omega_0 t} \iiint M(r, 0) e^{-t/T_2(r)} e^{-i\phi(r, t)} \, dr,$$

where the last approximation is essentially a narrowband approximation, which is reasonable because the time scales of relaxation and phase changes are much slower than the RF carrier frequency.

Next we demodulate the received signal to form a baseband signal. In practice this is done using I/Q channels just like in ultrasound. In the complex representation, demodulation is equivalent mathematically to multiplying by $e^{i\omega_0 t}$:

$$s(t) \triangleq \frac{e^{i\omega_0 t}}{-i\omega_0 B_{1xy}} s_r(t) = \iiint M(r, 0) e^{-t/T_2(r)} e^{-i\phi(r, t)} \, dr.$$

The original received signal oscillates in the 10-100 MHz range whereas the baseband signal bandwidth is typically 1-10 kHz.

Now we make some further simplifications to this signal equation for analysis.

- Assume the readout interval where we record $s(t)$, say $[t_1, t_2]$, is small compared to $T_2$. Typically somewhere in that interval will be a point in time where the signal is particularly large, called the echo time, denoted $T_E$.
- Approximate the relaxation term by its value at the echo time: $e^{-t/T_2(r)} \approx e^{-T_E/T_2(r)}$ for $t_1 \leq t \leq t_2$. Thus

$$s(t) \approx \iiint M(r, 0) e^{-T_E/T_2(r)} e^{-i\phi(r, t)} \, dr$$

$$= \iiint M(x, y, z, 0) e^{-T_E/T_2(x, y, z)} e^{-t/2\pi[z k_x(t) + y k_y(t) + z k_z(t)]} \, dx \, dy \, dz,$$

because we rewrite the gradient-induced phase as follows:

$$\phi(r, t) = \gamma \int_0^t \mathbf{r} \cdot \mathbf{G}(\tau) \, d\tau = \gamma \int_0^t x G_x(\tau) \, d\tau + \gamma \int_0^t y G_y(\tau) \, d\tau + \gamma \int_0^t z G_z(\tau) \, d\tau$$

$$= 2\pi [x k_x(t) + y k_y(t) + z k_z(t)],$$

where we define the k-space trajectory in terms of the gradient waveforms by:

$$k_x(t) = \oint G_x(\tau) \, d\tau, \quad k_y(t) = \oint G_y(\tau) \, d\tau, \quad k_z(t) = \oint G_z(\tau) \, d\tau.$$

- Now we focus on 2D imaging, for which $G_z = 0$ during the readout and hence $k_z(t) = 0$. (See refocusing lobe later; that is part of excitation, not readout.) In other words, $\phi(r, t)$ is independent of $z$ for $t_1 \leq t \leq t_2$. This assumption is reasonable provided that we have used selective excitation to excite just a slice or then slab of the object. When $G_z = 0$ we can rewrite the signal model above as follows:

$$s(t) = \iiint \left[ \int M(x, y, z, 0) e^{-t/T_2(x, y, z)} \, dz \right] e^{-t/2\pi[z k_x(t) + y k_y(t)]} \, dx \, dy.$$
To further simplify we define the 2D image $m(x, y)$ to be the bracketed term:

$$m(x, y) \triangleq \int M(x, y, z, 0) e^{-t/T_2(x, y, z)} \, dz \approx \int_{z_0 - \delta_z/2}^{z_0 + \delta_z/2} M(x, y, z, 0) e^{-T_E/T_2(x, y, z)} \, dz \approx \delta_z M(x, y, z_0, 0) e^{-T_E/T_2(x, y, z_0)}.$$  

The exponential factor is called $T_2$ weighting.

Note that $m(x, y)$ is some function of $T_1(r)$, $T_2(r)$, and $\rho(r)$. It also depends on excitation parameters, especially timing $T_R$, $T_E$, and the slice selection process.

With this definition, the baseband signal model (for the 2D case where $G_z = 0$), and under all of the above approximations and simplifications, simplifies to the signal equation:

$$s(t) = \int \int m(x, y) e^{-\gamma^2 \pi [k_X(t)x + k_Y(t)y]} \, dx \, dy = M(k_X(t), k_Y(t)).$$

where $M(u, v)$ or equivalently $M(k_X, k_Y)$ denotes the 2D FT of $m(x, y)$ and we define the 2D k-space trajectory by:

$$k_X(t) = \bar{\gamma} \int_0^t G_X(\tau) \, d\tau, \quad k_Y(t) = \bar{\gamma} \int_0^t G_Y(\tau) \, d\tau.$$  

In words: MRI directly measures information about the FT of the object’s transverse magnetization!

