## Chapter 4

## Properties of Analytical Tomographic Image Reconstruction

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### 4.1 Introduction (s,topo,intro)

Chapter 3 reviewed analytical tomographic image reconstruction methods that are based on integral formulations. Unlike Chapter 1, in which image restoration methods were formulated using cost functions, no cost functions appeared in Chapter 3. For a variety of purposes, including developing intuition, it can be instructive to analyze the properties of "analytical" methods for tomography that are based on cost functions. This chapter presents a somewhat non-traditional regularized least-squares formulation (cf. (1.8.8)) of the tomographic image reconstruction problem, and analyzes its properties. The key ingredients for this analysis are operators in suitable Hilbert spaces. The ideas in this chapter are fairly specialized and can be skipped for readers more interested in practical image reconstruction methods.

### 4.2 Operator formulation (s,tomo,op)

Chapter 1 extolled the virtues of using matrix-vector notation for statistical formulations of image restoration problems. Similarly, we can better understand the parallels between image restoration and image reconstruction problems if we also express the latter using the tools of linear algebra.

### 4.2.1 Forward projection

Because the Radon transform (3.2.4) is a linear operator, we can write it succinctly as

$$
p=\mathcal{P} f
$$

where ${ }^{1} \mathcal{P}: \mathcal{L}_{2}\left(\mathbb{R}^{2}\right) \rightarrow \mathcal{L}_{2}([0, \pi] \times \mathbb{R})$ denotes the Radon transform operator. The operator $\mathcal{P}$ is also called a forward projector, because it maps object space or image space into projection space or sinogram space. We also write

$$
p_{\varphi}=\mathcal{P}_{\varphi} f
$$

where $\mathcal{P}_{\varphi}: \mathcal{L}_{2}\left(\mathbb{R}^{2}\right) \rightarrow \mathcal{L}_{2}(\mathbb{R})$ is the projection operator that collapses a 2 D object to a 1 D function for each $\varphi$. We refer to $p_{\varphi}$ as the projection of $f$ at angle $\varphi$.

### 4.2.2 Back projection: the adjoint of forward projection

Chapter 1 showed that statistical solutions to image restoration problems, such as the PWLS estimator (1.8.9), involved both the system matrix $\boldsymbol{A}$ and its matrix transpose $\boldsymbol{A}^{\prime}$. To express such solutions in the context of continuous-space problems like analytical image reconstruction, we need a generalization of matrix transpose, which is called the adjoint in a general vector space for which a suitable inner product has been defined [1].

The Radon transform operator $\mathcal{P}$ maps 2D objects $f(x, y)$ into sinograms $p_{\varphi}(r)$. This operator is linear, so if we define inner products in sinogram space and in object space, then $\mathcal{P}$ will have an adjoint ${ }^{2}$. The natural inner product for object space is the usual inner product for $\mathcal{L}_{2}\left(\mathbb{R}^{2}\right)$ :

$$
\begin{equation*}
\langle f, g\rangle=\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) g^{*}(x, y) \mathrm{d} x \mathrm{~d} y, \quad \forall f, g \in \mathcal{L}_{2}\left(\mathbb{R}^{2}\right) \tag{4.2.1}
\end{equation*}
$$

[^0]The natural inner product for sinogram space is the usual inner product for $\mathcal{L}_{2}([0, \pi] \times \mathbb{R})$, i.e.,

$$
\begin{equation*}
\langle p, q\rangle=\int_{0}^{\pi} \int_{-\infty}^{\infty} p_{\varphi}(r)\left[q_{\varphi}(r)\right]^{*} \mathrm{~d} r \mathrm{~d} \varphi, \quad \forall p, q \in \mathcal{L}_{2}([0, \pi] \times \mathbb{R}) \tag{4.2.2}
\end{equation*}
$$

The adjoint of $\mathcal{P}$ is denoted $\mathcal{P}^{*}$, and is the operator that satisfies the following equality for any object $f \in \mathcal{L}_{2}\left(\mathbb{R}^{2}\right)$ and sinogram $q \in \mathcal{L}_{2}([0, \pi] \times \mathbb{R})$ :

$$
\langle\mathcal{P} f, q\rangle=\left\langle f, \mathcal{P}^{*} q\right\rangle
$$

For the choices of inner products given above, the adjoint of $\mathcal{P}$ is the backprojection operator defined in (3.3.1), i.e.,

$$
\left(\mathcal{P}^{*} q\right)(x, y)=\int_{0}^{\pi} q_{\varphi}(x \cos \varphi+y \sin \varphi) \mathrm{d} \varphi
$$

as shown by the following equalities:

$$
\begin{aligned}
\langle\mathcal{P} f, q\rangle & =\int_{0}^{\pi} \int_{-\infty}^{\infty}(\mathcal{P} f)(r, \varphi)\left[q_{\varphi}(r)\right]^{*} \mathrm{~d} r \mathrm{~d} \varphi \\
& =\int_{0}^{\pi} \int_{-\infty}^{\infty}\left[\iint f(x, y) \delta(x \cos \varphi+y \sin \varphi-r) \mathrm{d} x \mathrm{~d} y\right]\left[q_{\varphi}(r)\right]^{*} \mathrm{~d} r \mathrm{~d} \varphi \\
& =\iint f(x, y)\left[\int_{0}^{\pi} \int_{-\infty}^{\infty} \delta(x \cos \varphi+y \sin \varphi-r) q_{\varphi}(r) \mathrm{d} r \mathrm{~d} \varphi\right]^{*} \mathrm{~d} x \mathrm{~d} y \\
& =\iint f(x, y)\left[\int_{0}^{\pi} q_{\varphi}(x \cos \varphi+y \sin \varphi) \mathrm{d} \varphi\right]^{*} \mathrm{~d} x \mathrm{~d} y \\
& =\iint f(x, y)\left[\left(\boldsymbol{\mathcal { P }}^{*} q\right)(x, y)\right]^{*} \mathrm{~d} x \mathrm{~d} y=\left\langle f, \boldsymbol{P}^{*} q\right\rangle
\end{aligned}
$$

It follows from the properties of adjoints $[1$, p. 151$]$ that $\mathcal{P}^{*}$ is bounded, i.e., $\mathcal{P}^{*}: \mathcal{L}_{2}([0, \pi] \times \mathbb{R}) \rightarrow \mathcal{L}_{2}\left(\mathbb{R}^{2}\right)$.
We can also write the backprojection formula $b=\mathcal{P}^{*} q$ as follows:

$$
\begin{equation*}
b(x, y)=\int_{0}^{\pi} b_{\varphi}(x, y) \mathrm{d} \varphi \tag{4.2.3}
\end{equation*}
$$

where $b_{\varphi}=\mathcal{P}_{\varphi}^{*} q$ is defined by:

$$
\begin{equation*}
\left(\mathcal{P}_{\varphi}^{*} q\right)(x, y)=q_{\varphi}(x \cos \varphi+y \sin \varphi)=\int q_{\varphi}(r) \delta(r-[x \cos \varphi+y \sin \varphi]) \mathrm{d} r \tag{4.2.4}
\end{equation*}
$$

The projection operator $\mathcal{P}_{\varphi}$ converts a 2 D object into a 1D projection $p_{\varphi}(\cdot)$; the adjoint operator $\mathcal{P}_{\varphi}^{*}$ maps a 1D projection back into a 2D object by "smearing" that projection along the angle $\varphi$.

To illustrate the convenience of these expressions, it is helpful to define next a "diagonal" sinogram-space angular weighting operator $\mathcal{W}: \mathcal{L}_{2}([0, \pi] \times \mathbb{R}) \rightarrow \mathcal{L}_{2}([0, \pi] \times \mathbb{R})$ as follows

$$
\begin{equation*}
q=\mathcal{W} p \text { iff } q_{\varphi}(r)=w(\varphi) p_{\varphi}(r) \tag{4.2.5}
\end{equation*}
$$

for some $\pi$-periodic angular-weighting function $w(\varphi)$. Assuming that $w(\varphi)$ is real, the operator is self-adjoint, i.e., $\mathcal{W}^{*}=\mathcal{W}$, which is analogous to the symmetry of a real diagonal matrix. With this definition of $\mathcal{W}$, the angularlyweighted backprojection in (3.3.1) is expressed simply as $b=\mathcal{P}^{*} \mathcal{W} p$. More importantly, the $1 /|r|$ convolution relationship described by Theorem 3.3.1 is simply the following:

$$
\mathcal{P}^{*} \mathcal{W} \mathcal{P} f=f * * \frac{w(\varphi+\pi / 2)}{|r|}
$$

To help express this relationship in the frequency domain, we define next the 2D Fourier transform operator $\mathcal{F}_{2}: \mathcal{L}_{2}\left(\mathbb{R}^{2}\right) \rightarrow \mathcal{L}_{2}\left(\mathbb{R}^{2}\right)$ as follows:

$$
\begin{equation*}
\left(\mathcal{F}_{2} f\right)(u, v)=\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \mathrm{e}^{-\imath 2 \pi(x u+y v)} \mathrm{d} x \mathrm{~d} y \tag{4.2.6}
\end{equation*}
$$

We also define the "diagonal" polar Fourier-space operator $\mathcal{D}\left(\frac{w(\Phi)}{|\rho|}\right)$ by

$$
\begin{equation*}
P=\mathcal{D}\left(\frac{w(\Phi)}{|\rho|}\right) Q \text { iff } P(\rho, \Phi)=\frac{w(\Phi)}{|\rho|} Q(\rho, \Phi) \tag{4.2.7}
\end{equation*}
$$

Then the Fourier relationship (3.3.9) is simply

$$
\mathcal{P}^{*} \mathcal{W} \mathcal{P}=\mathcal{F}_{2}^{-1} \mathcal{D}\left(\frac{w(\Phi)}{|\rho|}\right) \mathcal{F}_{2}
$$

Using the above notation, we can express the BPF method (3.4.1) as follows:

$$
\hat{f}=\mathcal{F}_{2}^{-1} \mathcal{D}\left(\frac{|\rho|}{w(\Phi)}\right) \mathcal{F}_{2} \mathcal{P}^{*} \mathcal{W} p
$$

An even more succinct expression for BPF is the following:

$$
\begin{equation*}
\hat{f}=\left[\mathcal{P}^{*} \mathcal{W} \mathcal{P}\right]^{-1} \mathcal{P}^{*} \mathcal{W} p \tag{4.2.8}
\end{equation*}
$$

In this final form, we see that the BPF method has an interpretation that was not at all apparent in the original notation of $\S 3.4 .2$. Now we see that the reconstructed image $\hat{f}$ is the solution to the following weighted least-squares (WLS) minimization problem:

$$
\begin{equation*}
\hat{f}=\underset{f}{\arg \min }\|p-\mathcal{P} f\|_{\mathcal{W}^{1 / 2}}^{2} \tag{4.2.9}
\end{equation*}
$$

where for any sinogram $q$ we define

$$
\|q\|_{\mathcal{W}^{1 / 2}}^{2}=\langle\boldsymbol{\mathcal { W }} q, q\rangle=\int_{0}^{\pi} \int_{-\infty}^{\infty} w(\varphi)\left|q_{\varphi}(r)\right|^{2} \mathrm{~d} r \mathrm{~d} \varphi
$$

We emphasize that the WLS interpretation (4.2.9) holds only for angular weighting; if we were to consider a radially dependent weighting function, say $w(r, \varphi)$ in (4.2.5), then we could still write the solution (4.2.8) on paper, but the operator $\left[\mathcal{P}^{*} \mathcal{W} \mathcal{P}\right]^{-1}$ would no longer be shift-invariant in general, so a more complicated form of "deconvolution" would be needed, typically requiring an iterative algorithm. (See [3] for an interesting approximations.)

The FBP method (3.4.3) can be expressed

$$
\begin{equation*}
\hat{f}=\mathcal{P}^{*} \boldsymbol{V} p \tag{4.2.10}
\end{equation*}
$$

where $\mathcal{V}$ is the operator (3.4.2) that applies the ramp filter to each projection, i.e.,

$$
\mathcal{V}=\left(\boldsymbol{\mathcal { I }} \otimes \mathcal{F}_{1}^{-1}\right)[\mathcal{I} \otimes \mathcal{D}(|u|)]\left(\boldsymbol{\mathcal { I }} \otimes \mathcal{F}_{1}\right)=\boldsymbol{\mathcal { I }} \otimes\left(\mathcal{F}_{1}^{-1} \mathcal{D}(|u|) \mathcal{F}_{1}\right)
$$

where $\mathcal{I} \otimes \mathcal{F}_{1}$ takes the 1D FT of each projection view of a sinogram. There is no weighting $\mathcal{W}$ whatsoever in the FBP method, so FBP provides an unweighted least-squares solution to the minimization problem (4.2.9). (The same conclusion holds for BPF in the usual case where $w(\varphi)=1$.) This lack of weighting is one of the deficiencies of the FBP method (and of the BPF method in the usual unweighted case) because it treats all rays equally, whereas in practice different rays have different noise variances so should be weighted differently to reduce reconstructed image noise.

Comparing (4.2.10) to (4.2.8) with $\mathcal{W}=\mathcal{I}$, we can express the equivalence of the FBP and BPF methods using operators as follows:

$$
\mathcal{P}^{*} \mathcal{V}=\left[\mathcal{P}^{*} \mathcal{P}\right]^{-1} \mathcal{P}^{*}=\mathcal{P}^{*} \mathcal{P}\left[\mathcal{P}^{*} \mathcal{P}\right]^{-2} \mathcal{P}^{*}
$$

which establishes the following curious relationship: $\mathcal{V}=\mathcal{P}\left[\mathcal{P}^{*} \mathcal{P}\right]^{-2} \mathcal{P}^{*}$. It also follows that $\mathcal{P}^{*} \mathcal{V} \mathcal{P}=\mathcal{I}$.

### 4.2.3 Convolution property (s,topo,conv)

One can show (see [4, p. 11] and $\S 3.4 .7$ ) the following convolution property for any sinogram $q$ and 2D object $f$ :

$$
\begin{equation*}
\mathcal{P}^{*}\left\{q *_{r} \mathcal{P} f\right\}=\left\{\mathcal{P}^{*} q\right\} * * f \tag{4.2.11}
\end{equation*}
$$

where $*_{r}$ denotes 1D convolution with respect to the radial variable $r$.
Example 4.2.1 The choice $q_{\varphi}(r)=\delta(r)$ yields $\mathcal{P}^{*} \mathcal{P} f=\frac{1}{|r|} * * f$.

### 4.2.4 SVD of Radon transform (s,tomo,svd)

For some analyses, it can be useful to have a singular value decomposition (SVD) of the Radon transform operator $\mathcal{P}$. Natterer [4, p. 16] gives an SVD in terms of Jacobi and Gegenbauer polynomials, all terms of which are square integrable. We base the following "Fourier" description on the formulation given by Barrett and Myers [5, p. 1174]. However, we generalize that analysis to consider the Radon transform operator with blur:

$$
\begin{equation*}
\mathcal{A}=\mathcal{B P} \tag{4.2.12}
\end{equation*}
$$

where $\mathcal{B}$ denotes (possibly) angle-dependent radial blur:

$$
\begin{equation*}
p=\mathcal{B} q \Longleftrightarrow p_{\varphi}(r)=b_{\varphi}(r) * q_{\varphi}(r), \quad \forall \varphi \in[0, \pi] \tag{4.2.13}
\end{equation*}
$$

We first define the following three Hilbert spaces:

- $\mathcal{H}_{\square}=\mathcal{L}_{2}\left(\mathbb{R}^{2}\right)$ with the usual inner product (4.2.1),
- $\mathcal{H}_{\circ}=\mathcal{L}_{2}(\mathbb{R} \times[0, \pi])$ with the polar coordinates inner product

$$
\langle G, F\rangle=\int_{0}^{\pi} \int_{-\infty}^{\infty} G(\rho, \Phi) F^{*}(\rho, \Phi)|\rho| \mathrm{d} \rho \mathrm{~d} \Phi
$$

- and $\mathcal{H}_{\text {sino }}=\mathcal{L}_{2}([0, \pi] \times \mathbb{R})$ with the "sinogram" inner product (4.2.2).

Then we define the following three operators as mappings between these Hilbert spaces:

$$
\begin{align*}
& \mathcal{F}_{2}: \mathcal{H}_{\square} \rightarrow \mathcal{H}_{\square}, \quad F=\mathcal{F}_{2} f \quad \Longleftrightarrow \quad F(u, v)=\iint f(x, y) \mathrm{e}^{-\imath 2 \pi(x u+y v)} \mathrm{d} x \mathrm{~d} y \\
& \mathcal{D}: \mathcal{H}_{\square} \rightarrow \mathcal{H}_{\circ}, \quad G=\mathcal{D} F \quad \Longleftrightarrow \quad G(\rho, \varphi)=\frac{B_{\varphi}(\rho)}{\sqrt{|\rho|}} F(\rho \cos \varphi, \rho \sin \varphi)  \tag{4.2.14}\\
& \mathcal{U}: \mathcal{H}_{\circ} \rightarrow \mathcal{H}_{\text {sino }}, \quad p=\boldsymbol{\mathcal { U }} G \Longleftrightarrow p_{\varphi}(r)=\int_{-\infty}^{\infty} \sqrt{|\rho|} G(\rho, \varphi) \mathrm{e}^{\imath 2 \pi r \rho} \mathrm{~d} \rho,
\end{align*}
$$

where $B_{\varphi}(\nu)$ denotes the 1D Fourier transform of $b_{\varphi}(r)$ with respect to $r$, i.e.,

$$
B_{\varphi}(\nu) \triangleq \int b_{\varphi}(r) \mathrm{e}^{-\imath 2 \pi \nu r} \mathrm{~d} r
$$

Having defined these operators, we can write the blurred Radon transform operator $\mathcal{A}$ in the following "SVD like" form:

$$
\begin{equation*}
\mathcal{A}=\mathcal{U} \mathcal{D} \mathcal{F}_{2} \tag{4.2.15}
\end{equation*}
$$

Ignoring the blur, this form "reads" very much like the Fourier slice theorem: first take the 2D FT of the object, then convert to polar coordinates, and then take the inverse 1D FT for each angular view.

