Solutions to EECS 501 Final

1.

- $E = [1^6 e^5] \cup [01^6 e^4] \cup [e01^6 e^3] \cup [e^2 01^6 e^2] \cup [e^3 01^6 e] \cup [e^4 01^6]$, where e means either a 1 or a 0 is OK. So $P(E) = (1 \cdot 2^5 + 5 \cdot 2^4)/2^{11} = 7/2^7$. Thus the expected number of errors is $1024 \cdot 7/2^7 = 56$.

2.

- First, by taking the derivative of $F_X(x)$: $f_X(x) = (\theta - 1)x^{-\theta} u(x-1)$. So the log-likelihood is: $l(\theta) = \sum_{i=1}^n \log f_X(x_i) = \sum_{i=1}^n (\log(\theta - 1) - \theta \log x_i) = n \log(\theta - 1) - \theta \sum_{i=1}^n \log x_i$, so $l'(\theta) = \frac{n}{\theta - 1} - \sum_{i=1}^n \log x_i$. Thus $\hat{\theta} = 1 + \frac{n}{\sum_{i=1}^n \log x_i}$.

- $E[\hat{\alpha}] = E[\log X_i] = \int_1^\infty (\log x)(\theta - 1)x^{-\theta}\, dx = (\theta - 1)/(-\theta + 1)^2 = 1/(\theta - 1)$ (using integral given with $m = -\theta$). Thus $\hat{\alpha}$ is unbiased. $\mathrm{Var}(\hat{\alpha}) = \mathrm{Var}(\log X_i)/n \to 0$ as $n \to \infty$, if $\mathrm{Var}(\log X_i) < \infty$. But from integral table, $E[(\log X_i)^2] = \int_1^\infty (\log x)^2 (\theta - 1)x^{-\theta}\, dx = 2/(\theta - 1)^2$ is finite, so $\mathrm{Var}(\log X_i) = E[(\log X_i)^2] - (1/(\theta - 1))^2] = 1/(\theta - 1)^2$ is also finite, so $\hat{\alpha}$ is consistent, by the Theorem proved in class using Chebyshev's inequality.

3.

- $F_Y(y) = P[Y \le y | Z = 1]0.8 + P[Y \le y | Z = 0]0.2 = P[X \le y]0.8 + u(y)0.2 = 0.8F_X(y) + 0.2u(y)$, so $f_Y(y) = 0.8f_X(y) + 0.2\delta(y)$. By i.i.d.: $f_Y(y_1, \ldots, y_n) = \prod_{i=1}^n f_Y(y_i)$.

- $E[\hat{\mu}_X] = E[\frac{1}{n}\sum_{i=1}^n Y_i] = E[Y_i] = \int y(0.8f_X(y) + 0.2\delta(y))\, dy = 0.8\mu_X + 0.2 \cdot 0 = 0.8\mu_X$. Thus the bias is $E[\hat{\mu}_X] - \mu_X = -0.2\mu_X$.

- No Dirac deltas means X is continuous r.v., so $P[X = 0] = 0$, so estimate μ_X using $\hat{\mu}_X = (\sum_{i=1}^n Y_i 1_{\{Y_i \ne 0\}})/(\sum_{i=1}^n 1_{\{Y_i \ne 0\}})$. Alternatively, divide the sample mean by 0.8.

- Let $a = E[Y_{2i} + Y_{2i-1}] = 2E[Y] = 2(0.8 \cdot \mu_X + 0.2 \cdot 0) = 2 \cdot 4 = 8$.
  Note that $E[Y^2] = 0.8E[X^2] = 0.8(\mathrm{Var}(X) + \mu_X^2) = 0.8(10 + 5^2) = 0.8(35) = 28$.
  Thus $\mathrm{Var}(Y) = E[Y^2] - E[Y]^2 = 28 - 4^2 = 12$.
  $b^2 = \mathrm{Var}(Y_{2i} + Y_{2i-1}) = \mathrm{Var}(Y_{2i}) + \mathrm{Var}(Y_{2i-1}) = 2\mathrm{Var}(Y) = 2 \cdot 12 = 24$, so $b = \sqrt{24}$.
  Then for $c = 1/2$, by the CLT, $W_i \xrightarrow{d} N(0, 1)$.

- SSS since Y_i is i.i.d., WSS since SSS.

4.

- We want $P[X(5 + t) > 5100 | X(t) = 5200]$. (Since input is Gaussian and WSS, output is also WSS, so any t will do; just use $t = 0$.) Input is a Gaussian r.p., so output is a Gaussian r.p., hence $X(5)$ and $X(0)$ are jointly Gaussian, so the conditional pdf for $X(5)$ given $X(0) = 5200$ is also Gaussian with mean $E[X(5)|X(0) = 5200] = \mu_X(5) + \mathrm{Cov}(X(5), X(0))\mathrm{Var}(X(0))^{-1}(5200 - \mu_X(0)) = 5000 + K_X(5)K_X(0)^{-1}(5200 - 5000) = 5000 + K_X(5)K_X(0)^{-1}200$. Since $S_N(\omega) = 100$, $R_N(\tau) = 100\delta(\tau)$, so

$$R_X(\tau) = 100\delta(\tau) \star h(\tau) \star h(-\tau) = \begin{cases} 1000(1 - |\tau|/10), & |\tau| \le 10 \\ 0, & \text{otherwise} \end{cases}.$$

  Thus $K_X(0) = 1000$ and $K_X(5) = 1000(1 - 1/2) = 500$. Thus $E[X(5)|X(0) = 5200] = 5000 + 500/1000 \cdot 200 = 5100$. Thus $P[X(5) > 5100 | X(0) = 5200] = Q\left((5100 - 5100)/\mathrm{Var}(X(5)|X(0))\right) = 1/2$. $\mathrm{Var}(X(5)|X(0))$ not needed, but it is $1000 - 500/1000 \cdot 500 = 750$.

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