Name: ______________________________
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EECS 501 Final
Dec. 14, 1995

!! KEEP THIS PAGE FACE-UP UNTIL YOU ARE TOLD TO BEGIN !!

- This is an open book exam.
- There are 5 problems, worth a total of 100 points. The questions are not necessarily in order of increasing difficulty; read all before beginning.
- Box your final answer. You will be graded on both the final answer and the steps leading to it. Correct intermediate steps will help earn partial credit. For full credit, cross out any incorrect intermediate steps.
- The following integrals may be useful:
  $\int_0^\infty \frac{1}{a} e^{-x/a}\, dx = 1$
  $\int x^m \log x\, dx = x^{m+1}\left[\frac{\log x}{m+1} - \frac{1}{(m+1)^2}\right]$ for $m \neq -1$
  $\int_a^\infty x^m \log x\, dx = -a^{m+1}\left[\frac{\log a}{m+1} - \frac{1}{(m+1)^2}\right]$ for $m < -1$
  $\int_a^\infty (\log x)^2 x^m\, dx = -a^{m+1}\left[\frac{(\log a)^2}{m+1} - 2\frac{\log a}{(m+1)^2} + \frac{2}{(m+1)^3}\right]$ for $m < -1$
- Simplify your result when possible, but you need not compute factorials.
- Specify ranges when giving density functions or distribution functions.
- This exam has 6 pages. Make sure your copy is complete.
- Write the engineering honor pledge on your exam below and sign.
  (I have neither given nor received aid on this exam.)

1. (15 points)
Certain magnetic recording media will produce errors if one records too many bits of the same sign sequentially. Suppose we record binary data (0's and 1's) in 11-bit words. Assume that if an 11-bit word contains a run of 6 *or more* 1's in a row, then that word will be recorded erroneously. For example, the words 01111110100 or 11111111100 would be recorded erroneously, whereas the word 11111011111 would be recorded correctly. Assume 0's and 1's are equally likely.

- If we record $2^{10} = 1024$ words, what is the expected number of words that will be recorded erroneously?

2. (25 points)
Your company has invented a better light bulb. Instead of the usual exponential model, the design engineers claim that if X denotes the failure time of one of these light bulbs (in days), then the CDF of X is:

$$F_X(x) = (1 - x^{1-\theta})u(x-1),$$

where $\theta > 1$ is an unknown parameter.
(Note that $P[X \leq 1] = 0$, so these bulbs will last at least a year with probability 1.)
Marketing needs you to estimate θ so that they can advertise expected lifetimes etc.

- [10 points] You select n independent light bulbs and observe their failure times $x_1, \ldots, x_n$. Find the maximum likelihood estimate of θ from these observations.

- [15 points] Marketing changes their mind, and now they want an estimate of $\alpha = 1/(\theta - 1)$. Show that the following estimator is consistent for α.

$$\hat{\alpha} = \frac{1}{n} \sum_{i=1}^{n} \log X_i \ .$$

3. (30 points)
A device that you are "reverse engineering" emits an i.i.d. random sequence of voltages $\{X_i\}_{i=1}^{\infty}$ with mean μ_X and pdf $f_X(x)$. Unfortunately, the wires connecting the device to your measurement instrument are intermittent, and 20% of the time you record a 0 voltage, rather than the true value X_i. Specifically, you record (observe) a random sequence $\{Y_i\}$, where 80% of the time $Y_i = X_i$, and the other 20% of the time $Y_i = 0$.
Hint: let $Y_i = X_i Z_i$, where Z_i is a Bernoulli random sequence with $P[Z_i = 1] = 0.8$, and assume X_i and Z_i are independent, i.e. the state of the wires is independent of the output voltage. Note that $\{Y_i\}$ is i.i.d.

- [10 points] Find $f_Y(y_1, \ldots, y_n)$.

- [5 points] Suppose you estimate μ_X using the sample mean of n of the Y_i's. Show that the bias of this estimator is $-0.2\mu_X$.

- [5 points] Suppose the X_i's have a pdf containing no Dirac delta functions. How would you find an unbiased estimator for μ_X from the Y_i's? Explain briefly.

- [5 points] Now suppose the X_i's have mean 5 and variance 10. Find constants a, b, and c such that if we define a new random sequence by

$$W_i = \frac{Y_{2i} + Y_{2i+1} - a}{b\ i^c},$$

then W_i converges in distribution to a Normal(0,1) distribution. Explain your choice.

- [5 points] Is $\{Y_i\}$ strict-sense stationary, wide-sense stationary, or both? Explain.

4. (30 points)

Suppose an economist models a stock price by a random process $X(t)$ that is the sum of the constant 5000 plus the output of a linear time-invariant system driven by zero-mean white Gaussian noise input $N(t)$ with power spectral density $S_N(\omega) = 100$.

Specifically she assumes $X(t) = (h \star N)(t) + 5000$, where $\star$ denotes convolution and

$$h(t) = \begin{cases} 1, & 0 \leq t \leq 10 \\ 0, & \text{otherwise} \end{cases},$$

which corresponds to a moving average. The time t is measured in days.

- Under the above model, given that the stock price is 5200 at 8:00 AM on Dec. 14th, what is the probability of the event that at 8:00 AM on Dec. 19th the stock price will exceed 5100? (The final answer is a number.)

end