The importance of this form is that both $\mathcal{F}_{2}$ and $\mathcal{U}$ are unitary operators [6, p. 331], i.e., $\mathcal{F}_{2}^{-1}=\mathcal{F}_{2}^{*}$ and $\mathcal{U}^{-1}=\mathcal{U}^{*}$. (The purpose of the $\sqrt{|\rho|}$ factors is to make $\mathcal{U}$ be unitary.) The adjoints of $\mathcal{D}$ and $\mathcal{U}$ with respect to the inner products defined above can be shown to be as follows:

$$
\begin{array}{ll}
\mathcal{D}^{*}: \mathcal{H}_{\circ} \rightarrow \mathcal{H}_{\square}, & H=\mathcal{D}^{*} G \tag{4.2.16}
\end{array} \Longleftrightarrow \quad H(u, v)=\frac{B_{\angle_{\pi}(u, v)}^{*}\left(\rho_{ \pm}(u, v)\right)}{\sqrt{\left|\rho_{ \pm}(u, v)\right|}} G\left(\rho_{ \pm}(u, v), \angle_{\pi}(u, v)\right)
$$

where $\rho_{ \pm}(\cdot, \cdot)$ and $\angle_{\pi}(\cdot, \cdot)$ are defined as in (3.2.16).
In the absence of blur, i.e., when $\mathcal{B}=\mathcal{I}$, this SVD form allows the following concise expression for inverting the Radon transform:

$$
\mathcal{P}^{-1}=\mathcal{F}_{2}^{-1} \mathcal{D}^{-1} \mathcal{U}^{-1}=\mathcal{F}_{2}^{-1} \mathcal{D}^{-1} \mathcal{U}^{*}
$$

where

$$
F=\mathcal{D}^{-1} G \Longleftrightarrow F(u, v)=\sqrt{\left|\rho_{ \pm}(u, v)\right|} G\left(\rho_{ \pm}(u, v), \angle_{\pi}(u, v)\right)
$$

and

$$
F=\mathcal{D}^{-1} \mathcal{U}^{*} p \Longleftrightarrow F(u, v)=\int_{-\infty}^{\infty} p_{L_{\pi}(u, v)}(r) \mathrm{e}^{-\imath 2 \pi r \rho_{ \pm}(u, v)} \mathrm{d} r
$$

This is simply the direct Fourier inversion method.
For alternative SVD formulations, see [7, 8]. For the relation to FBP and pseudo-inverse solutions, see [9]. An SVD for the case of a finite number of projection angles has also been derived [10].

### 4.3 System blur, sampling, and noise (s,tomo,blur)

The preceding sections have considered the ideal line-integral model (3.2.4) for tomography. The measurements from real tomographic systems are degraded by blur, sampling, and noise. These effects are often ignored in classical treatments of analytical reconstruction methods. In this section we depart somewhat from classical treatments and attempt to analyze the effects of these degradations. Our purpose is not to derive reconstruction algorithms for practical use, but rather to provide insight into the properties of statistical image reconstruction methods.

For simplicity, we consider the effects of sampling in the radial dimension only; we continue to consider a continuum of projection angles $\varphi$. (The concepts generalize to finite sets of projection views; see Problem 4.10.) Let $\bar{y}_{\varphi}[n]$ denote the mean measurement for the $n$th radial sample at angle $\varphi$. The following model accounts for sampling and for shift-invariant blur [11]:

$$
\bar{y}_{\varphi}[n]=\left.p_{\varphi}(r) * b_{\varphi}(r)\right|_{r=r_{n}}
$$

$$
\begin{equation*}
=\iint f(x, y) b_{\varphi}\left(r_{n}-[x \cos \varphi+y \sin \varphi]\right) \mathrm{d} x \mathrm{~d} y, \quad n \in \mathbb{Z} \tag{4.3.1}
\end{equation*}
$$

where $p_{\varphi}=\mathcal{P}_{\varphi} f, b_{\varphi}(r)$ denotes the radial blur for each angle $\varphi$, the sample locations are given by $r_{n}=n \triangle_{\mathrm{R}}$, and $\triangle_{R}$ denotes the radial sample spacing. We allow the blur shape to depend on the projection angle $\varphi$ because PET and SPECT systems have such variations. To account for the effects of measurement errors, we assume a simple additive noise model:

$$
\begin{equation*}
y_{\varphi}[n]=\bar{y}_{\varphi}[n]+\varepsilon_{\varphi}[n], \tag{4.3.2}
\end{equation*}
$$

where $\varepsilon_{\varphi}[n]$ denotes zero-mean noise. Now the image reconstruction problem is to estimate $f$ from the collection of measurements

$$
\boldsymbol{y}=\left\{y_{\varphi}[n]: \varphi \in[0, \pi), n \in \mathbb{Z}\right\}
$$

This is still an idealized formulation because we are assuming a continuum of projection angles and an infinite number of radial samples per projection angle. Nevertheless, it is somewhat more realistic than the usual Radon transform model (3.2.4), and to my knowledge it is the most generality that is still conducive to shift-invariant solutions.

To formulate solutions, we again enlist the aid of operators. Define the blur operator $\mathcal{B}$ as in (4.2.13). Also define the 1D sampling operator $\mathcal{S}$ by

$$
\overline{\boldsymbol{y}}=\mathcal{S} q \text { iff } \bar{y}_{\varphi}[n]=q_{\varphi}\left(r_{n}\right), \quad \forall n \in \mathbb{Z}, \quad \forall \varphi \in[0, \pi]
$$

Then we express (4.3.1) using operators as follows

$$
\begin{equation*}
\overline{\boldsymbol{y}}=\mathcal{A} f, \text { where } \mathcal{A} \triangleq \mathcal{S B P} \tag{4.3.3}
\end{equation*}
$$

Here the system model $\mathcal{A}$ is the composition of the Radon transform, blur, and sampling operators.

### 4.3.1 WLS estimator

In light of the additive noise model (4.3.2) and the linear system model (4.3.3), a natural formulation of the problem of estimating the object $f$ from the measurements $\boldsymbol{y}$ would be the following weighted least-squares criterion:

$$
\begin{equation*}
\hat{f}=\underset{f}{\arg \min } \Psi(f), \quad \Psi(f)=\frac{1}{2}\|\boldsymbol{y}-\mathcal{A} f\|_{\mathcal{W}^{1 / 2}}^{2} \tag{4.3.4}
\end{equation*}
$$

where the "diagonal" weighting operator $\mathcal{W}$ is defined here by

$$
\begin{equation*}
\boldsymbol{p}=\mathcal{W} \boldsymbol{q} \text { iff } p_{\varphi}[n]=w(\varphi) q_{\varphi}[n], \quad \forall \varphi \in[0, \pi], \quad \forall n \in \mathbb{Z} \tag{4.3.5}
\end{equation*}
$$

for some user-selected, nonzero, $\pi$-periodic angular weighting function $w(\varphi)$.
From the theory of optimization in Hilbert spaces [1, p. 160], any $\hat{f}$ that minimizes the cost function $\Psi(f)$ in (4.3.4) must satisfy the following normal equations

$$
\begin{equation*}
\mathcal{A}^{*} \mathcal{W} \mathcal{A} \hat{f}=\mathcal{A}^{*} \mathcal{W} \boldsymbol{y} \tag{4.3.6}
\end{equation*}
$$

There may be multiple solutions or a unique solution to this linear system of equations, depending on the properties of $\mathcal{A}^{*} \mathcal{W} \mathcal{A}$.

Hereafter we assume that the object is band limited such that the sampling interval $\triangle_{R}$ satisfies the Nyquist condition. (This too is unrealistic because real objects are space limited, but it is convenient for analysis because it avoids consideration of aliasing effects. See §4.3.9.) Then using arguments similar to those in $\S 3.3$, one can show that

$$
\begin{equation*}
\mathcal{A}^{*} \mathcal{W} \mathcal{A}=\mathcal{F}_{2}^{-1} \mathcal{D}\left(\frac{w(\Phi)\left|B_{\Phi}(\rho)\right|^{2}}{|\rho|}\right) \mathcal{F}_{2}, \tag{4.3.7}
\end{equation*}
$$

where $B_{\varphi}(\cdot)$ denotes the 1D FT of the blur $b_{\varphi}(\cdot)$. If the transfer function $B_{\varphi}(\cdot)$ has any zeros in the interval $\left[-1 /\left(2 \triangle_{\mathrm{R}}\right), 1 /\left(2 \triangle_{\mathrm{R}}\right)\right]$, then the Gram operator $\mathcal{A}^{*} \mathcal{W} \mathcal{A}$ has a null space within the space of band-limited functions, so the normal equations will have multiple solutions $\hat{f}$, meaning that the cost function $\Psi$ is not sufficiently "selective" to identify a unique solution.

If the transfer function $B_{\varphi}(\cdot)$ has no zeros, then the unique solution to the normal equations within the space of band-limited functions is given by

$$
\begin{equation*}
\hat{f}=\left[\mathcal{A}^{*} \mathcal{W} \mathcal{A}\right]^{-1} \mathcal{A}^{*} \mathcal{W} \boldsymbol{y}=\mathcal{F}_{2}^{-1} \mathcal{D}\left(\frac{|\rho|}{w(\Phi)\left|B_{\Phi}(\rho)\right|^{2}}\right) \mathcal{F}_{2} \mathcal{A}^{*} \mathcal{W} \boldsymbol{y} \tag{4.3.8}
\end{equation*}
$$

This is a kind of BPF reconstruction method; first we perform a (blurred and angularly weighted) backprojection of the sinogram $\boldsymbol{y}$, and then deconvolve both the $1 /|r|$ effect of tomography and deconvolve the effects of the blur. The frequency response of the deconvolution filter has both the cone filter component $|\rho|$ seen previously for the BPF method (3.4.1) as well as the (squared!) inverse filter $1 /\left|B_{\Phi}(\rho)\right|^{2}$. Typically the transfer function $B_{\varphi}(\cdot)$ of the blur is a lowpass type, so both the cone filter and the inverse filter would amplify high spatial frequency components. Therefore, the WLS method described by (4.3.4) and (4.3.8) would yield unacceptably noisy images in practice.

### 4.3.2 Regularization

To control this noise, one could simply apodize the deconvolution filter as described for the BPF method in §3.4.2. Instead, to provide more insight into the statistical image reconstruction methods described in later chapters, we consider modifying the WLS cost function to include a regularizing penalty function $\mathrm{R}(f)$. The motivations are similar to those discussed in Chapter 1. Here we are analyzing a formulation in which the object $f$ is a continuousspace function, so we consider continuous-space penalty functionals as mentioned in (2.4.1) of §2.4.

Natural approaches to quantifying roughness involve the partial derivatives of $f$. Define $\mathcal{D}_{j}$ to be the differentiation operator with respect to the $j$ th spatial coordinate:

$$
\left(\mathcal{D}_{j} f\right)\left(x_{1}, x_{2}\right)=\frac{\partial}{\partial x_{j}} f\left(x_{1}, x_{2}\right), \quad j=1,2 .
$$

Then for the usual $\mathcal{L}_{2}$ norm we have

$$
\left\|\mathcal{D}_{j} f\right\|^{2}=\iint\left|\frac{\partial}{\partial x_{j}} f\left(x_{1}, x_{2}\right)\right|^{2} \mathrm{~d} x_{1} \mathrm{~d} x_{2}
$$

so these norms quantify the roughness of $f$. (To use such norms we restrict attention to the subspace of functions with square integrable derivatives.) It is useful to express the differentiation operators in the frequency domain:

$$
\mathcal{D}_{j}=\mathcal{F}_{2}^{-1} \mathcal{D}\left(\imath 2 \pi \nu_{j}\right) \mathcal{F}_{2}
$$

where $\left(\nu_{1}, \nu_{2}\right)$ denote the frequency domain Cartesian coordinates. For the usual inner product for $\mathcal{L}_{2}\left(\mathbb{R}^{2}\right)$ defined in (4.2.1), it is straightforward to show that the adjoint of $\mathcal{D}_{j}$ is $\mathcal{D}_{j}^{*}=\mathcal{F}_{2}^{-1} \mathcal{D}\left(-\imath 2 \pi \nu_{j}\right) \mathcal{F}_{2}$ and to show that

$$
\mathcal{D}_{j}^{*} \mathcal{D}_{j}=\mathcal{F}_{2}^{-1} \mathcal{D}\left(\left(2 \pi \nu_{j}\right)^{2}\right) \mathcal{F}_{2}
$$

The operators $\left\{\mathcal{D}_{j}\right\}$ and their adjoints all commute because they are essentially just linear shift-invariant "filters." Furthermore, we have

$$
\mathcal{D}_{1}^{*} \mathcal{D}_{1}+\mathcal{D}_{2}^{*} \mathcal{D}_{2}=\mathcal{F}_{2}^{-1}\left[\mathcal{D}\left(\left(2 \pi \nu_{1}\right)^{2}\right)+\mathcal{D}\left(\left(2 \pi \nu_{2}\right)^{2}\right)\right] \mathcal{F}_{2}=\mathcal{F}_{2}^{-1} \mathcal{D}\left((2 \pi \rho)^{2}\right) \mathcal{F}_{2}
$$

where $\rho=\sqrt{\nu_{1}^{2}+\nu_{2}^{2}}$.
As discussed in $\S 2.4 .2$, it seems natural to use isotropic measures of roughness. Towards this end, we define the following operator:

$$
\begin{equation*}
\mathcal{R}=\left(\mathcal{D}_{1}^{*} \mathcal{D}_{1}+\mathcal{D}_{2}^{*} \mathcal{D}_{2}\right)^{M_{\mathrm{R}}} \tag{4.3.9}
\end{equation*}
$$

for some nonnegative power $M_{\mathrm{R}}$. For example, the case $M_{\mathrm{R}}=2$ corresponds to the thin-plate spline energy (2.4.2). The corresponding frequency domain expression is

$$
\begin{equation*}
\mathcal{R}=\mathcal{F}_{2}^{-1} \mathcal{D}(R(\rho, \Phi)) \mathcal{F}_{2} \tag{4.3.10}
\end{equation*}
$$

where the frequency response of the regularizer (4.3.9) is ${ }^{3}$ :

$$
\begin{equation*}
R(\rho, \Phi)=(2 \pi \rho)^{2 M_{\mathrm{R}}} \tag{4.3.11}
\end{equation*}
$$

Using this operator, the following functional is a natural quadratic roughness measure that is convenient for Fourier analysis:

$$
\begin{equation*}
\mathrm{R}(f)=\frac{1}{2}\langle f, \boldsymbol{\mathcal { R }} f\rangle=\frac{1}{2}\left\|\mathcal{D}(\sqrt{R(\rho, \Phi)}) \mathcal{F}_{2} f\right\|^{2} \tag{4.3.12}
\end{equation*}
$$

using the usual $\mathcal{L}_{2}$ inner product (4.2.1). (Chu and Tam considered the case $M_{\mathrm{R}}=2$ in 1977 [12].)

### 4.3.3 QPWLS analytical reconstruction

Having defined a convenient quadratic roughness measure, we now analyze the effects of replacing the WLS cost function (4.3.4) with a quadratically penalized weighted least squares (QPWLS) cost function of the form

$$
\begin{equation*}
\Psi(f)=\frac{1}{2}\|\boldsymbol{y}-\mathcal{A} f\|_{\mathcal{W}^{1 / 2}}^{2}+\beta \mathrm{R}(f) \tag{4.3.13}
\end{equation*}
$$

Any minimizer $\hat{f}$ of this cost function must satisfy the following modified normal equations

$$
\left[\mathcal{A}^{*} \mathcal{W} \mathcal{A}+\beta \mathcal{R}\right] \hat{f}=\mathcal{A}^{*} \mathcal{W} \boldsymbol{y}
$$

[^1]In the frequency domain, combining (4.3.7) and (4.3.10) yields

$$
\begin{equation*}
\left[\mathcal{A}^{*} \mathcal{W} \mathcal{A}+\beta \mathcal{R}\right]=\mathcal{F}_{2}^{-1} \mathcal{D}\left(\frac{w(\Phi)\left|B_{\Phi}(\rho)\right|^{2}}{|\rho|}+\beta R(\rho, \Phi)\right) \mathcal{F}_{2} \tag{4.3.14}
\end{equation*}
$$

Clearly the only frequency component in the null space of $\mathcal{R}$ is $\mathrm{DC}(\rho=0)$, and this component will not be in the null space of $\mathcal{A}^{*} \mathcal{W} \mathcal{A}$ unless $B_{\varphi}(0)=0$, which would be quite unusual because $b_{\varphi}(r)$ is usually a lowpass filter. Assuming $B_{\varphi}(0) \neq 0$ hereafter, the solution to the modified normal equations above is given uniquely by

$$
\begin{align*}
\hat{f} & =\mathcal{F}_{2}^{-1} \mathcal{D}\left(\frac{w(\Phi)\left|B_{\Phi}(\rho)\right|^{2}}{|\rho|}+\beta R(\rho, \Phi)\right)^{-1} \mathcal{F}_{2} \mathcal{A}^{*} \mathcal{W} \boldsymbol{y}  \tag{4.3.15}\\
& =\mathcal{F}_{2}^{-1} \mathcal{D}\left(\frac{|\rho|}{w(\Phi)\left|B_{\Phi}(\rho)\right|^{2}+\beta|\rho| R(\rho, \Phi)}\right) \mathcal{F}_{2} \mathcal{A}^{*} \mathcal{W} \boldsymbol{y} \tag{4.3.16}
\end{align*}
$$

This estimator has the form of the BPF reconstruction method except that the cone filter has been "apodized" due to the regularization.

For further insight, we examine next the spatial resolution and noise properties of the QPWLS estimator (4.3.16).

### 4.3.4 Spatial resolution properties (s,tomo,blur,prop)

Using (4.3.3), the ensemble mean of the reconstructed image $\hat{f}$ is given by

$$
\begin{aligned}
\mathrm{E}[\hat{f}] & =\left[\mathcal{A}^{*} \mathcal{W} \mathcal{A}+\beta \boldsymbol{\mathcal { R }}\right]^{-1} \mathcal{A}^{*} \mathcal{W} \mathrm{E}[\boldsymbol{y}] \\
& =\left[\mathcal{A}^{*} \mathcal{W} \mathcal{A}+\beta \boldsymbol{R}\right]^{-1} \mathcal{A}^{*} \mathcal{W} \boldsymbol{A} f \\
& =\mathcal{F}_{2}^{-1} \mathcal{D}\left(\frac{|\rho|}{w(\Phi)\left|B_{\Phi}(\rho)\right|^{2}+\beta|\rho| R(\rho, \Phi)}\right) \mathcal{D}\left(\frac{w(\Phi)\left|B_{\Phi}(\rho)\right|^{2}}{|\rho|}\right) \mathcal{F}_{2} f \\
& =\mathcal{F}_{2}^{-1} \mathcal{D}\left(\frac{w(\Phi)\left|B_{\Phi}(\rho)\right|^{2}}{w(\Phi)\left|B_{\Phi}(\rho)\right|^{2}+\beta|\rho| R(\rho, \Phi)}\right) \mathcal{F}_{2} f
\end{aligned}
$$

Thus, the mean reconstructed image corresponds to the true image smoothed by a filter with frequency response [3, eqn. (10)]:

$$
\begin{equation*}
L(\rho, \Phi)=\frac{w(\Phi)\left|B_{\Phi}(\rho)\right|^{2}}{w(\Phi)\left|B_{\Phi}(\rho)\right|^{2}+\beta|\rho| R(\rho, \Phi)} \tag{4.3.17}
\end{equation*}
$$

Due to the $R(\rho, \Phi)$ term in the denominator (see (4.3.11)), this "filter" attenuates high spatial frequencies, with greater attenuation for large values of the regularization parameter $\beta$. See [3] for an approximation that leads to a closed-form solution to the corresponding local impulse response.