MR signal at time $t$ has amplitude (proportional to) the value of a spatial frequency component of the magnetization. Which spatial frequency component? $s(t) = M(k_X(t), k_Y(t))$, so which component is $(k_X(t), k_Y(t))$, which depends on the user-selected gradient signals $G_X(t)$ and $G_Y(t)$.

(A more general derivation including coil sensitivity effects and relaxation effects is given later in the notes.)
The following figure shows how the relative phases of the spins evolve as time progresses, when a $y$ field gradient is used.

In light of the above signal equation, we see that to form an image of $m(x, y)$ we must design the field gradient signals $G_X$ and $G_Y$ so that we “scan through $k$-space.” Of course, in a finite amount of time we cannot cover all of $k$-space, but we must at least cover enough of it to collect sufficient information to make an image of the transverse magnetization $m(x, y)$ by some type of inverse 2D FT.

**$k$-space trajectory**

Note that $k_X(0) = k_Y(0) = 0$. Thus the signal at time $t = 0$ is the “DC component” of the magnetization $m(x, y)$.

As time progresses, the signal value represents the values of the FT of the object along some trajectory in $k$ space. Note that

$$\frac{d}{dt} k_X(t) = \gamma G_X(t), \quad \frac{d}{dt} k_Y(t) = \gamma G_Y(t),$$

so the “velocity” at time $t$ along the $k$-space trajectory is proportional to the gradient strength. So stronger gradients can mean faster scans. Modern high-end scanners have more powerful gradient amplifiers to help accelerate scanning.

Fourier transform interpretation: a FT of an object is just a “weighted” integral of that object where the weights are phase terms varying linearly with spatial position. This is exactly what happens physically to the magnetization in an MR scanner with linear gradients.

Next we give some concrete examples of MR imaging pulse sequences to illustrate $k$-space trajectories.
5.6.1

Example 1: 2D Projection-reconstruction MR sequence

The above pulse sequence illustrates the sequence of excitation/readout phases. The sequence is repeated for many \( \theta \) values. For \( t \in [t_0, t_1] \), there is (ideally) no change in the transverse magnetization (for \( z = 0 \)) except for \( T_2 \) decay, which we have ignored in deriving the signal equation. So the baseband signal (which is not sampled during this time but could be) is just the DC component of \( m(x, y) \), the transverse magnetization pattern established by the RF excitation.

At time \( t = t_1 \), we apply the field gradients, and for \( t \in [t_1, t_2] \):

\[
\begin{align*}
  k_x(t) &= \bar{\gamma} \int_0^t G_x(\tau) \, d\tau = (t - t_1)\bar{\gamma} g \cos \theta \\
  k_y(t) &= \bar{\gamma} \int_0^t G_y(\tau) \, d\tau = (t - t_1)\bar{\gamma} g \sin \theta \\
  s(t) &= M(k_x(t), k_y(t)) = M((t - t_1)\bar{\gamma} g \cos \theta, (t - t_1)\bar{\gamma} g \sin \theta).
\end{align*}
\]

In other words, samples of the signal \( s(t) \) correspond to radial samples of the spectrum of \( m(x, y) \).

Using the Fourier-slice theorem, which is discussed when we cover tomography, we can also write

\[
s(t) = M_{\text{polar}}((t - t_1)\bar{\gamma} g, \theta) = G_\theta ((t - t_1)\bar{\gamma} g),
\]

where \( G_\theta = \mathcal{F}_\theta \{ g_\theta \} \) and \( g_\theta \) is the projection of \( m(x, y) \) at angle \( \theta \).

If we make \( |t_1| \) large, then there will be \( T_2 \) decay between the RF excitation and the readout, and we will get a \( T_2 \) weighted image, i.e., \( m(x, y) e^{-t_1/T_2(x,y)} \). So to be more precise, we are ignoring \( T_2 \) decay during the readout (and, as we will see later, during the excitation too), but we can still account for \( T_2 \) decay between the RF excitation and the readout. In fact, good contrast in MR images often requires such decay.