Properties of up-sampling and down-sampling

Impulse train function

Define

$$s_M[n] = \sum_{k=-\infty}^{\infty} \delta[n - kM] = \begin{cases} 1, & n \text{ an integer multiple of } M \\ 0, & \text{otherwise.} \end{cases}$$

By the DTFS (or by direct evaluation):

$$s_M[n] = \frac{1}{M} \sum_{k=0}^{M-1} e^{\jmath \frac{2\pi}{M} kn}.$$

Example. For $M = 2$:

$$s_2[n] = \{\underline{1}, 0\}_2 = \{\ldots, 1, 0, 1, 0, \underline{1}, 0, 1, 0, 1, \ldots\} = \frac{1}{2}(1 + (-1)^n).$$

Upsampling with zero insertion

$$y_0[n] = \begin{cases} x[n/M], & n \text{ an integer multiple of } M \\ 0, & \text{otherwise.} \end{cases}$$

z-transform:

$$Y_0(z) = \sum_{k=-\infty}^{\infty} x[k]\, z^{-kM} = X\left(z^M\right)$$

Fourier transform:

$$Y_0(\omega) = X(M\omega)$$

Example. For $M = 2$:

$$y_0[n] = \{\ldots, 0, x[-2], 0, x[-1], 0, x[0], 0, x[1], 0, x[2], 0, \ldots\}$$
$$Y_0(z) = X\left(z^2\right)$$
$$Y_0(\omega) = X(2\omega).$$

Upsampling with replication

$$y_1[n] = x\left[\frac{n - (n \bmod M)}{M}\right] = x[l],\ n = lM, lM+1, \ldots, (l+1)M - 1$$

z-transform:

$$Y_1(z) = \sum_{k=0}^{M-1} z^{-k}\, X\left(z^M\right) = \begin{cases} M\, X(1), & z = 1 \\ \dfrac{1 - z^{-M}}{1 - z^{-1}}\, X\left(z^M\right), & z \neq 1 \end{cases}$$

Fourier transform:

$$Y_1(\omega) = \sum_{k=0}^{M-1} e^{-\jmath\omega k}\, X(M\omega) = \begin{cases} M\, X(0), & \omega = 2\pi k,\ k \in \mathbb{Z} \\ \dfrac{1 - e^{-\jmath\omega M}}{1 - e^{-\jmath\omega}}\, X(M\omega), & \omega \neq 2\pi k,\ k \in \mathbb{Z} \end{cases}$$

Example. For $M = 2$:

$$y_1[n] = \{\ldots, x[-2], x[-2], x[-1], x[-1], x[0], x[0], x[1], x[1], x[2], x[2], \ldots\}$$
$$Y_1(z) = [1 + z^{-1}]\, X\left(z^2\right)$$
$$Y_1(\omega) = [1 + e^{-\jmath\omega}]\, X(2\omega).$$

Relationship between "replication" and "zero insertion" upsampling forms

$$y_1[n] = y_0[n] * \underbrace{\{\underline{1}, \ldots, 1\}}_{M \text{ ones}} = y_0[n] * \left(\sum_{k=0}^{M-1} \delta[n-k]\right) \Longrightarrow Y_1(\omega) = \left(\sum_{k=0}^{M-1} e^{-\jmath k\omega}\right) Y_0(\omega) = \left(\sum_{k=0}^{M-1} e^{-\jmath k\omega}\right) X(M\omega).$$

Instead of "replication" one can (and often will) use other **interpolators**.

"Downsampling" by zeroing

$$y[n] = \begin{cases} x[n], & n \text{ an integer multiple of } M \\ 0, & \text{otherwise.} \end{cases} = x[n]\, s_M[n] .$$

This is not really downsampling, since the zeros are retained, but it illustrates the analysis techniques.

z-transform:

$$Y(z) = \sum_{n=-\infty}^{\infty} x[n]\, s_M[n]\, z^{-n} = \sum_{n=-\infty}^{\infty} x[n] \left[\frac{1}{M} \sum_{k=0}^{M-1} \mathrm{e}^{-\jmath \frac{2\pi}{M} kn} \right] z^{-n} = \frac{1}{M} \sum_{k=0}^{M-1} \left[\sum_{n=-\infty}^{\infty} x[n]\, \mathrm{e}^{-\jmath \frac{2\pi}{M} kn}\, z^{-n} \right]$$

$$Y(z) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(\mathrm{e}^{-\jmath \frac{2\pi}{M} k}\, z \right)$$

Fourier transform:

$$Y(\omega) = \frac{1}{M} \sum_{k=0}^{M-1} X(\omega - 2\pi k / M)$$

<u>Example</u>. For $M = 2$:

$$y[n] = \{\ldots, 0, x[-4], 0, x[-2], 0, x[0], 0, x[2], 0, x[4], 0, \ldots\}$$

$$Y(z) = \frac{1}{2} [X(z) + X(-z)]$$

$$Y(\omega) = \frac{1}{2} [X(\omega) + X(\omega \pm \pi)] .$$

Downsampling by removing

$$y[n] = x[nM]$$

z-transform:

$$Y(z) = \sum_{k=-\infty}^{\infty} x[kM]\, z^{-k} = \sum_{n=-\infty}^{\infty} x[n]\, s_M[n]\, z^{-n/M} \quad \text{(note: not simply } n = kM \text{ !)}$$

$$= \sum_{n=-\infty}^{\infty} x[n] \left[\frac{1}{M} \sum_{k=0}^{M-1} \mathrm{e}^{\jmath \frac{2\pi}{M} kn} \right] z^{-n/M} = \frac{1}{M} \sum_{k=0}^{M-1} \left[\sum_{n=-\infty}^{\infty} x[n]\, \mathrm{e}^{\jmath \frac{2\pi}{M} kn}\, z^{-n/M} \right]$$

$$Y(z) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(\mathrm{e}^{-\jmath \frac{2\pi}{M} k}\, z^{1/M} \right)$$

Fourier transform:

$$Y(\omega) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(\frac{\omega - 2\pi k}{M} \right)$$

Example for $M = 2$:

$$y[n] = \{\ldots, x[-6], x[-4], x[-2], x[0], x[2], x[4], x[6], \ldots\}$$

$$Y(z) = \frac{1}{2} \left[X\left(\sqrt{z}\right) + X\left(-\sqrt{z}\right) \right]$$

$$Y(\omega) = \frac{1}{2} \left[X\left(\frac{\omega}{2}\right) + X\left(\frac{\omega}{2} \pm \pi\right) \right] .$$

The $X\left(\frac{\omega}{2} \pm \pi\right)$ term is a form of aliasing. In practice one usually would **filter** $x[n]$ *before* downsampling to reduce this aliasing.