Example 4.3.1 The (normalized) frequency response of a scintillating screen with attenuation coefficient $\mu$ and thickness $d$ is [13, p. 66]

$$
B(\nu)=\frac{1}{(1+2 \pi \nu / \mu)\left(1-\mathrm{e}^{-\mu d}\right)}\left[1-\mathrm{e}^{-\mu d(1+2 \pi \nu / \mu)}\right]
$$

Fig. 4.3.1 shows the resulting image-domain frequency response $L(\rho)$ for the unweighted case $w(\varphi)=1$, for $m=1$, $\mu=1.5 / \mathrm{mm}$, and $d=0.25 \mathrm{~mm}$. As $\beta$ increases, the knee of $L(\rho)$ moves towards lower spatial frequencies. Fig. 4.3.2 shows the corresponding point spread functions. As $\beta$ decreases, the PSF can exhibit negative sidelobes.

### 4.3.5 Noise properties

The QPWLS estimator (4.3.16) has two design variables: the angular weighting function $w(\varphi)$, and the regularizer (e.g., the regularization parameter $\beta$, and the regularization order $m$ ). To select these parameters appropriately, one should consider both the resolution and the noise properties of the resulting estimator $\hat{f}$.

As described in $\S 1.6 .1$, we can characterize the covariance properties of a random vector $\boldsymbol{z} \in \mathbb{R}^{n}$ using its covariance matrix, e.g.,

$$
\boldsymbol{\Pi}_{z}=\operatorname{Cov}\{\boldsymbol{z}\} \in \mathbb{R}^{n \times n}
$$

Such a matrix can be viewed as a linear operator from $\mathbb{R}^{n}$ to $\mathbb{R}^{n}$ in the following sense:

$$
\boldsymbol{u}=\boldsymbol{\Pi}_{\boldsymbol{z}} \boldsymbol{v} \Longleftrightarrow u_{j}=\sum_{k} \operatorname{Cov}\left\{z_{j}, z_{k}\right\} z_{k}, \quad j=1, \ldots, n
$$



Figure 4.3.1: Frequency response for QPWLS reconstructed images for a range of values of $\beta$.


Figure 4.3.2: Point spread functions corresponding to three of the frequency responses shown in Fig. 4.3.1.

To analyze the noise properties of the QPWLS estimator (4.3.16), we need to generalize the covariance property (1.9.3) to the case of continuous-space functions, an infinite dimensional vector space.

For continuous-space functions, we need a covariance operator rather than a covariance matrix. Because $\hat{f} \in$ $\mathcal{L}_{2}\left(\mathbb{R}^{2}\right)$, we define the covariance operator $\mathcal{K}_{\hat{f}}=\operatorname{Cov}\{\hat{f}\}$ as follows:

$$
\begin{align*}
g=\mathcal{K}_{\hat{f}} h \Longleftrightarrow g(\overrightarrow{\mathrm{x}}) & =\langle\operatorname{Cov}\{\hat{f}(\overrightarrow{\mathrm{x}}), \hat{f}(\cdot)\}, h(\cdot)\rangle \\
& =\iint \operatorname{Cov}\left\{\hat{f}(\overrightarrow{\mathrm{x}}), \hat{f}\left(\overrightarrow{\mathrm{x}}^{\prime}\right)\right\} h\left(\overrightarrow{\mathrm{x}}^{\prime}\right) \mathrm{d} \overrightarrow{\mathrm{x}}^{\prime} \tag{4.3.18}
\end{align*}
$$

where $\overrightarrow{\mathrm{x}}=(x, y)$. It follows from this definition that

$$
\operatorname{Cov}\left\{\hat{f}\left(\overrightarrow{\mathrm{x}}^{\prime}\right), \hat{f}(\overrightarrow{\mathrm{x}})\right\}=\left\langle\delta_{\overrightarrow{\mathrm{x}}^{\prime}}, \mathcal{K}_{\hat{f}} \delta_{\overrightarrow{\mathrm{x}}}\right\rangle,
$$

where $\delta_{\vec{x}}$ denotes the Dirac impulse located at position $\overrightarrow{\mathrm{x}}$.
Similarly, we define a covariance operator for $\boldsymbol{y}$ as follows:

$$
\boldsymbol{u}=\operatorname{Cov}\{\boldsymbol{y}\} \boldsymbol{v} \Longleftrightarrow u_{\varphi^{\prime}}\left[n^{\prime}\right]=\sum_{n=-\infty}^{\infty} \int_{0}^{\pi} \operatorname{Cov}\left\{y_{\varphi^{\prime}}\left[n^{\prime}\right], y_{\varphi}[n]\right\} v_{\varphi}[n] \mathrm{d} \varphi
$$

With these definitions, if $\hat{f}=\mathcal{L} \boldsymbol{y}$ for a linear operator $\mathcal{L}: \mathcal{L}_{2}([0, \pi] \times \mathbb{R}) \rightarrow \mathcal{L}_{2}\left(\mathbb{R}^{2}\right)$ having adjoint $\mathcal{L}^{*}$, then one can prove the following generalization of (1.9.3):

$$
\operatorname{Cov}\{\hat{f}\}=\operatorname{Cov}\{\boldsymbol{L} \boldsymbol{y}\}=\mathcal{L} \operatorname{Cov}\{\boldsymbol{y}\} \mathcal{L}^{*}
$$

Considering the linear form of (4.3.16), the covariance operator $\mathcal{K}_{\hat{f}}$ for the QPWLS estimator $\hat{f}$ is:

$$
\begin{aligned}
& \mathcal{K}_{\hat{f}}=\mathcal{F}_{2}^{-1} \mathcal{D}\left(\frac{|\rho|}{w(\Phi)\left|B_{\Phi}(\rho)\right|^{2}+\beta|\rho| R(\rho, \Phi)}\right) \mathcal{F}_{2} \mathcal{A}^{*} \mathcal{W} \operatorname{Cov}\{\boldsymbol{y}\} \\
& \quad \cdot \mathcal{W} \mathcal{A}_{\mathcal{F}}^{2} \\
& -\mathcal{D}\left(\frac{|\rho|}{w(\Phi)\left|B_{\Phi}(\rho)\right|^{2}+\beta|\rho| R(\rho, \Phi)}\right) \mathcal{F}_{2}
\end{aligned}
$$

where we have used the fact that $\mathcal{F}_{2}^{*}=\mathcal{F}_{2}^{-1}$. To facilitate analysis, hereafter we assume that the covariance of $\boldsymbol{y}$ depends only on the projection angle $\varphi$, and is not a function of the radial sample $n$, and is independent from angle to angle. In particular, we assume:

$$
\operatorname{Cov}\left\{y_{\varphi_{1}}\left[n_{1}\right], y_{\varphi_{2}}\left[n_{2}\right]\right\}= \begin{cases}c\left(\varphi_{1}\right) \delta\left(\varphi_{1}-\varphi_{2}\right), & n_{1}=n_{2}  \tag{4.3.19}\\ 0, & \text { otherwise }\end{cases}
$$

where $c(\cdot)$ is some $\pi$-periodic function. Thus if $\boldsymbol{u}=\operatorname{Cov}\{\boldsymbol{y}\} \boldsymbol{v}$, then $u_{\varphi}[n]=c(\varphi) v_{\varphi}[n]$. In the frequency domain, it follows from the Fourier-slice theorem that the middle portion of $\mathcal{K}_{\hat{f}}$ simplifies as follows:

$$
\begin{equation*}
\mathcal{A}^{*} \mathcal{W} \operatorname{Cov}\{\boldsymbol{y}\} \mathcal{W} \mathcal{A}=\mathcal{F}_{2}^{-1} \mathcal{D}\left(\frac{c(\Phi) w^{2}(\Phi)\left|B_{\Phi}(\rho)\right|^{2}}{|\rho|}\right) \mathcal{F}_{2} \tag{4.3.20}
\end{equation*}
$$

Combining with the preceding expression for $\mathcal{K}_{\hat{f}}$ yields the following covariance expression for the QPWLS estimator:

$$
\begin{equation*}
\mathcal{K}_{\hat{f}}=\mathcal{F}_{2}^{-1} \mathcal{D}\left(\frac{c(\Phi) w^{2}(\Phi)|\rho|\left|B_{\Phi}(\rho)\right|^{2}}{\left(w(\Phi)\left|B_{\Phi}(\rho)\right|^{2}+\beta|\rho| R(\rho, \Phi)\right)^{2}}\right) \mathcal{F}_{2} . \tag{4.3.21}
\end{equation*}
$$

The form of this expression indicates that under the noise model (4.3.19), $\hat{f}-\mathrm{E}[\hat{f}]$ is a wide-sense stationary random process for the QPWLS estimator, and the diagonal term reflects its power spectral density.

### 4.3.6 Optimization of angular weighting

The Gauss-Markov theorem [14, p. 141] of statistical estimation states that for a linear model of the form (4.3.3), the minimum-variance linear unbiased estimator of $f$ is given by

$$
\hat{f}=\left[\mathcal{A}^{*} \mathcal{C}^{-1} \mathcal{A}\right]^{-1} \mathcal{A} \mathcal{C}^{-1} \boldsymbol{y}
$$

where $\mathcal{C}$ denotes the covariance of $\boldsymbol{y}$, assuming the inverse exists. In other words, the minimum-variance estimator is of the WLS form (4.3.6) where the weighting operator $\mathcal{W}$ is chosen to be the inverse of the covariance operator:

$$
\begin{equation*}
\mathcal{W}=\mathcal{C}^{-1} \tag{4.3.22}
\end{equation*}
$$

When I was first deriving this section, I expected to arrive at an expression like (4.3.21) in which the optimal choice of the angular weighting would be given by $w(\varphi)=1 / c(\varphi)$, in analogy with (4.3.22). However, if we consider the unregularized case (where $\beta=0$ ), then the angular weighting $w(\varphi)$ in (4.3.21) becomes irrelevant! This conclusion would seem to contradict practical experience with iterative reconstruction showing that weighting can be highly relevant, e.g., [15]. Evidently the noise model (4.3.19) is over simplified. In PET, SPECT and X-ray CT, the noise variance changes both with projection angle and with radial bin position; the radial variations are ignored in (4.3.19) because including them would preclude the exact Fourier analysis result (4.3.21). (See $\S 4.4 .2$ for approximate analysis.) Unfortunately, Fourier analysis does not appear to provide insight into the importance of weighting.

### 4.3.7 Resolution-noise trade-offs

Consider the simplest case where there is no system blur: $B_{\varphi}(\cdot)=1$, the measurement variance is constant: $c(\cdot)=1$, and we choose uniform angular weights: $w(\cdot)=1$. Then the spatial resolution properties are governed by the frequency response (4.3.17) that simplifies to

$$
L(\rho)=\frac{1}{1+\beta|\rho||2 \pi \rho|^{2 M_{\mathrm{R}}}},
$$

using (4.3.11). The noise properties are governed by the diagonal of (4.3.21) that simplifies to

$$
\frac{|\rho|}{\left(1+\beta|\rho||2 \pi \rho|^{2 M_{\mathrm{R}}}\right)^{2}}=|\rho| L^{2}(\rho) .
$$

Clearly, to have good spatial resolution, we would like $\beta$ to be small, so that $L(\rho) \approx 1$. On the other hand, to have low image noise, we would like $\beta$ to be large, so that $L^{2}(\cdot) \approx 0$. These two expressions epitomize the resolution-noise trade-off for tomographic image reconstruction: as we try to recover higher object spatial frequency components, i.e., as $L(\rho) \rightarrow 1$, we will concurrently increase noise, particularly at high spatial frequencies where $|\rho|$ is large. This analysis is a continuous-space analog of the results in $\S 1.9$, and it is useful to be familiar with both the matrix-vector version and the continuous-space version.

### 4.3.8 Isotropic spatial resolution (s,tomo,blur,iso)

This subsection considers the case where the system blur is independent of angle, i.e., $b_{\varphi}(r)=b(r)$, and examines the anisotropy that results from angular weighting $w(\varphi)$.

If the angular weighting function $w(\varphi)$ is nonuniform, then when we use the isotropic regularizer (4.3.11), the resulting frequency response (4.3.17) has an angularly-dependent component, meaning that the the spatial resolution properties of $\hat{f}$ will be anisotropic. Of course, one way to eliminate this anisotropy would be to choose uniform angular weighting, e.g., $w(\varphi)=1$. However, under more realistic noise models than (4.3.19), uniform angular weighting might be suboptimal in terms of the estimator noise properties. An alternative way to eliminate (or at least reduce) the anisotropy would be to modify the regularization method by replacing (4.3.11) with an anisotropic operator. Considering (4.3.17), to provide isotropic spatial resolution, the desired regularization operator has the following anisotropic frequency response

$$
\begin{equation*}
R(\rho, \Phi)=w(\Phi)|2 \pi \rho|^{2 M_{\mathrm{R}}} \tag{4.3.23}
\end{equation*}
$$

so that the transfer function between $\mathrm{E}[\hat{f}]$ and $f$ becomes

$$
L(\rho, \Phi)=\frac{|B(\rho)|^{2}}{|B(\rho)|^{2}+\beta|\rho||2 \pi \rho|^{2 M_{\mathrm{R}}}}
$$

which is independent of $\Phi$ and hence isotropic. The power spectral density term in (4.3.21) becomes

$$
\frac{c(\Phi)|\rho||B(\rho)|^{2}}{\left(|B(\rho)|^{2}+\beta|\rho||2 \pi \rho|^{2 M_{\mathrm{R}}}\right)^{2}}
$$

For nonuniform noise variance $c(\varphi)$, this QPWLS estimator has anisotropic noise covariance. Apparently it is difficult, if not impossible, to achieve both isotropic spatial resolution and isotropic noise correlation, even under the oversimplified noise model (4.3.19).

Although one cannot attain (4.3.23) exactly in practice, $\S 5.2$ describes a practical procedure for designing a regularizer that leads to nearly isotropic spatial resolution.

An alternative approach to attaining isotropic spatial resolution would be to use the BPF method followed by isotropic smoothing:

$$
\begin{equation*}
\hat{f}=\boldsymbol{\mathcal { B }}_{\text {smooth }}\left(\mathcal{P}^{*} \boldsymbol{\mathcal { P }}\right)^{-1} \boldsymbol{\mathcal { P }}^{*} \boldsymbol{y} \tag{4.3.24}
\end{equation*}
$$

where $\mathcal{B}_{\text {smooth }}=\mathcal{F}_{2}^{-1} B_{\text {smooth }}(\rho) \mathcal{F}_{2}$. By analyses similar to those leading to (4.3.21), one can show that the covariance of this estimator is

$$
\operatorname{Cov}\{\hat{f}\}=\mathcal{F}_{2}^{-1} \mathcal{D}\left(c(\Phi)|\rho|\left|B_{\text {smooth }}(\rho)\right|^{2}\right) \mathcal{F}_{2}
$$

In particular, if we choose the post-reconstruction smoothing filter to have the following frequency response:

$$
B_{\mathrm{smooth}}(\rho)=\frac{B^{*}(\rho)}{|B(\rho)|^{2}+\beta|\rho||2 \pi \rho|^{2 M_{\mathrm{R}}}}
$$

then the post-smoothed BPF method has the exact same spatial resolution and noise properties as the QPWLS method with the regularization method (4.3.23). This conclusion is consistent with empirical studies that have found postsmooth maximum-likelihood reconstruction to have similar resolution-noise trade-offs as quadratically-penalized likelihood estimators for PET and SPECT [16, 17].

### 4.3.9 Aliasing effects due to radial sampling (s,topo,alias)

In general the objects that are imaged in tomography are space-limited, so they cannot be band-limited. So any reconstruction from sampled data like (4.3.1), even noiseless data, will be degraded by aliasing. This section provides a Fourier analysis of the aliasing effects of radial sampling, for conventional FBP reconstruction.

We first analyze the spectrum of the sampled sinogram data $\bar{y}_{\varphi}[n]$ defined in (4.3.1). Let $\bar{y}_{\varphi}[n]=\left.q_{\varphi}(r)\right|_{r=n \Delta_{\mathrm{R}}}$, where $q_{\varphi}(r) \triangleq p_{\varphi}(r) * b_{\varphi}(r)$. Then by the Fourier slice theorem and the convolution property of the 1D FT:

$$
q_{\varphi}(r) \stackrel{\mathrm{FT}}{\longleftrightarrow} Q_{\varphi}(\nu)=P_{\varphi}(\nu) B_{\varphi}(\nu)=F_{\circ}(\nu, \varphi) B_{\varphi}(\nu) .
$$

By the sampling theorem:

$$
\bar{y}_{\varphi}[n] \stackrel{\mathrm{DTFT}}{\longleftrightarrow} \bar{Y}_{\varphi}(\omega)=\frac{1}{\triangle_{\mathrm{R}}} \sum_{k=-\infty}^{\infty} Q_{\varphi}\left(\frac{\omega}{2 \pi \triangle_{\mathrm{R}}}-\frac{k}{\triangle_{\mathrm{R}}}\right)=\frac{1}{\triangle_{\mathrm{R}}} \sum_{k=-\infty}^{\infty} F_{\circ}\left(\frac{\omega}{2 \pi \triangle_{\mathrm{R}}}-\frac{k}{\triangle_{\mathrm{R}}}, \varphi\right) B_{\varphi}\left(\frac{\omega}{2 \pi \triangle_{\mathrm{R}}}-\frac{k}{\triangle_{\mathrm{R}}}\right) .
$$

This is aliasing of the object spectrum along the direction $\varphi$.
Ignoring noise, we apply a ramp filter to the sampled sinogram data $\bar{y}_{\varphi}[n]$. Specifically, we use samples of the impulse response of the band-limited ramp filter defined in (3.4.14), possibly with additional apodization $A(\nu)$. In other words:

$$
\check{q}_{\varphi}[n] \triangleq h[n] * \bar{y}_{\varphi}[n],
$$

where $h[n]=h_{\mathrm{A}}\left(n \triangle_{\mathrm{R}}\right)$ and $h_{\mathrm{A}}(r) \stackrel{\mathrm{FT}}{\longleftrightarrow} H_{\mathrm{A}}(\nu)=|\nu| A(\nu)$. Thus by the sampling theorem:

$$
h[n] \stackrel{\mathrm{DTFT}}{\longleftrightarrow} H(\omega)=\frac{1}{\triangle_{\mathrm{R}}} \sum_{k=-\infty}^{\infty} H_{\mathrm{A}}\left(\frac{\omega}{2 \pi \triangle_{\mathrm{R}}}-\frac{k}{\triangle_{\mathrm{R}}}\right) .
$$

