Continuous-Time Random Processes
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A continuous-time random process is an indexed collection of random variables $X(t, \omega)$ defined for each t in an index set $\mathcal{T}$. The ingredients are the following.

- A probability space $(\Omega, \mathcal{F}, P)$.
- An index set $\mathcal{T} \subseteq \mathbb{R}$.
- For each $t_0 \in \mathcal{T}$, a random variable $X(t_0)$ defined such that $\{\omega : X(t_0, \omega) \leq x\} \in \mathcal{F}$ for all $x \in \mathbb{R}$. Formally: $X : (\mathcal{T} \times \Omega) \to \mathbb{R}$.

There are two ways of looking at random processes.

- Hold t_0 fixed, then as ω varies, $x(t_0, \omega)$ is simply a random variable (a function that maps each outcome ω into a real number).
- Hold ω_0 fixed, then as t varies, $X(t, \omega)$ is called a sample path or realization of the random process.

Example (random telegraph). Here is one way to create a continuous-time random process from a random sequence. Assume $\{Y_i\}$ is an i.i.d. random sequence with $P[Y_i > 0] = 1$. Define

$$X(t) = (-1)^{\min\{n : \sum_{i=1}^n Y_i > t\}}.$$

A typical sample path looks like:

Unlike random sequences, which are *countable* collections of random variables, continuous-time random processes are *uncountable* collections of random variables. Thus the finite-dimensional distribution functions are in general insufficient to completely describe all probabilities of events of interest. We restrict attention to the class of random processes that are "sufficiently continuous" that their finite-dimensional distribution functions *are* adequate. This class is called separable random processes. In this class, probabilities, events, etc. can be described by (possibly limits of) finite collections of random variables, so we can apply all the tools we have already developed for such finite collections, (i.e. random vectors).

A random process $X(t)$, $t \in \mathcal{T}$ is separable iff for any countable set T_C that is dense in $\mathcal{T}$, and for any indexed collection of Borel sets B_t, $t \in \mathcal{T}$, if we define $N = \bigcap_{t \in T_C} [X_t \in B_t] - \bigcap_{t \in \mathcal{T}} [X_t \in B_t]$, then

- $N \in \mathcal{F}$, and
- $P(N) = 0$.

Note that if these conditions hold, then since

$$\bigcap_{t \in \mathcal{T}} [X_t \in B_t] \subseteq \bigcap_{t \in T_C} [X_t \in B_t],$$

we can define:

$$P(\bigcap_{t \in \mathcal{T}} [X_t \in B_t]) = P(\bigcap_{i=1}^{\infty} [X_{t_i} \in B_{t_i}]),$$

where, being countable, $T_C = \{t_1, t_2, \ldots\}$.

The nth-order finite-dimensional distribution functions is defined as:

$$F_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = P[X(t_1) \leq x_1, X(t_2) \leq x_2, \ldots, X(t_n) \leq x_n],$$

for appropriate values of the arguments, i.e. $x_1, \ldots, x_n \in \mathbb{R}$ and distinct $t_1, \ldots, t_n \in \mathcal{T}$, for $i = 1, \ldots, n$. Alternative shorthand notation with "double-duty" subscripts:

$$F_X(x_{t_1}, \ldots, x_{t_n}) = P[X(t_1) \leq x_{t_1}, X(t_2) \leq x_{t_2}, \ldots, X(t_n) \leq x_{t_n}]$$

We can then define the nth-order finite-dimensional density function:

$$f_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = \frac{\partial^n}{\partial x_1, \ldots, \partial x_n} F_X(x_1, \ldots, x_n; t_1, \ldots, t_n).$$

Moments

- Mean function:

$$\mu_X(t) = E[X(t)] = \int x f_X(x; t)\, dx$$

- Autocorrelation function:

$$R_X(t_1, t_2) = E[X(t_1) X^\star(t_2)] = \int\int x_1 x_2^\star f_X(x_1, x_2; t_1, t_2)\, dx_1 dx_2$$

- Autocovariance function:

$$K_X(t_1, t_2) = E[(X(t_1) - \mu_X(t_1))(X(t_2) - \mu_X(t_2))^\star]$$

Properties of Moments

- Hermitian:

$$R_X(t_1, t_2) = R_X^\star(t_2, t_1), \quad K_X(t_1, t_2) = K_X^\star(t_2, t_1)$$

- Variance Function:

$$\mathrm{Var}(X(t)) = K_X(t, t)$$

- Autocovariance and autocorrelation relationship:

$$K_X(t_1, t_2) = R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2)$$

- Nonnegative definiteness:

$$\sum_i \sum_j a_i a_j^\star K_X(t_i, t_j) \geq 0, \quad \sum_i \sum_j a_i a_j^\star R_X(t_i, t_j) \geq 0, \quad \forall a_i$$

