Example 2.19
A batch of 50 items contains 10 defective items.
Ten items are selected at random and tested.
What is the probability that exactly 5 of the items tested are defective?

Answer:

Note that $n = 50$ and $k = 10$ and we are sampling without replacement.
So the sample space consists of $n!/(n-k)! = 50!/40!$ equally likely outcomes, which is the "denominator," i.e.

$$P(E) = \frac{\#\text{ of outcomes in} E}{50!/40!}.$$

For the numerator, we must now answer the question, of all of those outcomes, how many of them are in E? I.e., how many ways are there to draw 5 good items and 5 defective items from a batch of 40 good items and 10 defective items, in any order?

To draw 5 good items from the 40 good items in the batch, there are $40 \cdot 39 \cdot \ldots \cdot 36 = 40!/(40-5)! = 40!/35!$ ways. Likewise, there are $10!/(10-5)! = 10!/5!$ ways to choose 5 defective items from the 10 defective items in the batch. For the event E, the good and defective items can be drawn in any order. How many ways are there to order 5 good and 5 defective items? We must choose 5 positions (for the good items) from the 10 possible positions, and there are $\binom{10}{5}$ ways to do this.

So the number of outcomes in event E is $\binom{10}{5}\frac{40!}{35!}\frac{10!}{5!}$ and thus

$$P(E) = \frac{\binom{10}{5}\frac{40!}{35!}\frac{10!}{5!}}{50!/40!} = \frac{10!\,40!\,10!\,40!}{5!\,5!\,35!\,5!\,50!},$$

which is the same answer as given in the text.

Summary

Urn: 4 Balls (Red, Green, Yellow, and Blue)

	Ordered	Unordered
With Replacement	$\begin{array}{cccc} (R,R) & (R,G) & (R,Y) & (R,B) \\ (G,R) & (G,G) & (G,Y) & (G,B) \\ (Y,R) & (Y,G) & (Y,Y) & (Y,B) \\ (B,R) & (B,G) & (B,Y) & (B,B) \end{array}$ $n^k = 4^2 = 16$	$\begin{array}{cccc} (2,0,0,0) & (1,1,0,0) & (1,0,1,0) & (1,0,0,1) \\ & (0,2,0,0) & (0,1,1,0) & (0,1,0,1) \\ & & (0,0,2,0) & (0,0,1,1) \\ & & & (0,0,0,2) \end{array}$ $\binom{n-1+k}{k} = \binom{5}{2} = 10$
Without Replacement	$\begin{array}{cccc} & (R,G) & (R,Y) & (R,B) \\ (G,R) & & (G,Y) & (G,B) \\ (Y,R) & (Y,G) & & (Y,B) \\ (B,R) & (B,G) & (B,Y) & \end{array}$ $\frac{n!}{(n-k)!} = \frac{4!}{2!} = 12$	$\begin{array}{cccc} & \{R,G\} & \{R,Y\} & \{R,B\} \\ & & \{G,Y\} & \{G,B\} \\ & & & \{Y,B\} \end{array}$ $\binom{n}{k} = \binom{4}{2} = 6$