By the (discrete-time) convolution property:

$$
\begin{aligned}
\check{q}_{\varphi}[n] \stackrel{\mathrm{DTFT}}{\longleftrightarrow} & \check{Q}_{\varphi}(\omega)=H(\omega) \bar{Y}_{\varphi}(\omega) \\
& =\left[\frac{1}{\triangle_{\mathrm{R}}} \sum_{k=-\infty}^{\infty} H_{\mathrm{A}}\left(\frac{\omega}{2 \pi \triangle_{\mathrm{R}}}-\frac{k}{\triangle_{\mathrm{R}}}\right)\right]\left[\frac{1}{\triangle_{\mathrm{R}}} \sum_{k=-\infty}^{\infty} F_{\circ}\left(\frac{\omega}{2 \pi \triangle_{\mathrm{R}}}-\frac{k}{\triangle_{\mathrm{R}}}, \varphi\right) B_{\varphi}\left(\frac{\omega}{2 \pi \triangle_{\mathrm{R}}}-\frac{k}{\triangle_{\mathrm{R}}}\right)\right] .
\end{aligned}
$$

Prior to backprojection, we use some interpolation kernel $c(r)$ to interpolate the filtered projections $\check{q}_{\varphi}[n]$ :

$$
\hat{q}_{\varphi}(r)=\sum_{n=-\infty}^{\infty} \check{q}_{\varphi}[n] c\left(r-n \triangle_{\mathrm{R}}\right) \stackrel{\mathrm{FT}}{\longleftrightarrow} \hat{Q}_{\varphi}(\nu)=C(\nu) \sum_{n=-\infty}^{\infty} \check{q}_{\varphi}[n] \mathrm{e}^{-\imath 2 \pi n \triangle_{\mathrm{R}} \nu}=\left.C(\nu) \check{Q}_{\varphi}(\omega)\right|_{\omega=2 \pi \triangle_{\mathrm{R}} \nu} .
$$

Thus the spectra of the filtered and interpolated projections are:

$$
\hat{Q}_{\varphi}(\nu)=C(\nu)\left[\frac{1}{\triangle_{\mathrm{R}}} \sum_{k=-\infty}^{\infty} H_{\mathrm{A}}\left(\nu-\frac{k}{\triangle_{\mathrm{R}}}\right)\right]\left[\frac{1}{\triangle_{\mathrm{R}}} \sum_{k=-\infty}^{\infty} F_{\circ}\left(\nu-\frac{k}{\triangle_{\mathrm{R}}}, \varphi\right) B_{\varphi}\left(\nu-\frac{k}{\triangle_{\mathrm{R}}}\right)\right] .
$$

Applying (3.3.10), which considers a continuum of projection angles $\varphi$, the spectrum of the FBP reconstruction is:

$$
F_{\mathrm{b}}(\rho, \Phi)=\frac{1}{|\rho|} C(\rho)\left[\frac{1}{\triangle_{\mathrm{R}}} \sum_{k=-\infty}^{\infty} H_{\mathrm{A}}\left(\rho-\frac{k}{\triangle_{\mathrm{R}}}\right)\right]\left[\frac{1}{\triangle_{\mathrm{R}}} \sum_{k=-\infty}^{\infty} F_{\circ}\left(\rho-\frac{k}{\triangle_{\mathrm{R}}}, \Phi\right) B_{\Phi}\left(\rho-\frac{k}{\triangle_{\mathrm{R}}}\right)\right] .
$$

In particular, if $H_{\mathrm{A}}(\nu)=|\nu| \operatorname{rect}\left(\nu \triangle_{\mathrm{R}}\right) A(\nu)$, then the spectrum of the FBP reconstruction is:

$$
\begin{equation*}
F_{\mathrm{b}}(\rho, \Phi)=C(\rho) A(\rho) \operatorname{rect}\left(\rho \triangle_{\mathrm{R}}\right)\left[\frac{1}{\triangle_{\mathrm{R}}} \sum_{k=-\infty}^{\infty} F_{\circ}\left(\rho-\frac{k}{\triangle_{\mathrm{R}}}, \Phi\right) B_{\Phi}\left(\rho-\frac{k}{\triangle_{\mathrm{R}}}\right)\right] \tag{4.3.25}
\end{equation*}
$$

The $k \neq 0$ terms correspond to aliasing due to radial sampling. (See $\S 3.7 .2$ and Problem 3.6 for angular sampling.) Although generally blur is considered undesirable, the blur spectrum $B_{\Phi}(\rho)$ within the summation can have the beneficial effect of helping to reduce aliasing.

### 4.4 Local shift invariance (s,tomo,local)

The preceding sections' analyses of spatial resolution properties focused on shift-invariant situations for which Fourier methods are applicable easily. For linear systems with shift-varying PSFs, often we can apply local shift-invariance approximations to formulate useful predictions of resolution and noise properties.

### 4.4.1 Local impulse response

Consider a linear operator $\mathcal{L}$ defined by the superposition integral

$$
\begin{equation*}
g=\mathcal{L} f \Longleftrightarrow g(\overrightarrow{\mathrm{x}})=\int h\left(\overrightarrow{\mathrm{x}}, \overrightarrow{\mathrm{x}}^{\prime}\right) f\left(\overrightarrow{\mathrm{x}}^{\prime}\right) \mathrm{d} \overrightarrow{\mathrm{x}}^{\prime} \tag{4.4.1}
\end{equation*}
$$

If this integral described a shift-invariant operator, then its kernel $h$ would satisfy

$$
h\left(\overrightarrow{\mathrm{x}}, \overrightarrow{\mathrm{x}}^{\prime}\right)=h\left(\overrightarrow{\mathrm{x}}-\overrightarrow{\mathrm{x}}^{\prime}, 0\right)
$$

and the superposition integral would become a convolution integral. For shift-varying operators, an exact convolution expression is unattainable. But often the position dependence of the PSF $h\left(\cdot, \vec{x}^{\prime}\right)$ varies slowly as a function of $\vec{x}^{\prime}$, in which case we can define a local shift invariance property as follows.

Definition 4.4.1 A linear operator of the form (4.4.1) with kernel $h\left(\overrightarrow{\mathrm{x}}, \overrightarrow{\mathrm{x}}^{\prime}\right)$ is said to be locally shift invariant near a point $\overrightarrow{\mathrm{x}}_{0}$ if

$$
\overrightarrow{\mathrm{x}}^{\prime} \approx \overrightarrow{\mathrm{x}}_{0} \text { and } \vec{\tau} \approx \overrightarrow{0} \Longrightarrow h\left(\overrightarrow{\mathrm{x}}^{\prime}+\vec{\tau}, \overrightarrow{\mathrm{x}}^{\prime}\right) \approx h\left(\overrightarrow{\mathrm{x}}_{0}+\vec{\tau}, \overrightarrow{\mathrm{x}}_{0}\right)
$$

i.e.,

$$
h\left(\overrightarrow{\mathrm{x}}, \overrightarrow{\mathrm{x}}^{\prime}\right) \approx h\left(\overrightarrow{\mathrm{x}}-\overrightarrow{\mathrm{x}}^{\prime}+\overrightarrow{\mathrm{x}}_{0}, \overrightarrow{\mathrm{x}}_{0}\right) .
$$

When this property holds near some spatial location $\overrightarrow{\mathrm{x}}_{0}$ of interest, we define the following local impulse response:

$$
h_{0}(\vec{\tau})=h\left(\vec{\tau}+\overrightarrow{\mathrm{x}}_{0}, \overrightarrow{\mathrm{x}}_{0}\right)
$$

If a linear operator is locally shift invariant near $\vec{x}_{0}$, and if an object $f_{0}(\cdot)$ is spatially "concentrated" around $\vec{x}_{0}$, i.e., $\left|f_{0}(\overrightarrow{\mathrm{x}})\right| \ll\left|f_{0}\left(\overrightarrow{\mathrm{x}}_{0}\right)\right|$ for $\overrightarrow{\mathrm{x}}$ far from $\overrightarrow{\mathrm{x}}_{0}$, then we can approximate the superposition integral as follows:

$$
\begin{aligned}
g(\overrightarrow{\mathrm{x}}) & =\int h\left(\overrightarrow{\mathrm{x}}, \overrightarrow{\mathrm{x}}^{\prime}\right) f_{0}\left(\overrightarrow{\mathrm{x}}^{\prime}\right) \mathrm{d} \overrightarrow{\mathrm{x}}^{\prime}=\int h\left(\overrightarrow{\mathrm{x}}^{\prime}+\left(\overrightarrow{\mathrm{x}}-\overrightarrow{\mathrm{x}}^{\prime}\right), \overrightarrow{\mathrm{x}}^{\prime}\right) f_{0}\left(\overrightarrow{\mathrm{x}}^{\prime}\right) \mathrm{d} \overrightarrow{\mathrm{x}}^{\prime} \\
& \approx \int h_{0}\left(\overrightarrow{\mathrm{x}}-\overrightarrow{\mathrm{x}}^{\prime}\right) f_{0}\left(\overrightarrow{\mathrm{x}}^{\prime}\right) \mathrm{d} \overrightarrow{\mathrm{x}}^{\prime}=h_{0}(\overrightarrow{\mathrm{x}}) * f_{0}(\overrightarrow{\mathrm{x}}) .
\end{aligned}
$$

This approximation is exact if $f_{0}$ is "perfectly" concentrated around $\vec{x}_{0}$, meaning that $f_{0}(\overrightarrow{\mathrm{x}})=\delta\left(\overrightarrow{\mathrm{x}}-\overrightarrow{\mathrm{x}}_{0}\right)$, because in this case $h_{0}(\overrightarrow{\mathrm{x}}) * f_{0}(\overrightarrow{\mathrm{x}})=h_{0}\left(\overrightarrow{\mathrm{x}}-\overrightarrow{\mathrm{x}}_{0}\right)=h\left(\overrightarrow{\mathrm{x}}, \overrightarrow{\mathrm{x}}_{0}\right)$, agreeing with (4.4.1). In other words, for a linear operator that is locally shift invariant near $\overrightarrow{\mathrm{x}}_{0}$ and an object $f_{0}$ that is spatially localized around that point, e.g., a small lesion, the superposition integral is approximately the convolution of the object with the local impulse response. This local shift invariance property is very useful for analyzing the resolution and noise properties of image formation methods. In the frequency domain, we write

$$
\mathcal{L} \approx \mathcal{F}_{d}^{-1} \mathcal{D}\left(H_{0}(\vec{\nu})\right) \mathcal{F}_{d}
$$

where $H_{0}(\cdot)$ denotes the $\bar{d}$-dimensional FT of $h_{0}(\cdot)$. Often it is left implicit that this approximation is accurate only for objects that are spatially localized around $\overrightarrow{\mathrm{x}}_{0}$. See also Chapter 22.

### 4.4.2 Radially-dependent weighting and position-dependent blur

Previous sections focused on weighting functions $w(\varphi)$ that depend only on the angle $\varphi$. In practice, the desired weighting function will also depend on the radial position $r$, i.e., the weighting function $\mathcal{W}$ in (4.2.5) or (4.3.5) should have a "diagonal" of the form $w_{\varphi}(r)$. Furthermore, tomographic systems can have position-dependent blur, such as the effects of crystal penetration in PET, e.g., $[18,19]$, and depth-dependent detector response in SPECT. So the system model operator $\mathcal{A}$ in (4.3.3) should be generalized to an integral of the form

$$
q=\mathcal{A} f \Longleftrightarrow q_{\varphi}(r)=\int a(r, \varphi ; x, y) f(x, y) \mathrm{d} x \mathrm{~d} y
$$

where $a(r, \varphi ; x, y)$ denotes the response ${ }^{4}$ of the system to an impulse at $(x, y)$. For the usual $\mathcal{L}_{2}$ inner products, the adjoint of this operator is

$$
\left(\mathcal{A}^{*} q\right)(x, y)=\int_{0}^{\pi} \int_{-\infty}^{\infty} a^{*}(r, \varphi ; x, y) q_{\varphi}(r) \mathrm{d} r \mathrm{~d} \varphi
$$

Strictly speaking, the analyses of the preceding section do not apply directly to these generalizations. However, often the system response functions $a(\cdot)$ are locally shift invariant, and in some cases the weighting function $w_{\varphi}(r)$ varies slowly with $r$. Then because the PSF of QPWLS estimators ( $c f$. (4.3.17)) are spatially concentrated, as are the autocorrelation functions ( $c f$. (4.3.21)), we can develop useful approximations to the local impulse response functions and autocorrelation functions of the QPWLS estimator even for the generalized $\mathcal{A}$ and $\mathcal{W}$ above.

The key to any such approximations is to analyze the Gram operator $\mathcal{A}^{*} \mathcal{W} \mathcal{A}$ that appears in (4.3.6). (We ignore sampling here for simplicity.) Consider a certain spatial location of interest, say $\left(x_{0}, y_{0}\right)$, and define $\delta_{0}(x, y)=$ $\delta_{2}\left(x-x_{0}, y-y_{0}\right)$. Then the Gram operator corresponds to a superposition integral with kernel

$$
h\left(x, y ; x_{0}, y_{0}\right)=\left(\mathcal{A}^{*} \mathcal{W} \mathcal{A} \delta_{0}\right)(x, y)
$$

where

$$
\left(\mathcal{A} \delta_{0}\right)(r, \varphi)=a\left(r, \varphi ; x_{0}, y_{0}\right) .
$$

Using the above adjoint $\mathcal{A}^{*}$, we see

$$
\begin{equation*}
h\left(x, y ; x_{0}, y_{0}\right)=\int_{0}^{\pi} \int_{-\infty}^{\infty} a^{*}(r, \varphi ; x, y) w_{\varphi}(r) a\left(r, \varphi ; x_{0}, y_{0}\right) \mathrm{d} r \mathrm{~d} \varphi \tag{4.4.2}
\end{equation*}
$$

This is a shift-varying kernel for which we would like to find a locally shift-invariant approximation. One could imagine a variety of criteria for developing such an approximation. In the spirit of our previous work [20, 21], we adopt the criterion that we would like the approximation to the inner integral to be exact when $(x, y)=\left(x_{0}, y_{0}\right)$, because this is where the kernels usually have the largest values.

With some hindsight, we define the following "certainty" function:

$$
\begin{equation*}
\kappa_{\varphi}(x, y)=\sqrt{\frac{\int_{-\infty}^{\infty}|a(r, \varphi ; x, y)|^{2} w_{\varphi}(r) \mathrm{d} r}{\int_{-\infty}^{\infty}|a(r, \varphi ; x, y)|^{2} \mathrm{~d} r}} \tag{4.4.3}
\end{equation*}
$$

which usually satisfies $\kappa_{\varphi}(x, y) \approx w_{\varphi}(x \cos \varphi+y \sin \varphi)$ for 2D parallel-beam tomography. We also approximate the kernel as follows

$$
h\left(x, y ; x_{0}, y_{0}\right) \approx \int_{0}^{\pi} \kappa_{\varphi}(x, y) \kappa_{\varphi}\left(x_{0}, y_{0}\right)\left[\int_{-\infty}^{\infty} a^{*}(r, \varphi ; x, y) a\left(r, \varphi ; x_{0}, y_{0}\right) \mathrm{d} r\right] \mathrm{d} \varphi .
$$

To further simplify, define the following "local detector response" function for the object point $\left(x_{0}, y_{0}\right)$ :

$$
b_{0}(\tau, \varphi) \triangleq a\left(\tau+\left[x_{0} \cos \varphi+y_{0} \sin \varphi\right], \varphi ; x_{0}, y_{0}\right)
$$

so that

$$
a\left(r, \varphi ; x_{0}, y_{0}\right)=b_{0}\left(r-\left[x_{0} \cos \varphi+y_{0} \sin \varphi\right], \varphi\right)
$$

Now assume that the detector response functions $a(\cdot)$ are locally shift invariant in the following sense ${ }^{5}$ :

$$
\begin{equation*}
a(r, \varphi ; x, y) \approx b_{0}(r-[x \cos \varphi+y \sin \varphi], \varphi), \tag{4.4.4}
\end{equation*}
$$

[^2]for $(x, y) \approx\left(x_{0}, y_{0}\right)$. The assumption (4.4.4) leads to the following approximation for the inner integral:
\[

$$
\begin{aligned}
\int_{-\infty}^{\infty} & a^{*}(r, \varphi ; x, y) a\left(r, \varphi ; x_{0}, y_{0}\right) \mathrm{d} r \\
& \approx \int_{-\infty}^{\infty} b_{0}^{*}(r-[x \cos \varphi+y \sin \varphi], \varphi) b_{0}\left(r-\left[x_{0} \cos \varphi+y_{0} \sin \varphi\right], \varphi\right) \mathrm{d} r \\
& =\int_{-\infty}^{\infty} b_{0}^{*}(\tau, \varphi) b_{0}\left(\tau+\left(x-x_{0}\right) \cos \varphi+\left(y-y_{0}\right) \sin \varphi, \varphi\right) \mathrm{d} \tau \\
& =\left(b_{0} \star b_{0}\right)\left[\left(x-x_{0}\right) \cos \varphi+\left(y-y_{0}\right) \sin \varphi\right]
\end{aligned}
$$
\]

where $\star$ denotes 1D autocorrelation (with respect to $r$ ). This relationship leads to our final approximations for the kernel of the Gram operator:

$$
\begin{align*}
h\left(x, y ; x_{0}, y_{0}\right) & \approx \int_{0}^{\pi} \kappa_{\varphi}(x, y) \kappa_{\varphi}\left(x_{0}, y_{0}\right)\left(b_{0} \star b_{0}\right)\left[\left(x-x_{0}\right) \cos \varphi+\left(y-y_{0}\right) \sin \varphi\right] \mathrm{d} \varphi  \tag{4.4.5}\\
& \approx \int_{0}^{\pi} \kappa_{\varphi}^{2}\left(x_{0}, y_{0}\right)\left(b_{0} \star b_{0}\right)\left[\left(x-x_{0}\right) \cos \varphi+\left(y-y_{0}\right) \sin \varphi\right] \mathrm{d} \varphi \tag{4.4.6}
\end{align*}
$$

This final shift-invariant approximation agrees with the exact expression (4.4.2) "along the diagonal" of $\mathcal{A}^{*} \mathcal{W} \mathcal{A}$, i.e., when $(x, y)=\left(x_{0}, y_{0}\right)$. And it agrees everywhere if $w_{\varphi}(r)$ is independent of $r$ and if the detector response functions $a(\cdot)$ are shift invariant in the sense that (4.4.4) holds exactly.

Because $\kappa_{\varphi}^{2}$ depends only on the angle $\varphi$ and not on the radial position $r$, it has been described as a "radiallyconstant" approximation [21].