- Schwarz:

$$|R_X(t_1, t_2)| \leq \sqrt{R_X(t_1, t_1) R_X(t_2, t_2)} \text{ since } |E[X(t_1)X(t_2)]| \leq \sqrt{E[|X(t_1)|^2] E[|X(t_2)|^2]}$$

Classes of Random Processes

- Markov

$$f_X(x_n;t_n|x_{n-1},\ldots,x_1;t_{n-1},\ldots,t_1) = f_X(x_n;t_n|x_{n-1};t_{n-1}) \text{ for } t_n > t_{n-1} > \ldots > t_1$$

  or in shorthand:

$$f_X(x_{t_n}|x_{t_{n-1}},\ldots,x_{t_1}) = f_X(x_{t_n}|x_{t_{n-1}})$$

- Independent Increments

$$X(t_1),\ X(t_2)-X(t_1),\ \ldots,\ X(t_n)-X(t_{n-1}) \text{ mutually independent for } t_n > t_{n-1} > \ldots > t_1$$

- Strict-Sense Stationary

$$f_X(x_1,\ldots,x_n;t_1,\ldots,t_n) = f_X(x_1,\ldots,x_n;\tau+t_1,\ldots,\tau+t_n),\ \forall\tau\in\mathbb{R}$$

  In particular, for $\tau = -t_1$:

$$f_X(x_1,\ldots,x_n;t_1,\ldots,t_n) = f_X(x_1,\ldots,x_n;0,t_2-t_1,\ldots,t_n-t_1)$$

- Wide-Sense Stationary

$$E[X(t)] = \mu_X,\ \forall t\in\mathbb{R}, \quad R_X(t_1,t_2) = R_X(\tau+t_1,\tau+t_2),\ \forall\tau\in\mathbb{R}$$

  In particular, for $\tau = -t_1$:

$$R_X(t_1,t_2) = R_X(0,t_2-t_1) \stackrel{\triangle}{=} R_X(t_2-t_1)$$

- Gaussian

$$\begin{bmatrix} X(t_1) \\ \vdots \\ X(t_n) \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} E[X(t_1)] \\ \vdots \\ E[X(t_n)] \end{bmatrix}, \begin{bmatrix} K_X(t_1,t_1) & \cdots & K_X(t_1,t_n) \\ \vdots & & \vdots \\ K_X(t_n,t_1) & \cdots & K_X(t_n,t_n) \end{bmatrix} \right)$$

Fact: if $X(t)$ is SSS, then it is WSS.
The converse is not true in general; an exception is when $X(t)$ is a Gaussian r.p.

Pairs of Random Processes

- Joint finite-dimensional distribution function:

$$F_{XY}(x_1,x_2,\ldots,x_k,y_1,y_2,\ldots,y_l;t_1,t_2,\ldots,t_k,s_1,s_2,\ldots,s_l)$$
$$= P[X(t_1)\le x_1,\ldots,X(t_k)\le x_k, Y(s_1)\le y_1,\ldots,Y(s_l)\le y_l]$$

- Mutual Independence:

$$F_{XY}(x_1,\ldots,x_k,y_1,\ldots,y_l;t_1,\ldots,t_k,s_1,\ldots,s_l) = F_X(x_1,\ldots,x_k;t_1,\ldots,t_k)F_Y(y_1,\ldots,y_l;s_1,\ldots,s_l)$$

- Jointly strict-sense stationary. $\forall\tau\in\mathbb{R}$:

$$F_{XY}(x_1,\ldots,x_k,y_1,\ldots,y_l;t_1,\ldots,t_k,s_1,\ldots,s_l) = F_{XY}(x_1,\ldots,x_k,y_1,\ldots,y_l;\tau+t_1,\ldots,\tau+t_k,\tau+s_1,\ldots,\tau+s_l)$$

- If $X(t)$ and $Y(t)$ are jointly strict-sense stationary, then they are individually strict-sense stationary. The reverse is not true in general.

