Signals and Systems: Summary 1
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Circuits: $v(t) = Ri(t)$, $v(t) = L\frac{d}{dt}i(t)$, $i(t) = C\frac{d}{dt}v(t)$

Notation: $x(t) = \begin{cases} e^{-t}, & t > 2, \\ 0, & \text{otherwise} \end{cases} = e^{-t}u(t-2)$

Time transformation: $x\left(\frac{t-t_0}{w}\right)$.
First scale according to w, then shift according to t_0.

Integrator system $y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau = x(t) * u(t)$

Even symmetry: $x(-t) = x(t)$
Odd symmetry: $x(-t) = -x(t)$
$\text{Ev}\{x(t)\} = \frac{1}{2}(x(t) + x(-t))$
$\text{Od}\{x(t)\} = \frac{1}{2}(x(t) - x(-t))$
$x(t) = \text{Ev}\{x(t)\} + \text{Od}\{x(t)\}$.

Average value: $A \triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\,dt$
Energy: $E \triangleq \int_{-\infty}^{\infty} |x(t)|^2\,dt$
Average power: $P \triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2\,dt$
Energy signal: $E < \infty$, $P = 0$.
Power signal: $E = \infty$, $0 < P < \infty$.
Power of periodic signal: $P = \frac{1}{T_0}\int_{0}^{T_0} |x(t)|^2\,dt$

Step function: $u(t) = 1$ for $t > 0$.
Rect function: $\text{rect}(t) = 1$ for $-1/2 < t < 1/2$
$\text{rect}(t) = u(t+1/2) - u(t-1/2) =$

Impulse functions
- Sifting property: $\int_{-\infty}^{\infty} x(t)\delta(t-t_0)\,dt = x(t_0)$ if $x(t)$ is continuous at t_0
- Sampling property: $x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$ if $x(t)$ is continuous at t_0
- unit area property: $\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$ for any t_0
- scaling property: $\delta(at+b) = \frac{1}{|a|}\delta(t+b/a)$ for $a \neq 0$.
- symmetry property: $\delta(t) = \delta(-t)$
- support property: $\delta(t-t_0) = 0$ for $t \neq t_0$
- relationships with unit step function: $\delta(t) = \frac{d}{dt}u(t)$, $u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau$

Continuous-time system properties
- Stability (BIBO): all bounded input signals produce bounded output signals
- Invertibility: each output signal is the response to only one input signal
- Causal: output signal value $y(t)$ at any time t depends only on present and past input signal values.
- Static (memoryless): output at any time only depends on input signal at the same time. (otherwise dynamic)
- Time invariant: $x(t) \xrightarrow{\mathcal{T}} y(t)$ implies that $x(t-t_0) \xrightarrow{\mathcal{T}} y(t-t_0)$

Linear systems:
- superposition property: $\mathcal{T}[\sum_k a_k x_k(t)] = \sum_k a_k \mathcal{T}[x_k(t)]$
- additivity property: $\mathcal{T}[x_1(t) + x_2(t)] = \mathcal{T}[x_1(t)] + \mathcal{T}[x_2(t)]$
- scaling property: $\mathcal{T}[ax(t)] = a\mathcal{T}[x(t)]$

LTI systems

input-output relationship described by convolution integral:

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)\,d\tau$$

Properties:
- Commutative property: $x(t) * h(t) = h(t) * x(t)$
- Associative property: $[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$
- Distributive property: $x(t) * [h_1(t) + h_2(t)] = [x(t) * h_1(t)] + [x(t) * h_2(t)]$
- The order of serial connection of LTI systems does not affect the overall impulse response.
- $x(t) * \delta(t) = x(t)$
- Delay property: $x(t) * \delta(t-t_0) = x(t-t_0)$
- $\delta(t-t_0) * \delta(t-t_1) = \delta(t-t_0-t_1)$
- Time-invariance: If $y(t) = x(t) * h(t)$, then $x(t-t_0) * h(t-t_1) = y(t-t_0-t_1)$

LTI system properties
- causal: $h(t) = 0$ for all $t < 0$
- static: $h(t) = k\delta(t)$, otherwise dynamic
- stable: $\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$
- invertible: $h(t) * h_i(t) = \delta(t)$ for some $h_i(t)$
  If $h(t) * x(t) = 0$ for some nonzero signal $x(t)$, then not invertible
- step response: $h(t) = \frac{d}{dt}s(t)$, where $u(t) \xrightarrow{\text{LTI}} s(t)$

Linear, constant coefficient, differential equation systems
- LTI, causal, dynamic unless $N = M = 0$
- homogenous solution, natural response: $y_h(t) = \sum_l C_l e^{s_l t}$, where s_l's are the N roots of the characteristic polynomial $\sum_{k=0}^{N} a_k s^k = 0$.
- particular solution, forced response: $y_p(t) = P_0 x(t) + P_1 \frac{d}{dt}x(t) + \cdots$