The following "local" approximation to the Gram operator then follows from (4.3.7) and (4.4.6):

$$
\begin{equation*}
\mathcal{A}^{*} \mathcal{W} \mathcal{A} \approx \mathcal{F}_{2}^{-1} \mathcal{D}\left(\frac{w_{0}(\Phi)\left|B_{0}(\rho, \Phi)\right|^{2}}{|\rho|}\right) \mathcal{F}_{2} \tag{4.4.7}
\end{equation*}
$$

where $B_{0}(\cdot, \varphi)$ denotes the 1 D FT of $b_{0}(\cdot, \varphi)$, and

$$
w_{0}(\varphi) \triangleq \kappa_{\varphi}^{2}\left(x_{0}, y_{0}\right)=\frac{\int_{-\infty}^{\infty}\left|a\left(r, \varphi ; x_{0}, y_{0}\right)\right|^{2} w_{\varphi}(r) \mathrm{d} r}{\int_{-\infty}^{\infty}\left|a\left(r, \varphi ; x_{0}, y_{0}\right)\right|^{2} \mathrm{~d} r}
$$

Therefore, generalizing (4.3.17), the local frequency response (the 2D FT of the local impulse response) near $\left(x_{0}, y_{0}\right)$ of the QPWLS estimator is

$$
\begin{equation*}
L_{0}(\rho, \Phi) \approx \frac{w_{0}(\Phi)\left|B_{0}(\rho, \Phi)\right|^{2}}{w_{0}(\Phi)\left|B_{0}(\rho, \Phi)\right|^{2}+\beta|\rho| R(\rho, \Phi)} \tag{4.4.8}
\end{equation*}
$$

We apply this result to regularizer design in $\S 5.2$.

### 4.4.3 Radially-dependent noise

The covariance (4.3.21) for QPWLS was derived under the assumption that the measurement variance depended only on the projection angle. A more realistic model than (4.3.19) would also include a radial dependence:

$$
\operatorname{Cov}\left\{y_{\varphi_{1}}\left[n_{1}\right], y_{\varphi_{2}}\left[n_{2}\right]\right\}= \begin{cases}c\left(r_{n_{1}}, \varphi_{1}\right) \delta\left(\varphi_{1}-\varphi_{2}\right), & n_{1}=n_{2} \\ 0, & \text { otherwise }\end{cases}
$$

In this case, the expression (4.3.20) has the following (local) approximation

$$
\mathcal{A}^{*} \mathcal{W} \operatorname{Cov}\{\boldsymbol{y}\} \mathcal{W} \mathcal{A} \approx \mathcal{F}_{2}^{-1} \mathcal{D}\left(\frac{\tilde{w}_{0}(\Phi)\left|B_{0}(\rho, \Phi)\right|^{2}}{|\rho|}\right) \mathcal{F}_{2}
$$

where

$$
\tilde{w}_{0}(\varphi)=\frac{\int\left|a\left(r, \varphi ; x_{0}, y_{0}\right)\right|^{2} w_{\varphi}^{2}(r) c(r, \varphi) \mathrm{d} r}{\int\left|a\left(r, \varphi ; x_{0}, y_{0}\right)\right|^{2} \mathrm{~d} r}
$$

In the usual case where $w_{\varphi}(r)=1 / c(r, \varphi)$, we have $\tilde{w}_{0}(\varphi)=w_{0}(\varphi)$. The expression (4.3.21) for the covariance the QPWLS estimator has the following local approximation:

$$
\mathcal{K}_{\hat{f}} \approx \mathcal{F}_{2}^{-1} \mathcal{D}\left(\frac{\tilde{w}_{0}(\Phi)|\rho|\left|B_{0}(\rho, \Phi)\right|^{2}}{\left(w_{0}(\Phi)\left|B_{0}(\rho, \Phi)\right|^{2}+\beta|\rho| R(\rho, \Phi)\right)^{2}}\right) \mathcal{F}_{2}
$$

The inner term is the local noise power spectrum (NPS) of the QPWLS estimator. When $\beta=0$, the inner expression is proportional to

$$
\frac{\int\left|a\left(r, \varphi ; x_{0}, y_{0}\right)\right|^{2} w_{\varphi}^{2}(r) c(r, \varphi) \mathrm{d} r}{\left(\int\left|a\left(r, \varphi ; x_{0}, y_{0}\right)\right|^{2} w_{\varphi}(r) \mathrm{d} r\right)^{2}}
$$

In contrast to the discussion in $\S 4.3 .6$, here the weighting $w_{\varphi}(r)$ does not cancel out. Using the Cauchy-Schwarz inequality (26.4.2), one can show that this expression is minimized when we choose $w_{\varphi}(r)=1 / c(r, \varphi)$, which is the result expected from the Gauss-Markov theorem [14, p. 141].

Our experiences with similar approximations derived from matrix-vector formulations is that they can be quite accurate and are useful for tasks such as the design of regularizers (see $\S 5.2$ ) and the prediction of noise properties [16, 20-24].

### 4.4.4 Isotropic resolution revisited

Suppose we can choose the regularizer $R(\rho, \Phi)$ so that the local impulse response is approximately isotropic, i.e., $L_{0}(\rho, \Phi) \approx L_{0}(\rho)$, which is independent of angle $\Phi$. Then from (4.4.8), apparently

$$
\frac{B_{0}^{*}(\rho, \Phi)}{w_{0}(\Phi)\left|B_{0}(\rho, \Phi)\right|^{2}+\beta|\rho| R(\rho, \Phi)} \approx \frac{L_{0}(\rho)}{w_{0}(\Phi) B_{0}(\rho, \Phi)}
$$

and

$$
\mathcal{K}_{\hat{f}} \approx \mathcal{F}_{2}^{-1} \mathcal{D}\left(|\rho| \frac{1}{w_{0}(\Phi)}\left|\frac{L_{0}(\rho)}{B_{0}(\rho, \Phi)}\right|^{2}\right) \mathcal{F}_{2}
$$

so again we have anisotropic noise in the reconstructed image.
Can we again achieve comparable results with a BPF method like (4.3.24)? Choosing

$$
\mathcal{B}_{\text {smooth }}(\rho, \Phi)=\frac{L_{0}(\rho)}{B_{0}(\rho, \Phi)}
$$

and making the (somewhat questionable) approximation

$$
\mathcal{P}^{*} \operatorname{Cov}\{\boldsymbol{y}\} \mathcal{P} \approx \mathcal{F}_{2}^{-1} \mathcal{D}\left(|\rho| c_{0}(\Phi)\right) \mathcal{F}_{2}
$$

where $c_{0}(\Phi)=c\left(x_{0} \cos \Phi+y_{0} \sin \Phi, \Phi\right)$, leads to the following BPF covariance approximation

$$
\mathcal{K}_{\hat{f}_{\mathrm{BPF}}} \approx \mathcal{F}_{2}^{-1} \mathcal{D}\left(|\rho| c_{0}(\Phi)\left|\frac{L_{0}(\rho)}{B_{0}(\rho, \Phi)}\right|^{2}\right) \mathcal{F}_{2}
$$

Although $w_{\varphi}(r)=1 / c(r, \varphi)$, in general $w_{0}(\Phi) \neq 1 / c_{0}(\Phi)$. However, if the variance $c(r, \varphi)$ varies slowly with $r$, then $w_{0}(\Phi) \approx 1 / c_{0}(\Phi)$. On the other hand, if there are substantial ray-to-ray fluctuations in variance, due to nonuniform detector efficiencies for example, then it seems that here there is a hint of a possibility that the QPWLS method could have lower noise than BPF at matched isotropic spatial resolution. Furthermore, the so-called BPF method above would require a different inverse filter for every pixel, spoiling some of the computational advantage of analytical methods over iterative methods. (However, see [3] for clever approximations.)

Interestingly, the noise in this BPF method seems to depend directly on the measurement noise covariance $c(r, \varphi)$, where as the (approximate) noise of the QPWLS method depends on the reciprocal of this average:

$$
\int\left|a\left(r, \varphi ; x_{0}, y_{0}\right)\right|^{2} \frac{1}{c(r, \varphi)} \mathrm{d} r
$$

assuming $w_{\varphi}(r)=1 / c(r, \varphi)$. Perhaps this averaging helps reduce the influence of particularly noisy rays that cause streaks in FBP images.

### 4.5 2D fan beam geometry (s.topo,fan)

This section analyzes the Gram operator of the WLS cost function for the case where the projection operator $\mathcal{P}$ corresponds to the fan-beam geometry considered in (3.9.11). The primary motivation for this section is its relevance to regularization design [25].

The usual inner product for fan-beam projection space is

$$
\left\langle p_{1}, p_{2}\right\rangle=\int_{0}^{\beta_{\max }} \int_{-s_{\max }}^{s_{\max }} p_{1}(s, \beta) p_{2}(s, \beta) \mathrm{d} s \mathrm{~d} \beta
$$

This is the natural inner product when considering the usual case of samples that are equally-spaced in arc length $s$ and in source angle $\beta$. For this inner product, the adjoint of $\mathcal{P}$ is given by

$$
\left(\mathcal{P}^{*} p\right)(x, y)=\int_{0}^{\beta_{\max }} \int_{-s_{\max }}^{s_{\max }} \delta(x \cos \varphi(s, \beta)+y \sin \varphi(s, \beta)-r(s)) p(s, \beta) \mathrm{d} s \mathrm{~d} \beta
$$

where $r(s)$ and $\varphi(s, \beta)$ were defined in (3.9.7).
Define a "diagonal" weighting operator $\mathcal{W}$ in fan-beam sinogram space by

$$
(\mathcal{W} p)(s, \beta)=w_{2 \pi}(s, \beta) p(s, \beta)
$$

where $w_{2 \pi}(s, \beta)$ is a user-selected nonnegative weighting function. Following (4.3.13), the natural QPWLS estimator for this problem has the form

$$
\begin{equation*}
\hat{f}=\underset{f}{\arg \min }\|p-\mathcal{P} f\|_{\mathcal{W}^{1 / 2}}^{2}+\mathrm{R}(f) \tag{4.5.1}
\end{equation*}
$$

where R was defined in (4.3.12).
We assume hereafter that $w_{2 \pi}(s, \beta)$ is chosen such that $w_{2 \pi}(s, \beta)=0$ when $\beta>\beta_{\max }$. Thus we can assume $\beta_{\max }=2 \pi$ for the analysis, yet the results are still applicable to "short" scans provided $w_{2 \pi}(s, \beta)$ is chosen appropriately. To analyze the impulse response of the Gram operator $\mathcal{P}^{*} \mathcal{W} \mathcal{P}$, consider an impulse object $\delta_{0}(x, y)=\delta\left(x-x_{0}, y-y_{0}\right)$ as follows:

$$
\begin{aligned}
h\left(x, y ; x_{0}, y_{0}\right)= & \left(\mathcal{P}^{*} \mathcal{W} \mathcal{P} \delta_{0}\right)(x, y) \\
= & \int_{0}^{2 \pi} \int_{-s_{\max }}^{s_{\max }} \delta(x \cos \varphi(s, \beta)+y \sin \varphi(s, \beta)-r(s)) \\
& \delta\left(x_{0} \cos \varphi(s, \beta)+y_{0} \sin \varphi(s, \beta)-r(s)\right) w_{2 \pi}(s, \beta) \mathrm{d} s \mathrm{~d} \beta
\end{aligned}
$$

For convenience, we may also express the point $\left(x_{0}, y_{0}\right)$ in polar coordinates $\left(r_{0}, \varphi_{0}\right)$.
In the spirit of local shift invariance described in $\S 4.4$, consider the following local impulse response:

$$
\begin{aligned}
h_{0}(r, \varphi) \triangleq & h\left(x_{0}+r \cos \varphi, y_{0}+r \sin \varphi ; x_{0}, y_{0}\right) \\
= & \int_{0}^{2 \pi} \int_{-s_{\max }}^{s_{\max }} \delta\left(\left(x_{0}+r \cos \varphi\right) \cos \varphi(s, \beta)+\left(y_{0}+r \sin \varphi\right) \sin \varphi(s, \beta)-r(s)\right) \\
& \delta\left(x_{0} \cos \varphi(s, \beta)+y_{0} \sin \varphi(s, \beta)-r(s)\right) w_{2 \pi}(s, \beta) \mathrm{d} s \mathrm{~d} \beta
\end{aligned}
$$

Applying the sampling property of the (second) Dirac impulse and simplifying with trigonometry yields:

$$
h_{0}(r, \varphi)=\int_{0}^{2 \pi} \int_{-s_{\max }}^{s_{\max }} \delta(r \cos (\varphi(s, \beta)-\varphi)) \delta\left(r_{0} \cos \left(\varphi(s, \beta)-\varphi_{0}\right)-r(s)\right) w_{2 \pi}(s, \beta) \mathrm{d} s \mathrm{~d} \beta
$$

Convert from fan to parallel coordinates by making the change of variables $r^{\prime}=D_{\mathbf{s} 0} \sin \gamma(s), \varphi^{\prime}=\beta+\gamma(s)$ as defined in (3.9.7), assume hereafter that $r_{\text {off }}=0$ for simplicity. The local impulse response simplifies to

$$
h_{0}(r, \varphi)=\int_{0}^{2 \pi} \int_{-r_{\max }}^{r_{\max }} \delta\left(r \cos \left(\varphi^{\prime}-\varphi\right)\right) \delta\left(r_{0} \cos \left(\varphi^{\prime}-\varphi_{0}\right)-r^{\prime}\right) w_{1}\left(r^{\prime}, \varphi^{\prime}\right) \mathrm{d} r^{\prime} \mathrm{d} \varphi^{\prime}
$$

incorporating the Jacobian determinant in (3.9.19) into the following modified weighting function (cf. (3.9.15)):

$$
\left.w_{1}(r, \varphi) \triangleq \frac{w_{2 \pi}(s, \beta)}{J(s)}\right|_{s=s(r), \beta=\beta(r, \varphi)}=\frac{w_{2 \pi}\left(\gamma^{-1}\left(\arcsin \left(r / D_{\mathrm{s} 0}\right)\right), \varphi-\arcsin \left(r / D_{\mathrm{s} 0}\right)\right)}{J\left(\gamma^{-1}\left(\arcsin \left(r / D_{\mathrm{s} 0}\right)\right)\right)}
$$

Applying the sifting property to the second Dirac impulse and the scaling property to the first Dirac impulse yields

$$
h_{0}(r, \varphi)=\frac{1}{|r|} \int_{0}^{2 \pi} \delta\left(\cos \left(\varphi^{\prime}-\varphi\right)\right) w_{2}\left(r_{0} \cos \left(\varphi^{\prime}-\varphi_{0}\right), \varphi^{\prime}\right) \mathrm{d} \varphi^{\prime}
$$

where we define

$$
w_{2}(r, \varphi) \triangleq w_{1}(r, \varphi) \mathbb{I}_{\left\{|r| \leq r_{\max }\right\}}
$$

The preceding integral depends only on $\varphi$ and point location $\left(x_{0}, y_{0}\right)$. Thus, similar to the parallel-beam case in Theorem 3.3.1, for the fan-beam case the local impulse response of the Gram operator has the following form:

$$
\begin{equation*}
h_{0}(r, \varphi)=\frac{1}{|r|} w_{0}(\varphi+\pi / 2) \tag{4.5.2}
\end{equation*}
$$

where here the weighting function is

$$
\begin{aligned}
w_{0}(\varphi) & \triangleq \int_{0}^{2 \pi} \delta\left(\cos \left(\varphi^{\prime}-\varphi+\pi / 2\right)\right) w_{2}\left(r_{0} \cos \left(\varphi^{\prime}-\varphi_{0}\right), \varphi^{\prime}\right) \mathrm{d} \varphi^{\prime} \\
& =\int_{0}^{2 \pi} \delta\left(\sin \left(\varphi-\varphi^{\prime}\right)\right) w_{2}\left(r_{0} \cos \left(\varphi^{\prime}-\varphi_{0}\right), \varphi^{\prime}\right) \mathrm{d} \varphi^{\prime}
\end{aligned}
$$

Simplifying by applying the sifting property (3.3.8) yields

$$
\begin{aligned}
w_{0}(\varphi) & =w_{2}\left(r_{0} \cos \left(\varphi-\varphi_{0}\right), \varphi\right)+w_{2}\left(r_{0} \cos \left(\varphi-\varphi_{0}+\pi\right), \varphi+\pi\right) \\
& =w_{2}\left(r_{0}(\varphi), \varphi\right)+w_{2}\left(-r_{0}(\varphi), \varphi+\pi\right)
\end{aligned}
$$

where we define

$$
r_{0}(\varphi) \triangleq r_{0} \cos \left(\varphi-\varphi_{0}\right)
$$

Using (3.9.2) define

$$
\begin{aligned}
& \gamma_{0}(\varphi) \triangleq \arcsin \left(r_{0}(\varphi) / D_{\mathrm{s} 0}\right) \\
& s_{0}(\varphi) \triangleq \gamma^{-1}\left(\gamma_{0}(\varphi)\right) .
\end{aligned}
$$

Then because $J(-s)=J(s)$ we have our final expression for the weighting:

$$
\begin{equation*}
w_{0}(\varphi)=\frac{1}{J\left(s_{0}(\varphi)\right)}\left[w_{2 \pi}\left(s_{0}(\varphi), \varphi-\gamma_{0}(\varphi)\right)+w_{2 \pi}\left(-s_{0}(\varphi), \varphi+\pi+\gamma_{0}(\varphi)\right)\right] \mathbb{I}_{\left\{\left|r_{0}(\varphi)\right| \leq r_{\max }\right\}} \tag{4.5.3}
\end{equation*}
$$

In other words, $w_{0}(\varphi)$ is the sum of the weights in the sinogram $w_{2 \pi}(s, \beta)$ corresponding to the two rays that intersect the point $\left(x_{0}, y_{0}\right)$ at angle $\varphi$, adjusted by the Jacobian of the fan-to-parallel coordinate transformation. It follows then from Theorem 3.3.2 that the local frequency response of the Gram operator is

$$
H_{0}(\rho, \Phi)=\frac{1}{|\rho|} w_{0}(\Phi)
$$

It is interesting that the local impulse response and local frequency response have the same form in the fan-beam and parallel-beam cases, disregarding detector blur. Generalizing the above analysis to consider detector blur is an open problem. (Detector response has an anisotropic effect in the fan-beam geometry due to the distant-dependent magnification.)

In the equiangular case, where $D_{\mathrm{fs}}=0$ and $\gamma(s)=s / D_{\mathrm{sd}}$, we have the following simplifications:

$$
\begin{align*}
s_{0}(\varphi) & =D_{\mathrm{sd}} \arcsin \left(r_{0}(\varphi) / D_{\mathrm{s} 0}\right)  \tag{4.5.4}\\
J\left(s_{0}(\varphi)\right) & =D_{\mathrm{s} 0} \cos \left(\arcsin \left(r_{0}(\varphi) / D_{\mathrm{s} 0}\right)\right) \frac{1}{D_{\mathrm{sd}}}=\frac{D_{\mathrm{s} 0}}{D_{\mathrm{sd}}} \sqrt{1-\left(\frac{r_{0}(\varphi)}{D_{\mathrm{s} 0}}\right)^{2}} . \tag{4.5.5}
\end{align*}
$$

For typical fan-beam geometries, the FOV is much smaller than $D_{\mathrm{s} 0}$, so the Jacbian factor is nearly uniform.
The design of regularization methods for the fan-beam case has been explored using the above relationships [25].

### 4.6 3D tomography (s,3d,intro)

The preceding sections focused on the case of 2D objects $f(x, y)$. This section considers the 3D case. There are a variety of methods for parameterizing line integrals of 3D objects. The various parameterizations describe the same information for noiseless continuous measurements, but correspond to different sampling patterns and have differing noise properties for discrete, noisy measurements.