Moments of Pairs of Random Processes

- Cross-correlation Function

$$R_{XY}(t_1,t_2) = E[X(t_1)Y^\star(t_2)] = \iint xy^\star f_{XY}(x,y;t_1,t_2)\ dxdy$$

- Cross-covariance Function

$$K_{XY}(t_1,t_2) = E[(X(t_1)-\mu_X(t_1))(Y(t_2)-\mu_Y(t_2))^\star]$$

Properties of Moments of Pairs of Random Processes

- Hermitian symmetry:

$$R_{XY}(t_1,t_2) = R^\star_{YX}(t_2,t_1), \quad K_{XY}(t_1,t_2) = K^\star_{YX}(t_2,t_1)$$

- Autocorrelation function from cross-correlation function

$$R_X(t_1,t_2) = R_{XX}(t_1,t_2)$$

- Cross-covariance / cross-correlation relationship:

$$K_{XY}(t_1,t_2) = R_{XY}(t_1,t_2) - E[X(t_1)]E[Y^\star(t_2)]$$

- Schwarz inequality for cross-correlation:

$$|R_{XY}(t_1,t_2)| \le \sqrt{R_X(t_1,t_1)R_Y(t_2,t_2)}$$

- Mutual independence and cross-covariance:

$$K_{XY}(t_1,t_2) = 0 \text{ if } X(t) \text{ and } Y(t) \text{ mutually independent}$$

We say $X(t)$ and $Y(t)$ are <u>jointly Gaussian random processes</u> iff all of their joint finite-dimensional density functions have the normal form with the appropriate mean and covariance, i.e.

$$\begin{bmatrix} X(t_1) \\ \vdots \\ X(t_k) \\ Y(s_1) \\ \vdots \\ Y(s_l) \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} E[X(t_1)] \\ \vdots \\ E[X(t_k)] \\ E[Y(s_1)] \\ \vdots \\ E[Y(s_l)] \end{bmatrix}, \begin{bmatrix} K_X(t_1,t_1) & \cdots & K_X(t_1,t_k) & K_{XY}(t_1,s_1) & \cdots & K_{XY}(t_1,s_l) \\ \vdots & & \vdots & \vdots & & \vdots \\ K_X(t_k,t_1) & \cdots & K_X(t_k,t_k) & K_{XY}(t_k,s_1) & \cdots & K_{XY}(t_k,s_l) \\ K_{YX}(s_1,t_1) & \cdots & K_{YX}(s_1,t_k) & K_Y(s_1,s_1) & \cdots & K_Y(s_1,s_l) \\ \vdots & & \vdots & \vdots & & \vdots \\ K_{YX}(s_l,t_1) & \cdots & K_{YX}(s_l,t_k) & K_Y(s_l,s_1) & \cdots & K_Y(s_l,s_l) \end{bmatrix} \right)$$

for all appropriate values of the indices.
We say $X(t)$ and $Y(t)$ are <u>jointly wide-sense stationary</u> iff

- Each of $X(t)$ and $Y(t)$ are individually WSS, and
- $R_{XY}(t_1,t_2) = R_{XY}(\tau+t_1,\tau+t_2)$, $\forall t_1,t_2,\tau \in \mathbb{R}$, i.e. $R_{XY}(t_1,t_2) = R_{XY}(t_2-t_1)$

If $X(t)$ and $Y(t)$ are jointly strict-sense stationary, then they are jointly wide-sense stationary. In general, the reverse is not true. An exception is jointly Gaussian, jointly WSS random processes.

Properties of Autocorrelation Function for WSS Random Processes

- $R_X(\tau) = R_X^*(-\tau)$
- $R_X(0) = E[|X(t)|^2]$
- $|R_X(\tau)| \leq R_X(0)$ since $|R_X(\tau, 0)| \leq \sqrt{R_X(\tau,\tau)R_X(0,0)}$
- If $R_X(\tau)$ is continuous at $\tau = 0$, then $R_X(\tau)$ is continuous over all of $\mathbb{R}$.
  Proof: $|R_X(\tau+\delta) - R_X(\tau)| = |E[(X(\tau+\delta) - X(\tau))X(0)]| \leq \sqrt{E[|X(\tau+\delta) - X(\tau)|^2]E[|X(0)|^2]} = \sqrt{2(R_X(0) - R_X(\delta))R_X(0)}$, so if $|R_X(0) - R_X(\delta)| \to 0$ as $\delta \to 0$, then $|R_X(\tau+\delta) - R_X(\tau)| \to 0$ as $\delta \to 0$ for any τ.

WSS Random Processes and LSI systems

- For BIBO LSI system, WSS input yields WSS output, and input and output are jointly WSS.
- Power spectral density: $S_X(\omega) = \int R_X(t)e^{-j\omega t}\,dt$. (Fourier transform of autocorrelation function.)
- For LSI system with impulse response $h(t)$ and transfer function $H(\omega) = \int h(t)e^{-j\omega t}\,dt$, the input-output relationship is $S_Y(\omega) = |H(\omega)|^2 S_X(\omega)$.