Homework #1, EECS 516, F09. Due **Due Fri. Sep. 18**, by 1:30PM

Notes

- To enroll in this class (and to have this homework graded) you must complete the one-page student information form and sign the "Statement of Informed Participation," available on the web site.
- You must write the Engineering Honor Pledge on your exam(s) in this class for them to be graded. To review the honor pledge, visit `http://www.engin.umich.edu/org/ehc/`

Fourier transforms

1. [5] Find the Hankel transform of the following signals.
   (a) [0] $\delta(r - r_0)$ where $r_0 > 0$.
   (b) [0] $g(ar)$ where $g(r) \overset{\mathcal{F}_2}{\longleftrightarrow} G(\rho)$ and $a \neq 0$
   (c) [5] $\mathrm{rect}\left(\frac{r-a}{b}\right)$, where $a > b > 0$. Hint: no integration is needed; there are no complex exponentials in the solution.

2. [10] Prove the following FT relations.
   (a) [5] $\mathcal{F}[\mathcal{F}[g(x,y)]] = g(-x,-y)$
   (b) [5] $\mathcal{F}[f(x,y) ** h(x,y)] = F(u,v)\, H(u,v)$

3. [15] In MR imaging (simplified to 1D here), one measures samples of the Fourier transform of the image $g(x)$:

$$G_k = G(k/l) = \int_{-\infty}^{\infty} g(x)\, \mathrm{e}^{-\imath 2\pi x k/l}\, \mathrm{d}x,$$

where the object is space limited, *i.e.*, $g(x) = 0$ for $|x| \geq l/2$.
We cannot acquire *all* the samples, but rather only the finite set $k = -m, -m+1, \ldots, m-1, m$.
One natural approach to approximating g from the samples is the truncated Fourier series:

$$\tilde{g}(x) = \frac{1}{l} \sum_{k=-m}^{m} G_k\, \mathrm{e}^{\imath 2\pi k x/l}\, \mathrm{rect}(x/l)\,.$$

(a) [10] Manipulate the expression for $\tilde{g}(x)$ to show that

$$\tilde{g}(x) = (g * h)(x) \cdot \mathrm{rect}(x/l), \quad \text{where} \quad h(x) = \frac{1}{l} \frac{\sin((2m+1)\pi x/l)}{\sin(\pi x/l)}.$$

(This is called the Dirichlet kernel. It is also important in ultrasound imaging with (finite) linear arrays.)
(b) [5] Briefly describe qualitatively the relationship between g and $\tilde{g}(x)$.
(c) [0] Describe what happens as $m \to \infty$.
(d) [0] Plot h to see how it changes with m and l.
(e) [0] Describe what you would do if you had G_k only for $k = 0, 1 \ldots, m$.

4. [15] In practice an odd number of samples is inconvenient, because FFT algorithms prefer powers of 2. Repeat the previous problem in the case where an even number N of k-space (Fourier) samples are measured: $G_k = G((k + 1/2)/l)$ for $k = -N/2, \ldots, N/2 - 1$. Modify the formulae for $\tilde{g}$ and h as necessary.

Matlab and images

5. [10] Use MATLAB's `ndgrid` and `imagesc` commands to make a grayscale image of the following function, over the range $x \in [-100, 100]$ $y \in [-130, 130]$:

$$\begin{aligned} f(x, y) \;=\; & 100\, g(x/85, y/115) - 80\, g(x/30, (y + 60)/20) + 20\, g((x - 30)/25, (y - 20)/35) \\ & + 20\, g((x + 30)/25, (y - 20)/35) - 100\, g((x - 35)/7, (y - 25)/7) - 100\, g((x + 15)/7, (y - 25)/7) \\ & - 50\, g(x/60, (y - 75)/15), \end{aligned}$$

where $g(x, y) = 1_{\{x^2+y^2 \leq 1\}}$. For full credit your MATLAB code should have *no loops*. Make sure you label your axes. Use the `axis image` and `axis xy` commands to correct the displayed coordinate system.
Hint: to save typing, cut and paste the above equation from the pdf file and use an anonymous function in MATLAB. (See `help function_handle` for details.) For example, the command `h = @(x) abs(x) <= 1/2;` creates an anonymous function that is a rect function.
In general, for problems where you use MATLAB, turn in both your m-file and the results.

Probability and random variables

6. [5] If $X \sim \mathsf{N}(2, \sigma^2)$, find $\mathsf{E}[(X + a)^2]$ and $\mathsf{E}[(X - 1)^3]$. Hint: $X - 1 = X - 2 + 1$.

7. [10] Suppose a radioactive sample emits 16 γ-photons per second (on average).
   (a) [10] Use the central limit theorem to find a *simple* approximate value for the probability that more than 1700 emissions will occur in a 100 second interval.
   (b) [0] Compare your answer with `poisscdf` in MATLAB.

Linear systems

8. [0] Show that $\delta_2(ax, by) = \frac{1}{|ab|}\, \delta_2(x, y)$ if $a, b \neq 0$.

9. [10] Consider the 2D pinhole imaging system described in the lecture notes with an input plane separated by a distance d_1 from a pinhole aperture $a(x, y)$ that is separated by a distance d_2 from a output plane. Relate the spectrum $G(u, v)$ of the output image $g(x, y)$ to the spectrum $F(u, v)$ of the input image $f(x, y)$, assuming that the pinhole is a circle of radius R, centered at the origin. Hint: begin with the 2D space domain relationship derived in the lecture notes. Your final answer should involve jinc.

10. [0] The Society for Sloppy Notation believes that if $h(t) = f_1(t) * f_2(t)$, then $h(t - \tau) = f_1(t - \tau) * f_2(t - \tau)$. Or more precisely, they believe that $h(t - \tau) = (g_1 * g_2)(t)$, where $g_k(t) = f_k(t - \tau)$, $k = 1, 2$. Show that the Society is *completely wrong*. (This is why $f_1(t - \tau) * f_2(t - \tau)$ and the like is dangerously ambiguous notation.)

11. [0] Optional challenge problem.
We observe a radioactive sample with (unknown) emission rate λ (units decays per second). We record x_1 decays in time interval $[0, t_1]$, and x_2 decays in time interval $[t_1, t_2]$, etc., up to x_n decays in time interval $[t_{n-1}, t_n]$.
Determine the ML estimate of λ.