### 4.6.1 Parallel beam geometry ( $\mathrm{s}, 3 \mathrm{~d}, \mathrm{par}$ )

### 4.6.1.1 Definition of 3D parallel-beam X-ray transform

For 3D tomography with parallel rays, there are two projection angles: the azimuthal angle $\varphi$, and a polar angle $\theta$. We use a coordinate system in which $\varphi \in[-\pi, \pi]$, and $\theta \in \mathcal{T} \subset[-\pi / 2, \pi / 2]$. Define $\vec{\gamma}=(\varphi, \theta)$ and define the following orthogonal unit vectors

$$
\vec{e}(\vec{\gamma})=\left[\begin{array}{r}
-\sin \varphi \cos \theta  \tag{4.6.1}\\
\cos \varphi \cos \theta \\
\sin \theta
\end{array}\right], \quad \vec{e}_{1}(\vec{\gamma})=\left[\begin{array}{c}
\cos \varphi \\
\sin \varphi \\
0
\end{array}\right], \quad \vec{e}_{2}(\vec{\gamma})=\left[\begin{array}{r}
\sin \varphi \sin \theta \\
-\cos \varphi \sin \theta \\
\cos \theta
\end{array}\right]
$$

and the following point in $\mathbb{R}^{3}$ :

$$
\begin{equation*}
\vec{p}=\vec{p}(u, v ; \vec{\gamma})=u \vec{e}_{1}+v \vec{e}_{2}=(u \cos \varphi+v \sin \varphi \sin \theta, u \sin \varphi-v \cos \varphi \sin \theta, v \cos \theta) \tag{4.6.2}
\end{equation*}
$$

Letting $(u, v)$ denote the coordinates on any 2D projection plane, define the (X-ray) projection of a 3D object $f(\overrightarrow{\mathrm{x}})$ as

$$
\begin{equation*}
p(u, v ; \vec{\gamma})=\int f(\vec{p}+\ell \vec{e}) \mathrm{d} \ell=\int_{-\infty}^{\infty} f(\vec{p}(u, v ; \vec{\gamma})+\ell \vec{e}(\vec{\gamma})) \mathrm{d} \ell \tag{4.6.3}
\end{equation*}
$$

i.e., the line integrals along the lines $\{\vec{p}+\ell \vec{e}: \ell \in \mathbb{R}\}$, where $u, v \in \mathbb{R}$. Another way of writing this operation is

$$
p(u, v ; \vec{\gamma})=\int f\left(\boldsymbol{T}_{\vec{\gamma}}\left[\begin{array}{l}
u  \tag{4.6.4}\\
\ell \\
v
\end{array}\right]\right) \mathrm{d} \ell
$$

where $\boldsymbol{T}_{\vec{\gamma}}=\left[\vec{e}_{1}(\vec{\gamma}) \vec{e}(\vec{\gamma}) \vec{e}_{2}(\vec{\gamma})\right]$ is a unitary matrix, so $\boldsymbol{T}^{-1}=\boldsymbol{T}^{\prime}=\left[\begin{array}{ccc}\cos \varphi & \sin \varphi & 0 \\ -\sin \varphi \cos \theta & \cos \varphi \cos \theta & \sin \theta \\ \sin \varphi \sin \theta & -\cos \varphi \sin \theta & \cos \theta\end{array}\right]$. The matrix $\boldsymbol{T}$ is the product of two 3D rotation matrices [26, p. 100]:

$$
\boldsymbol{T}=\boldsymbol{R}_{12}(\varphi) \boldsymbol{R}_{23}(\theta), \quad \boldsymbol{R}_{12}(\varphi)=\left[\begin{array}{ccc}
\cos \varphi & -\sin \varphi & 0 \\
\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{array}\right], \quad \boldsymbol{R}_{23}(\theta)=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{array}\right]
$$

Problem 4.14 gives examples of 3D X-ray transform pairs. The goal in 3D tomography is to reconstruct $f$ from a collection of (noisy) samples of $p(u, v ; \vec{\gamma})$. See $\S 3.10$. Here, we focus on analyzing the properties of this 3D transform, rather than describing analytical 3D reconstruction methods.

### 4.6.1.2 Properties

- Central plane property

If $\theta=0$, then

$$
\begin{equation*}
p(u, v ; \varphi, 0)=\int_{-\infty}^{\infty} f(u \cos \varphi-\ell \sin \varphi, u \sin \varphi+\ell \cos \varphi, v) \mathrm{d} \ell \tag{4.6.5}
\end{equation*}
$$

which generalizes (3.2.4) to 3D.

- Symmetry property

The 3D projection (4.6.3) satisfies the following symmetry property:

$$
p(-u, v ; \varphi \pm \pi,-\theta)=p(u, v ; \varphi, \theta)
$$

So in the usual case where $\mathcal{T}=-\mathcal{T}$, it would suffice to restrict $\varphi$ to the range $\varphi \in[0, \pi]$. However, this restriction would complicate some of the analysis below so we focus on the case where $\varphi \in[-\pi, \pi]$.

- Shift property

The 3D projection operation also satisfies the following shift property:

$$
\begin{align*}
& f(\overrightarrow{\mathrm{x}}) \stackrel{\text { 3D Xray }}{\longleftrightarrow} p(u, v ; \varphi, \theta) \Longrightarrow \\
& f(\overrightarrow{\mathrm{x}}-(a, b, c)) \stackrel{\text { 3D Xray }}{\longleftrightarrow} p(u-(a \cos \varphi+b \sin \varphi), v-(a \sin \varphi \sin \theta-b \cos \varphi \sin \theta+c \cos \theta) ; \vec{\gamma}) . \tag{4.6.6}
\end{align*}
$$

- Spherical symmetry property

If $f(\overrightarrow{\mathrm{x}})$ is spherically symmetric, then $p(u, v ; \vec{\gamma})$ is a function only of $u^{2}+v^{2}$, i.e., all the projection views are identical and are circularly symmetric.

- Scaling property

If $f(\overrightarrow{\mathrm{x}}) \stackrel{3 \mathrm{D} \text { Xray }}{\longleftrightarrow} p(u, v ; \varphi, \theta)$, then we have the following (isotropic) scaling property:

$$
\begin{equation*}
f(x / a, y / a, z / a) \stackrel{3 \mathrm{D} \text { Xray }}{\longleftrightarrow}|a| p(u / a, v / a ; \vec{\gamma}) . \tag{4.6.7}
\end{equation*}
$$

### 4.6.1.3 Fourier slice theorem in 3D

For the definition (4.6.3), one can establish the following projection integral theorem:

$$
\begin{aligned}
& \iint p(u, v ; \vec{\gamma}) h(u, v) \mathrm{d} u \mathrm{~d} v \\
& \quad=\iiint f(\vec{p}+\ell \vec{e}) h(u, v) \mathrm{d} u \mathrm{~d} v \mathrm{~d} \ell=\iiint f(\overrightarrow{\mathrm{x}}) h\left(\overrightarrow{\mathrm{x}} \cdot \vec{e}_{1}(\vec{\gamma}), \overrightarrow{\mathrm{x}} \cdot \vec{e}_{2}(\vec{\gamma})\right) \mathrm{d} \overrightarrow{\mathrm{x}} \\
& \quad=\iiint f(\overrightarrow{\mathrm{x}}) h(x \cos \varphi+y \sin \varphi, x \sin \varphi \sin \theta-y \cos \varphi \sin \theta+z \cos \theta) \mathrm{d} \overrightarrow{\mathrm{x}},
\end{aligned}
$$

which holds for any (suitably regular) function $h(u, v)$. To show this equality, simply make the change of variables $\overrightarrow{\mathrm{x}}=\boldsymbol{T}\left[\begin{array}{l}u \\ \ell \\ v\end{array}\right]$ where $\boldsymbol{T}$ was defined above.

Using this 3D projection integral theorem, one can establish the following 3D Fourier slice theorem simply by substituting $h(u, v)=\mathrm{e}^{-u 2 \pi\left(u \nu_{1}+v \nu_{2}\right)}$ into (4.6.8), yielding

$$
\begin{equation*}
P\left(\nu_{1}, \nu_{2} ; \vec{\gamma}\right)=F\left(\nu_{1} \vec{e}_{1}(\vec{\gamma})+\nu_{2} \vec{e}_{2}(\vec{\gamma})\right) \tag{4.6.9}
\end{equation*}
$$

where $F(\vec{\nu})$ denotes the 3D FT of $f(\overrightarrow{\mathrm{x}})$, and the 2D FT of the projection view at angle $\vec{\gamma}$ is

$$
P\left(\nu_{1}, \nu_{2} ; \vec{\gamma}\right)=\iint p(u, v ; \vec{\gamma}) \mathrm{e}^{-\imath 2 \pi\left(u \nu_{1}+v \nu_{2}\right)} \mathrm{d} u \mathrm{~d} v
$$

The vectors $\vec{e}_{1}$ and $\vec{e}_{2}$ define a plane in $\mathbb{R}^{3}$, so (4.6.9) says that the 2D FT of a projection view taken at angle $\vec{\gamma}$ equals a planar slice of the 3D FT of the object.

### 4.6.1.4 System blur

Rather than considering the idealized line integrals described by (4.6.3), a more realistic model includes the effects of detector blur. For simplicity, we assume the blur is shift-invariant within each view, but allow it to vary between views as follows:

$$
\begin{equation*}
q(u, v ; \vec{\gamma})=\iint b_{0}\left(u-u^{\prime}, v-v^{\prime} ; \vec{\gamma}\right) p\left(u^{\prime}, v^{\prime} ; \vec{\gamma}\right) \mathrm{d} u^{\prime} \mathrm{d} v^{\prime} \tag{4.6.10}
\end{equation*}
$$

Taking the 2D FT of both sides yields

$$
Q\left(\nu_{1}, \nu_{2} ; \vec{\gamma}\right)=B_{0}\left(\nu_{1}, \nu_{2} ; \vec{\gamma}\right) P\left(\nu_{1}, \nu_{2} ; \vec{\gamma}\right)=B_{0}\left(\nu_{1}, \nu_{2} ; \vec{\gamma}\right) F\left(\nu_{1} \vec{e}_{1}(\vec{\gamma})+\nu_{2} \vec{e}_{2}(\vec{\gamma})\right)
$$

where $B_{0}\left(\nu_{1}, \nu_{2} ; \vec{\gamma}\right)$ is the 2D frequency response corresponding to blur $b_{0}(u, v ; \vec{\gamma})$. We write this in operator notation as follows:

$$
q=\mathcal{A} f, \quad \mathcal{A}=\boldsymbol{\mathcal { B }} \mathcal{P}_{3}
$$

where $\mathcal{B}$ denotes the blur operator in (4.6.10), and $\mathcal{P}_{3}$ denotes the 3D projection operator in (4.6.3).

### 4.6.1.5 SVD for 3D case ( $\mathrm{s}, 3 \mathrm{~d}, \mathrm{svd}$ )

For analyzing properties like spatial resolution, we want to write $\mathcal{A}$ in the SVD -like representation $\mathcal{A}=\mathcal{U} \mathcal{D} \mathcal{F}_{3}$, where $\mathcal{U}$ and $\mathcal{F}_{3}$ are unitary operators (with respect to suitable Hilbert spaces) and where $\mathcal{D}$ is something like a "diagonal" operator. Toward that end, define the following Hilbert spaces.

- $\mathcal{H}_{\square}=\mathcal{L}_{2}\left(\mathbb{R}^{3}\right)$ with the usual inner product $\left\langle f_{1}, f_{2}\right\rangle=\iiint f_{1}(\overrightarrow{\mathrm{x}}) f_{2}^{*}(\overrightarrow{\mathrm{x}}) \mathrm{d} \overrightarrow{\mathrm{x}}$.
- $\mathcal{H}_{4}=\mathcal{L}_{2}\left(\mathbb{R}^{2} \times[0, \pi] \times \mathcal{T}\right)$ with the following inner product ${ }^{6}$ :

$$
\left\langle P_{1}, P_{2}\right\rangle_{4}=\int_{\mathcal{T}} \int_{-\pi}^{\pi} \iint P_{1}\left(\nu_{1}, \nu_{2} ; \vec{\gamma}\right) P_{2}^{*}\left(\nu_{1}, \nu_{2} ; \vec{\gamma}\right) w_{4}\left(\nu_{1}, \nu_{2} ; \vec{\gamma}\right) \mathrm{d} \nu_{1} \mathrm{~d} \nu_{2} \mathrm{~d} \varphi \mathrm{~d} \theta
$$

[^3]- $\mathcal{H}_{\mathrm{proj}}=\mathcal{L}_{2}\left(\mathbb{R}^{2} \times[0, \pi] \times \mathcal{T}\right)$ with the following inner product

$$
\left\langle p_{1}, p_{2}\right\rangle_{\text {proj }}=\int_{\mathcal{T}} \int_{-\pi}^{\pi} \iint p_{1}(u, v ; \vec{\gamma}) p_{2}^{*}(u, v ; \vec{\gamma}) w_{p}(u, v ; \vec{\gamma}) \mathrm{d} u \mathrm{~d} v \mathrm{~d} \varphi \mathrm{~d} \theta
$$

where $w_{4}$ and $w_{p}$ are real, positive weighting functions that depend on the type of sampling used. Define the following operators

$$
\begin{gathered}
\mathcal{F}_{3}: \mathcal{H}_{\square} \rightarrow \mathcal{H}_{\square}, \quad F=\mathcal{F}_{3} f \Longleftrightarrow F(\vec{\nu})=\iiint f(\overrightarrow{\mathrm{x}}) \mathrm{e}^{-\imath 2 \pi \overrightarrow{\mathrm{x}} \cdot \vec{\nu}} \mathrm{~d} \vec{\nu} \\
\mathcal{D}: \mathcal{H}_{\square} \rightarrow \mathcal{H}_{4}, \quad G=\mathcal{D} F \Longleftrightarrow G\left(\nu_{1}, \nu_{2} ; \vec{\gamma}\right)=w_{2}\left(\nu_{1}, \nu_{2} ; \vec{\gamma}\right) B_{0}\left(\nu_{1}, \nu_{2} ; \vec{\gamma}\right) F\left(\nu_{1} \vec{e}_{1}(\vec{\gamma})+\nu_{2} \vec{e}_{2}(\vec{\gamma})\right) \\
\mathcal{U}: \mathcal{H}_{4} \rightarrow \mathcal{H}_{\mathrm{proj}}, \quad q=\mathcal{U} G \Longleftrightarrow q(u, v ; \vec{\gamma})=\iint \frac{1}{w_{2}\left(\nu_{1}, \nu_{2} ; \vec{\gamma}\right)} G\left(\nu_{1}, \nu_{2} ; \vec{\gamma}\right) \mathrm{e}^{\imath 2 \pi\left(u \nu_{1}+v \nu_{2}\right)} \mathrm{d} \nu_{1} \mathrm{~d} \nu_{2},
\end{gathered}
$$

where $w_{2}$ is some weighting function to be determined below. One can verify easily that $\mathcal{A}=\mathcal{U} \mathcal{D} \mathcal{F}_{3}$.
The inverse of $\mathcal{U}$ is clearly

$$
G=\mathcal{U}^{-1} q \Longleftrightarrow G\left(\nu_{1}, \nu_{2} ; \vec{\gamma}\right)=w_{2}\left(\nu_{1}, \nu_{2} ; \vec{\gamma}\right) \iint q(u, v ; \vec{\gamma}) \mathrm{e}^{-\imath 2 \pi\left(u \nu_{1}+v \nu_{2}\right)} \mathrm{d} u \mathrm{~d} v
$$

One can show that the adjoint of $\mathcal{U}$ with respect to the above Hilbert spaces is given by

$$
\begin{equation*}
G=\mathcal{U}^{*} p \Longleftrightarrow G\left(\nu_{1}, \nu_{2} ; \vec{\gamma}\right)=\frac{1}{w_{2}\left(\nu_{1}, \nu_{2} ; \vec{\gamma}\right) w_{4}\left(\nu_{1}, \nu_{2} ; \vec{\gamma}\right)} \iint w_{p}(u, v ; \vec{\gamma}) p(u, v ; \vec{\gamma}) \mathrm{e}^{-\imath 2 \pi\left(u \nu_{1}+v \nu_{2}\right)} \mathrm{d} u \mathrm{~d} v \tag{4.6.11}
\end{equation*}
$$

It follows that $\mathcal{U}$ will be unitary if $w_{p}$ is independent of $(u, v)$ and if we choose $w_{2}=\frac{w_{p}}{w_{2} w_{4}}$ or equivalently that

$$
\begin{equation*}
w_{2}\left(\nu_{1}, \nu_{2} ; \vec{\gamma}\right)=\sqrt{\frac{w_{p}(\vec{\gamma})}{w_{4}\left(\nu_{1}, \nu_{2} ; \vec{\gamma}\right)}} \tag{4.6.12}
\end{equation*}
$$

To determine the adjoint of $\mathcal{D}$, note that

$$
\begin{aligned}
\langle G, \mathcal{D} F\rangle_{4} & =\int_{\mathcal{T}} \int_{-\pi}^{\pi} \iint w_{4}\left(u_{1}, u_{2} ; \vec{\gamma}\right) G\left(u_{1}, u_{2} ; \vec{\gamma}\right) \\
& {\left[w_{2}\left(u_{1}, u_{2} ; \vec{\gamma}\right) B_{0}\left(u_{1}, u_{2} ; \vec{\gamma}\right) F\left(u_{1} \vec{e}_{1}(\vec{\gamma})+u_{2} \vec{e}_{2}(\vec{\gamma})\right)\right]^{*} \mathrm{~d} u_{1} \mathrm{~d} u_{2} \mathrm{~d} \varphi \mathrm{~d} \theta }
\end{aligned}
$$

For any given $\theta$, make the change of variables ${ }^{7}$

$$
\begin{align*}
u_{1} & =u_{1}(\vec{\nu}, \theta)=\operatorname{sgn}\left(\nu_{2}\right) \sqrt{\nu_{1}^{2}+\nu_{2}^{2}-\nu_{3}^{2} \tan ^{2} \theta}  \tag{4.6.13}\\
u_{2} & =u_{2}(\vec{\nu}, \theta)=\nu_{3} / \cos \theta  \tag{4.6.14}\\
\varphi & =\varphi(\vec{\nu}, \theta)=\operatorname{sgn}\left(\nu_{2}\right) \arccos \left(\frac{-\nu_{3} \tan \theta}{\sqrt{\nu_{1}^{2}+\nu_{2}^{2}}}\right)-\pi / 2+\angle\left(\nu_{1}, \nu_{2}\right) \tag{4.6.15}
\end{align*}
$$

for which one can verify that

$$
\begin{aligned}
\vec{\nu} & =u_{1} \vec{e}_{1}(\vec{\gamma})+u_{2} \vec{e}_{2}(\vec{\gamma}) \\
& =\left(u_{1} \cos \varphi+u_{2} \sin \varphi \sin \theta, u_{1} \sin \varphi-u_{2} \cos \varphi \sin \theta, u_{2} \cos \theta\right)
\end{aligned}
$$

This transformation is well defined when

$$
\begin{equation*}
\left|\nu_{3} \tan \theta\right| \leq \sqrt{\nu_{1}^{2}+\nu_{2}^{2}} \tag{4.6.16}
\end{equation*}
$$

Its Jacobian is

$$
\boldsymbol{J}=\left[\begin{array}{ccc}
\cos \varphi & \sin \varphi \sin \theta & -u_{1} \sin \varphi+u_{2} \cos \varphi \sin \theta \\
\sin \varphi & -\cos \varphi \sin \theta & u_{1} \cos \varphi+u_{2} \sin \varphi \sin \theta \\
0 & \cos \theta & 0
\end{array}\right]
$$

and its Jacobian determinant is

$$
|\operatorname{det} \boldsymbol{J}|=\left|u_{1} \cos \theta\right|=\cos \theta \sqrt{\nu_{1}^{2}+\nu_{2}^{2}-\nu_{3}^{2} \tan ^{2} \theta}
$$

So by defining

$$
\vec{u}(\vec{\nu}, \theta) \triangleq\left(u_{1}(\vec{\nu}, \theta), u_{2}(\vec{\nu}, \theta)\right)
$$

[^4]\[

$$
\begin{align*}
\vec{\gamma}(\vec{\nu}, \theta) & \triangleq(\varphi(\vec{\nu}, \theta), \theta) \\
w_{2}(\vec{\nu}, \theta) & \triangleq w_{2}(\vec{u}(\vec{\nu}, \theta) ; \vec{\gamma}(\vec{\nu}, \theta)) \\
w_{\theta}(\vec{\nu}) & \triangleq \frac{1}{\cos \theta \sqrt{\nu_{1}^{2}+\nu_{2}^{2}-\nu_{3}^{2} \tan ^{2} \theta}} \mathbb{I}_{\left\{\left|\nu_{3} \tan \theta\right| \leq \sqrt{\nu_{1}^{2}+\nu_{2}^{2}}\right\}}, \tag{4.6.17}
\end{align*}
$$
\]

we have shown

$$
\langle G, \mathcal{D} F\rangle_{4}=\iiint\left[\int_{\mathcal{T}} w_{\theta}(\vec{\nu}) w_{2}(\vec{\nu}, \theta) B_{0}^{*}(\vec{u}(\vec{\nu}, \theta) ; \vec{\gamma}(\vec{\nu}, \theta)) G(\vec{u}(\vec{\nu}, \theta) ; \vec{\gamma}(\vec{\nu}, \theta)) \mathrm{d} \theta\right] F^{*}(\vec{\nu}) \mathrm{d} \vec{\nu}
$$

Thus the adjoint of $\mathcal{D}$ with respect to the above Hilbert spaces is given by

$$
F_{2}=\mathcal{D}^{*} G \Longleftrightarrow F_{2}(\vec{\nu})=\int_{\mathcal{T}} w_{\theta}(\vec{\nu}) w_{2}(\vec{\nu}, \theta) B_{0}^{*}(\vec{u}(\vec{\nu}, \theta) ; \vec{\gamma}(\vec{\nu}, \theta)) G(\vec{u}(\vec{\nu}, \theta) ; \vec{\gamma}(\vec{\nu}, \theta)) \mathrm{d} \theta
$$

### 4.6.1.6 The Gram operator

Let $\mathcal{W}: \mathcal{H}_{\text {proj }} \rightarrow \mathcal{H}_{\text {proj }}$ denote an angle-dependent weighting operator defined by

$$
p_{2}=\mathcal{W} p_{1} \Longleftrightarrow p_{2}(u, v ; \vec{\gamma})=w_{\varepsilon}(\vec{\gamma}) p_{1}(u, v ; \vec{\gamma})
$$

Then when (4.6.12) is satisfied, one can show that $\mathcal{U}^{*} \mathcal{W} \mathcal{U}=\mathcal{W}$, i.e.,

$$
G_{2}=\mathcal{U}^{*} \mathcal{W} \mathcal{U} G \Longleftrightarrow G_{2}\left(u_{1}, u_{2} ; \vec{\gamma}\right)=w_{\varepsilon}(\vec{\gamma}) G\left(u_{1}, u_{2} ; \vec{\gamma}\right) .
$$

Having analyzed the above adjoints, we can analyze the properties of 3 D reconstruction by examining the Gram operator of the WLS cost function as follows:

$$
\mathcal{A}^{*} \mathcal{W} \mathcal{A}=\mathcal{F}_{3}^{-1} \mathcal{D}^{*} \mathcal{U}^{*} \mathcal{W} \mathcal{U} \mathcal{D} \mathcal{F}_{3}=\mathcal{F}_{3}^{-1} \mathcal{D}^{*} \mathcal{W} \mathcal{D} \mathcal{F}_{3}
$$

where $F_{2}=\mathcal{D}^{*} \mathcal{W} \mathcal{D} F$ if

$$
\begin{aligned}
F_{2}(\vec{\nu})= & \int_{\mathcal{T}} w_{\theta}(\vec{\nu}) w_{2}(\vec{\nu}, \theta) B_{0}^{*}(\vec{u}(\vec{\nu}, \theta) ; \vec{\gamma}(\vec{\nu}, \theta)) w_{\varepsilon}(\vec{\gamma}(\vec{\nu}, \theta)) \\
& \cdot\left[w_{2}(\vec{u}(\vec{\nu}, \theta) ; \vec{\gamma}(\vec{\nu}, \theta)) B_{0}(\vec{u}(\vec{\nu}, \theta) ; \vec{\gamma}(\vec{\nu}, \theta)) F(\vec{\nu})\right] \mathrm{d} \theta \\
= & H(\vec{\nu}) F(\vec{\nu})
\end{aligned}
$$

and where the general form of the frequency response of the Gram operator of the WLS cost function is

$$
\begin{align*}
H(\vec{\nu}) & \triangleq \int_{\mathcal{T}} w_{\theta}(\vec{\nu}) w_{2}^{2}(\vec{\nu}, \theta) w_{\varepsilon}(\varphi(\vec{\nu}, \theta), \theta)\left|B_{0}(\vec{u}(\vec{\nu}, \theta) ; \vec{\gamma}(\vec{\nu}, \theta))\right|^{2} \mathrm{~d} \theta \\
& =\int_{\mathcal{T}} \frac{w_{\theta}(\vec{\nu}) w_{p}(\varphi(\vec{\nu}, \theta), \theta)}{w_{4}\left(u_{1}(\vec{\nu}, \theta), u_{2}(\vec{\nu}, \theta) ; \varphi(\vec{\nu}, \theta), \theta\right) w_{\varepsilon}(\varphi(\vec{\nu}, \theta), \theta)}\left|B_{0}(\vec{u}(\vec{\nu}, \theta) ; \vec{\gamma}(\vec{\nu}, \theta))\right|^{2} \mathrm{~d} \theta \tag{4.6.18}
\end{align*}
$$

(Compare to (4.3.7) in the 2D case.)
Now express this frequency response in the following spherical coordinates

$$
\vec{\nu}=(\varrho \cos \Phi \cos \Theta, \varrho \sin \Phi \cos \Theta, \varrho \sin \Theta), \quad \begin{array}{ll}
\Theta & \Phi \in[-\pi / 2, \pi / 2] \\
& \varrho \in[0,2 \pi) \\
& \varrho \in[0, \infty)
\end{array}
$$

for which $\sqrt{\nu_{1}^{2}+\nu_{2}^{2}}=\varrho \cos \Theta$. Hence the weighting function (4.6.17) becomes

$$
\begin{aligned}
w_{\theta} & =\frac{1}{\varrho \cos \Theta \cos \theta \sqrt{1-\tan ^{2} \theta \tan ^{2} \Theta}} \mathbb{I}_{\{|\tan \theta| \leq 1 /|\tan \Theta|\}} \\
& =\frac{1}{\varrho \cos \theta} \frac{1}{\sqrt{1-\left(\frac{\sin \theta}{\cos \Theta}\right)^{2}}}\left(\frac{\cos \theta}{\cos \Theta}\right) \mathbb{I}_{\left\{|\theta| \leq \frac{\pi}{2}-|\Theta|\right\}} .
\end{aligned}
$$

Hereafter we assume that $\mathcal{T}=\left[-\theta_{\max }, \theta_{\max }\right]$, so the frequency response in spherical coordinates is

$$
\begin{aligned}
H(\varrho, \Phi, \Theta)= & \frac{1}{\varrho} \int_{-\min \left(\theta_{\max }, \frac{\pi}{2}-|\Theta|\right)}^{\min \left(\theta_{\max }, \frac{\pi}{2}-|\Theta|\right)} \frac{\left(\frac{\cos \theta}{\cos \Theta}\right)}{\sqrt{1-\left(\frac{\sin \theta}{\cos \Theta}\right)^{2}}} \\
& \quad \cdot \frac{w_{p}(\varphi(\vec{\nu}, \theta), \theta) w_{\varepsilon}(\varphi(\vec{\nu}, \theta), \theta)}{\cos \theta w_{4}\left(u_{1}(\vec{\nu}, \theta), u_{2}(\vec{\nu}, \theta) ; \varphi(\vec{\nu}, \theta), \theta\right)}\left|B_{0}(\vec{u}(\vec{\nu}, \theta) ; \vec{\gamma}(\vec{\nu}, \theta))\right|^{2} \mathrm{~d} \theta
\end{aligned}
$$

4.6.1.6.1 Unweighted case Suppose for simplicity that the weighting functions have the form

$$
\frac{w_{p} w_{\varepsilon}}{w_{4}}=\frac{1}{2} \cos \theta
$$

and that $\mathcal{T}=\left[-\theta_{\max }, \theta_{\max }\right]$ where $0<\theta_{\max }<\pi / 2$. Then in the absence of blur, i.e., when $B_{0}=1$, the frequency response integral (4.6.18) becomes

$$
\begin{aligned}
H(\varrho, \Phi, \Theta) & =\frac{1}{\varrho} \int_{0}^{\min \left(\theta_{\max }, \frac{\pi}{2}-|\Theta|\right)} \frac{1}{\sqrt{1-\left(\frac{\sin \theta}{\cos \Theta}\right)^{2}}}\left(\frac{\cos \theta}{\cos \Theta}\right) \mathrm{d} \theta \\
& =\left.\frac{1}{\varrho} \sin ^{-1}\left(\frac{\sin \theta}{\cos \Theta}\right)\right|_{\theta=\min \left(\theta_{\max }, \frac{\pi}{2}-|\Theta|\right)}=\frac{1}{\varrho} \sin ^{-1}\left(\frac{\min \left(\sin \theta_{\max }, \cos \Theta\right)}{\cos \Theta}\right)
\end{aligned}
$$

This frequency response expression agrees with [27].
4.6.1.6.2 SPECT case For $360^{\circ}$ SPECT scans, we have $w_{p}=\delta(\theta)$ and $w_{4}=1$ so $w_{0}=\frac{1}{\sqrt{\nu_{1}^{2}+\nu_{2}^{2}}}$. Thus the frequency response of the Gram operator is

$$
H\left(\rho \cos \Phi, \rho \sin \Phi \nu_{3}\right)=\frac{1}{|\rho|} w_{\varepsilon}(\Phi, 0)\left|B_{0}(\rho, 0 ; \Phi, 0)\right|^{2}
$$

or equivalently in spherical coordinates

$$
H(\varrho, \Phi, \Theta)=\frac{1}{|\varrho| \cos \Theta} w_{\varepsilon}(\Phi, 0)\left|B_{0}(\varrho \cos \Theta, 0 ; \Phi, 0)\right|^{2}
$$

4.6.1.6.3 Extensions The local frequency response of the Gram matrix for the cone-beam case is derived in [28] and has been applied to predicting noise for axial [29] and helical [30] CT scans with iterative reconstruction. Such local frequency response expressions have also been used for designing 3D regularization methods [31, 32].

### 4.6.2 3D cylindrical PET (s,3d,cyl)

This section considers a 3D parameterization that is suitable for cylindrical PET scanners [33]. We parameterize the line-integral projections, $p=\mathcal{P} f$, of a 3D object $f(\overrightarrow{\mathrm{x}})=f(x, y, z)$ as follows:

$$
\begin{align*}
p(r, \tilde{z} ; \varphi, \tau)= & \int_{-\infty}^{\infty} f\left(r \cos \varphi-\frac{\ell}{\sqrt{1+\tau^{2}}} \sin \varphi, r \sin \varphi+\frac{\ell}{\sqrt{1+\tau^{2}}} \cos \varphi, \tilde{z}+\frac{\ell}{\sqrt{1+\tau^{2}}} \tau\right) \mathrm{d} \ell \\
= & \sqrt{1+\tau^{2}} \int_{-\infty}^{\infty} f(r \cos \varphi-t \sin \varphi, r \sin \varphi+t \cos \varphi, \tilde{z}+t \tau) \mathrm{d} t \\
= & \sqrt{1+\tau^{2}} \iiint f(x, y, z) \delta(x \cos \varphi+y \sin \varphi-r) \\
& \delta(z-(-x \sin \varphi+y \cos \varphi) \tau-\tilde{z}) \mathrm{d} x \mathrm{~d} y \mathrm{~d} z \tag{4.6.19}
\end{align*}
$$

where $\tau$ denotes the tangent of the angle between the line and the transaxial plane at axial position $\tilde{z}$, and the variable of integration $t$ is along the projection of the line in that plane. The angle of that projected line with respect to the $y$ axis is $\varphi \in[0, \pi)$. The variable $r$ is the signed distance of the point where that line intersects the plane to the origin. Specifically: $\left[\begin{array}{l}r \\ t\end{array}\right]=\left[\begin{array}{rr}\cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi\end{array}\right]\left[\begin{array}{l}x \\ y\end{array}\right]$. In typical multi-ring PET scanners, $\tau$ is proportional to the difference between the ring indices [33]. The goal of 3D reconstruction is to estimate $f(x, y, z)$ from (noisy) samples of $p(r, \tilde{z} ; \varphi, \tau)$.

### 4.6.2.1 Backprojection

We consider the hypothetical case of an infinitely long cylindrical scanner with an infinite radius, so $r, \tilde{z} \in \mathbb{R}$. However, practical systems only accept coincidences for a maximum ring difference, so we limit $\tau$ to lie in an interval $\left[-\tau_{\max }, \tau_{\max }\right]$. (If $\tau_{\max }=0$, then the problem reverts to the ordinary 2D Radon transform for each slice.)

The projection operation (4.6.19) satisfies the following shift property. If $f_{2}(\overrightarrow{\mathrm{x}})=f_{1}\left(\overrightarrow{\mathrm{x}}-\overrightarrow{\mathrm{x}}_{0}\right), p_{1}=\mathcal{P} f_{1}$, and $p_{2}=\mathcal{P} f_{2}$, then

$$
p_{2}(r, \tilde{z} ; \varphi, \tau)=p_{1}\left(r-\left(x_{0} \cos \varphi+y_{0} \sin \varphi\right), \tilde{z}-z_{0}+\left(-x_{0} \sin \varphi+y_{0} \cos \varphi\right) ; \varphi, \tau\right) .
$$

Natural inner products for this parameterization of projection space have the form

$$
\left\langle p_{1}, p_{2}\right\rangle=\int_{0}^{\pi} \iiint p_{1}(r, \tilde{z} ; \varphi, \tau) p_{2}(r, \tilde{z} ; \varphi, \tau) \mathrm{d} r \mathrm{~d} \tilde{z} \frac{1}{\sqrt{1+\tau^{2}}} w_{0}(\varphi, \tau) \mathrm{d} \tau \mathrm{~d} \varphi
$$

where $w_{0}(\varphi, \tau)$ is a user-selectable weighting function that is positive on $\left[-\tau_{\max }, \tau_{\max }\right]$ and zero otherwise. For this inner product, and the usual $\mathcal{L}_{2}\left(\mathbb{R}^{3}\right)$ inner product for object space, the adjoint of the projection operator $\mathcal{P}$ is given by the backprojection operator $b=\mathcal{P}^{*} p$ defined as follows:

$$
\begin{aligned}
b(x, y, z)= & \int_{0}^{\pi} \iiint p(r, \tilde{z} ; \varphi, \tau) \delta(x \cos \varphi+y \sin \varphi-r) \\
& \delta(z-(-x \sin \varphi+y \cos \varphi) \tau-\tilde{z}) \mathrm{d} r \mathrm{~d} \tilde{z} w_{0}(\varphi, \tau) \mathrm{d} \tau \mathrm{~d} \varphi \\
= & \int_{0}^{\pi} \int p(x \cos \varphi+y \sin \varphi, z-(-x \sin \varphi+y \cos \varphi) \tau ; \varphi, \tau) w_{0}(\varphi, \tau) \mathrm{d} \tau \mathrm{~d} \varphi
\end{aligned}
$$

The combined projection/backprojection operation $\mathcal{P}^{*} \mathcal{P}$ is shift invariant, i.e., if $f_{2}(\vec{x})=f_{1}\left(\vec{x}-\vec{x}_{0}\right), b_{2}=\mathcal{P}^{*} \mathcal{P} f_{2}$, and $b_{1}=\mathcal{P}^{*} \mathcal{P} f_{1}$, then $b_{2}(\overrightarrow{\mathrm{x}})=b_{1}\left(\overrightarrow{\mathrm{x}}-\overrightarrow{\mathrm{x}}_{0}\right)$, because

$$
\begin{aligned}
& b_{1}\left(\overrightarrow{\mathrm{x}}-\overrightarrow{\mathrm{x}}_{0}\right) \\
&=\int_{0}^{\pi} \int p_{1}\left(\left(x-x_{0}\right) \cos \varphi+\left(y-y_{0}\right) \sin \varphi\right. \\
&\left.z-z_{0}-\left(-\left(x-x_{0}\right) \sin \varphi+\left(y-y_{0}\right) \cos \varphi\right) \tau ; \varphi, \tau\right) w_{0}(\varphi, \tau) \mathrm{d} \tau \mathrm{~d} \varphi \\
&= \int_{0}^{\pi} \int p_{1}\left(x \cos \varphi+y \sin \varphi-\left(x_{0} \cos \varphi+y_{0} \sin \varphi\right),\right. \\
&\left.z-(-x \sin \varphi+y \cos \varphi) \tau-z_{0}+\left(-x_{0} \sin \varphi+y_{0} \cos \varphi\right) \tau ; \varphi, \tau\right) w_{0}(\varphi, \tau) \mathrm{d} \tau \mathrm{~d} \varphi \\
&= \int_{0}^{\pi} \int p_{2}(x \cos \varphi+y \sin \varphi, z-(-x \sin \varphi+y \cos \varphi) \tau ; \varphi, \tau) w_{0}(\varphi, \tau) \mathrm{d} \tau \mathrm{~d} \varphi \\
&= b_{2}(x, y, z)
\end{aligned}
$$

The weighted projection/backprojection operator $\mathcal{P}^{*} \mathcal{W} \mathcal{P}$ is shift invariant if $\mathcal{W}$ depends only on $\varphi$ and $\tau$, i.e.

$$
p_{2}=\mathcal{W} p_{1} \Longrightarrow p_{2}(r, \tilde{z} ; \varphi, \tau)=w_{1}(\varphi, \tau) p_{1}(r, \tilde{z} ; \varphi, \tau),
$$

where $w_{1}(\varphi, \tau)$ is another user-selectable nonnegative weighting function. As in $\S 3.3$, due to this shift invariance we can examine the behavior of $b=\mathcal{P}^{*} \mathcal{W} \mathcal{P} f$ at the origin:

$$
\begin{aligned}
b(0,0,0) & =\int_{0}^{\pi} \int w_{1}(\varphi, \tau) p(0,0 ; \varphi, \tau) w_{0}(\varphi, \tau) \mathrm{d} \tau \mathrm{~d} \varphi \\
& =\int_{0}^{\pi} \int w_{1}(\varphi, \tau) \sqrt{1+\tau^{2}} \int f(0 \cos \varphi-\ell \sin \varphi, 0 \sin \varphi+\ell \cos \varphi, 0+\ell \tau) \mathrm{d} \ell w_{0}(\varphi, \tau) \mathrm{d} \tau \mathrm{~d} \varphi \\
& =\iiint w\left(厶_{\pi}(-t, s), \tau\right) f\left(-s,-t, \sqrt{s^{2}+t^{2}} \tau\right) \frac{1}{\sqrt{s^{2}+t^{2}}} \mathrm{~d} s \mathrm{~d} t \mathrm{~d} \tau \\
& =\iiint w\left(\angle(s, t)+\pi / 2, \frac{-z^{\prime}}{\sqrt{s^{2}+t^{2}}}\right) f\left(-s,-t,-z^{\prime}\right) \frac{1}{\left(\sqrt{s^{2}+t^{2}}\right)^{2}} \mathrm{~d} z^{\prime} \mathrm{d} s \mathrm{~d} t
\end{aligned}
$$

where (cf. (3.3.5)) $s=\ell \sin \varphi, t=-\ell \cos \varphi, \varphi=\angle_{\pi}(-t, s), \ell= \pm \sqrt{s^{2}+t^{2}}, z^{\prime}=-\sqrt{s^{2}+t^{2}} \tau$, and we define

$$
w(\varphi, \tau) \triangleq \sqrt{1+\tau^{2}} w_{1}(\varphi, \tau) w_{0}(\varphi, \tau)
$$

Applying shift invariance, it follows that

$$
\begin{equation*}
b(x, y, z)=f(x, y, z) * * * h(r, \varphi, z), \text { where } h(r, \varphi, z)=w\left(\varphi+\frac{\pi}{2}, \frac{-z}{|r|}\right) \frac{1}{r^{2}} \tag{4.6.20}
\end{equation*}
$$

where $r= \pm \sqrt{x^{2}+y^{2}}$.
Converting to spherical coordinates using $r_{3}=r / \cos \theta$ and $\tan \theta=z /|r|$, where $\theta \in[-\pi / 2, \pi / 2]$, we have:

$$
h\left(r_{3}, \varphi, \theta\right)=\frac{1}{r_{3}^{2}} w\left(\varphi+\frac{\pi}{2},-\tan \theta\right) \frac{1}{\cos ^{2} \theta} .
$$

In the case where $w(\varphi, \tau)=1 /\left(1+\tau^{2}\right)=\cos ^{2} \theta$, and when all possible rays are considered, i.e., $\tau_{\max }=\infty$, then this impulse response is simply $1 / r_{3}^{2}$, which is the classical PSF for a spherical system. At the other extreme, if $w(\varphi, \tau)=\frac{1}{2 \tau_{\max }} \operatorname{rect}\left(\frac{\tau}{2 \tau_{\max }}\right)$, then as $\tau_{\max } \rightarrow 0$, this impulse response approaches the expected response of the 2D case: $\frac{1}{r} \delta(z)$.

### 4.6.2.2 Frequency response

Hereafter we assume that the weighting $w(\varphi, \tau)$ has the form

$$
w(\varphi, \tau)=w(\varphi) \frac{1}{1+\tau^{2}} \operatorname{rect}\left(\frac{\tau}{2 \tau_{\max }}\right)
$$

where $w(\varphi)$ is $\pi$-periodic. (This form means that any statistical weighting depends only on $\varphi$, rather than on $\tau$, which seems to be a reasonable starting point for systems having modest acceptance angles, i.e., small $\tau_{\text {max }}$.) The frequency response corresponding to $(4.6 .20)$ is

$$
\begin{aligned}
H(\rho, \Phi, \zeta) & =\iint_{0}^{\pi} \int h(r, \varphi, z) \mathrm{e}^{-\imath 2 \pi \rho r \cos (\varphi-\Phi)} \mathrm{e}^{-\imath 2 \pi z \zeta} r \mathrm{~d} r \mathrm{~d} \varphi \mathrm{~d} z \\
& =\iint^{\operatorname{rect}\left(\frac{z / r}{2 \tau_{\max }}\right) \frac{1}{1+(z / r)^{2}} \int_{0}^{\pi} w\left(\varphi+\frac{\pi}{2}\right) \frac{1}{r^{2}} \mathrm{e}^{-\imath 2 \pi z \zeta} \mathrm{e}^{-\imath 2 \pi \rho r \cos (\varphi-\Phi)} r \mathrm{~d} r \mathrm{~d} \varphi \mathrm{~d} z} \\
& =\int_{-\tau_{\max }}^{\tau_{\max }} \frac{1}{1+\tau^{2}} \int_{-\pi / 2-\Phi}^{\pi / 2-\Phi} w\left(\varphi^{\prime}+\Phi\right)\left[\int \mathrm{e}^{-\imath 2 \pi r \tau \zeta} \mathrm{e}^{\imath 2 \pi \rho r \sin \varphi^{\prime}} \mathrm{d} r\right] \mathrm{d} \varphi^{\prime} \mathrm{d} \tau \\
& =\int_{-\tau_{\max }}^{\tau_{\max }} \frac{1}{1+\tau^{2}} \delta\left(\tau \zeta-\rho \sin \varphi^{\prime}\right) \int_{-\pi / 2-\Phi}^{\pi / 2-\Phi} w\left(\varphi^{\prime}+\Phi\right) \mathrm{d} \varphi^{\prime} \mathrm{d} \tau \\
& =\frac{1}{|\rho|} \int_{-\tau_{\max }}^{\tau_{\max }} \frac{1}{1+\tau^{2}} \int_{-\pi / 2-\Phi}^{\pi / 2-\Phi} w\left(\Phi+\varphi^{\prime}\right) \delta\left(\sin \varphi^{\prime}-\frac{\tau \zeta}{\rho}\right) \mathrm{d} \varphi^{\prime} \mathrm{d} \tau \\
& =\frac{1}{|\rho|} \int_{-\tau_{\max }}^{\tau_{\max }} \frac{1}{1+\tau^{2}} w\left(\Phi+\sin ^{-1}(\tau \zeta / \rho)\right) \frac{1}{\sqrt{1-(\tau \zeta / \rho)^{2}}} \mathbb{I}_{\{|\tau| \leq|\rho / \zeta|\}} \mathrm{d} \tau \\
& \approx \frac{w(\Phi)}{|\rho|} \int_{-\min \left(\tau_{\max },|\rho / \zeta|\right)}^{\min \left(\tau_{\max },|\rho / \zeta|\right)} \frac{1}{1+\tau^{2}} \frac{1}{\sqrt{1-(\tau \zeta / \rho)^{2}}} \mathrm{~d} \tau \\
& =\left.\frac{w(\Phi)}{|\rho| \sqrt{1+(\zeta / \rho)^{2}}} 2 \sin ^{-1}\left(\frac{\tau \sqrt{1+(\zeta / \rho)^{2}}}{\sqrt{1+\tau^{2}}}\right)\right|_{\tau=\min \left(\tau_{\max },|\rho / \zeta|\right)}
\end{aligned}
$$

where $\tau=z / r$ and $\varphi^{\prime}=\varphi-\Phi-\pi / 2$. The approximation relies on the fact that $\tau_{\max }$ is small; the final expression is exact if $w(\varphi)$ is a constant.

Now we express this frequency response in spherical coordinates where $\varrho=\rho / \cos \Theta$ and $\tan \Theta=\zeta /|\rho|$. So $\sqrt{1+(\zeta / \rho)^{2}}=1 /|\cos \Theta|$. Thus our final expression for the frequency response of the Gram operator is

$$
\begin{equation*}
H(\varrho, \Phi, \Theta)=\frac{1}{|\varrho|} Q(\Phi, \Theta), \text { where } Q(\Phi, \Theta)=w(\Phi) 2 \sin ^{-1}\left(\frac{\min \left(\sin \theta_{\max }, \cos \Theta\right)}{\cos \Theta}\right) \tag{4.6.21}
\end{equation*}
$$

where $\tan \theta_{\max }=\tau_{\max }$. This frequency response expression agrees with [27] for the case $w(\varphi)=1$.

### 4.6.3 General 3D tomography

For any shift-invariant system having an impulse response of the form

$$
h\left(r_{3}, \varphi, \theta\right)=\frac{1}{r_{3}^{2}} q(\varphi, \theta)
$$

in spherical coordinates, the corresponding frequency response has the following form in spherical coordinates [34]:

$$
H(\varrho, \Phi, \Theta)=\frac{1}{|\varrho|} Q(\Phi, \Theta)
$$

where $Q(\Phi, \Theta)$ is related by a 1D integral to $q(\varphi, \theta)$. The result (4.6.21) is a special case. In other words,

$$
\mathcal{P}^{*} \mathcal{W} \mathcal{P}=\mathcal{F}_{3}^{-1} \mathcal{D}(Q(\Phi, \Theta)) \mathcal{F}_{3}
$$

where $\mathcal{F}_{3}$ denotes the 3D Fourier transform operator and $\mathcal{D}$ is defined as in (4.2.7).

### 4.6.4 Regularization in 3D (s,3d,reg)

Generalizing (4.3.10), if we define a 3D regularizer for which

$$
\mathcal{R}=\frac{1}{2} \mathcal{F}_{3}^{-1} \mathcal{D}\left((2 \pi \varrho)^{2 M_{\mathrm{R}}} R_{3}(\Phi, \Theta)\right) \mathcal{F}_{3}
$$

then the QPWLS estimator that generalizes (4.3.13) has ensemble mean

$$
\mathrm{E}[\hat{f}]=\mathcal{F}_{3}^{-1} \mathcal{D}(L(\varrho, \Phi, \Theta)) \mathcal{F}_{3}
$$

where (cf. (4.3.17)) ignoring detector blur, the frequency response is:

$$
\begin{equation*}
L(\varrho, \Phi, \Theta) \approx \frac{Q(\Phi, \Theta)}{Q(\Phi, \Theta)+\beta|\varrho|(2 \pi \varrho)^{2 M_{\mathrm{R}}} R_{3}(\Phi, \Theta)} . \tag{4.6.22}
\end{equation*}
$$

To produce isotropic spatial resolution, evidently we would like to design the regularization method so that

$$
\begin{equation*}
R_{3}(\Phi, \Theta) \approx Q(\Phi, \Theta) \tag{4.6.23}
\end{equation*}
$$

In 2D, such a design is simple in the statistically "unweighted" case where $w(\varphi)=1$. In realistic 3D systems, i.e., excluding a spherical detector, even when $w(\varphi)=1$ the other terms in $Q(\Phi, \Theta)$ related to the geometry greatly increase the challenge in designing $\mathcal{R}$ for isotropic spatial resolution. $\S 5.2 .2$ describes a practical design method.

### 4.6.5 The 'long object'" problem (s,3d,long)

The preceding description assumes that the axial field of view of the scanner is sufficiently long that all "cross plane" rays (those for which $\tau \neq 0$ ) are measured over the entire object. In practice, in most scanners the axial field of view is shorter than the physical length of the object, so $\tau_{\max }$ varies linearly from some maximum value at the axial center of the scanner down to zero at the ends of the scanner. This limitation is known as the long object problem, and is present in both 3D PET scanners and in cone-beam X-ray CT scanners. A solution to this problem was proposed in N. Pelc's thesis [35, p. 140].
[We can] first reconstruct the entire volume using only the in-plane rays. From that reconstruction we can calculate values for the missing cross-plane rays that would make the blurring function invariant over the entire field. By including these "forward projected" values into the set of measured cross-plane rays the data set can be made into one that resembles that of a circularly symmetric spatially invariant geometry with a field of view equal to the volume covered by the in-plane rays.
Apparently this idea was never published elsewhere until it was again proposed by Roger's et al. [36], and then finally implemented and evaluated by Kinahan et al. [37, 38]. Today it is known as the reprojection method or 3DRP algorithm.

Such projection completion steps are unnecessary when iterative algorithms are used. Nevertheless, the long object problem also complicates iterative reconstruction of CT scans because one must reconstruct extra slices at each end of the object to account for all possible attenuation along the rays [39].

### 4.6.6 Rebinning to 2D sinograms ( $\mathrm{s}, 3 \mathrm{~d}$, rebin)

Rather than performing fully 3D image reconstruction, which can be computationally expensive, an alternative is to first rebin the projection measurements into a stack of 2D sinograms, and then perform 2D reconstruction for each slice. Such methods have been proposed for a variety of imaging geometries including cylindrical PET [33, 40-42] and cone-beam CT [43-48]. In practice, approximate rebinning methods are used frequently, the accuracy of which decrease as the acceptance angle increases [49]. The most popular method for PET currently is Fourier rebinning (FORE) [33]. Both 2D FBP and 2D iterative methods have been used to reconstruct the rebinned 2D sinograms.

### 4.6.7 Helical cone-beam CT geometry ( $\mathrm{s}, 3 \mathrm{~d}$, helix)

In helical cone-beam CT, the X-ray source traverses a helical trajectory described by

$$
\vec{p}_{0}=\left(D_{\mathrm{s} 0} \sin \beta,-D_{\mathrm{s} 0} \cos \beta, \operatorname{pitch} \beta\right),
$$

where pitch denotes a pitch parameter with typical units $\mathrm{cm} /$ radian. The above expression is for a constant pitch; variable pitch methods also have been investigated [50].

It is an open problem to analyze the local impulse response or local frequency response of a helical cone-beam CT system. For such analysis it may be useful to consider the starting angle to be uniformly distributed over $[0,2 \pi)$. See [28].

### 4.7 Summary (s.topo,summ)

This chapter has considered analytical methods for tomographic image reconstruction that are based on a quadraticallypenalized weighted least-squares (QPWLS) reconstruction method. The methods were analyzed using Hilbert-space operators. The QPWLS method becomes equivalent to the BPF method as the regularization parameter shrinks to zero. The QPWLS "method" presented here may be of limited practical use, but the operator formulation serves as a bridge between the integral formulation of Chapter 3 and the matrix-vector formulations that will be the focus of all subsequent chapters. In fact, many of the main ingredients are here already: a system model, a cost function that includes regularization and statistical (angular) weighting, an estimator that minimizes that cost function, and analysis of the mean (spatial resolution) and variance (noise) of that estimator.

### 4.8 Problems (s,topoprob)

Problem 4.1 Reconcile the term projection used in tomography for $\mathcal{P}_{\varphi}$ with a standard orthogonal projection $\mathcal{P}$ used in Hilbert spaces. Hint. Consider functions on the unit disk that are are constant along rays at angle $\varphi$.

Problem 4.2 Prove the backprojection/convolution property (4.2.11).
(Solve?)
Problem 4.3 Verify the adjoints shown in (4.2.16).
Problem 4.4 Verify the SVD of $\mathcal{A}$ given in (4.2.15)
(Need typed.)
Problem 4.5 Let $d(|\rho|)$ denote any (suitably regular) function of $\rho$, and define the following two operators:

$$
\begin{aligned}
g=\mathcal{Q}_{2} f & \Longleftrightarrow G(u, v)=d\left(\sqrt{u^{2}+v^{2}}\right) F(u, v) \\
q=\mathcal{Q}_{1} p & \Longleftrightarrow Q_{\varphi}(\nu)=d(|\nu|) P_{\varphi}(\nu) .
\end{aligned}
$$

Use the SVD of $\mathcal{P}$ to show the following "commutative" property:

$$
\mathcal{Q}_{1} \mathcal{P}=\mathcal{P} \mathcal{Q}_{2} .
$$

When $d(\nu)=|\nu|^{\alpha}$, it is called a Riesz potential [4, p. 5]. One can use this result to derive the FBP method [4, p. 11]. (Need typed.)

Problem 4.6 Generalize the $2 D$ SVD analysis in $\S 4.2 .4$ by using weighted inner products for $\mathcal{H}_{0}$ and $\mathcal{H}_{\text {sino }}$, mimicking the weighted inner products used in $\S 4.6 .1 .5$. Determine the frequency response of the Gram operator $\mathcal{A}^{*} \mathcal{W} \mathcal{A}$ for the WLS cost function, generalizing (4.3.7). In the 3D case this frequency response depends on the inner product weights, as seen in (4.6.18). Does it in the 2D case?

Problem 4.7 As noted in [51], for 2D parallel-beam tomography, the FWHM of the PSF is proportional to $\beta^{1 / 3}$, which can provide a useful rule of thumb for choosing the regularization parameter.
Starting with the local frequency response expression (4.4.8), assume equal statistical weighting $w_{0}(\Phi)=1$, no detector blur $B_{0}=1$, and $R(\rho)=\rho^{2}$, so that $L_{0}(\rho)=\frac{1}{1+\beta \rho^{3}}$. Determine the RMS bandwidth $\rho_{\mathrm{RMS}}=\sqrt{\frac{\int_{0}^{\infty} \rho^{2}\left|L_{0}(\rho)\right|^{2} \rho \mathrm{~d} \rho}{\int_{0}^{\infty}\left|L_{0}(\rho)\right|^{2} \rho \mathrm{~d} \rho}}$ as a function of $\beta$.

Problem 4.8 Prove the equality (4.3.1).
Problem 4.9 Establish an SVD decomposition for a continuous-to-discrete system operator $\mathcal{A}$ like that in (4.3.3). (Solve?)

Problem 4.10 Generalize the analysis in $\S 4.3$ by considering a measurement model with a finite number of "angular samples" having the following general form

$$
\bar{y}_{l}[n]=\left.\int_{0}^{\pi} \eta_{l}(\varphi) p_{\varphi}(r) * b_{\varphi}(r) \mathrm{d} \varphi\right|_{r=r_{n}}, \quad l=1, \ldots, L
$$

where $\eta_{l}(\varphi)$ characterizes the angular sampling. For example, in a "step and shoot" system we would have $\eta_{l}(\varphi)=$ $\delta\left(\varphi-\varphi_{l}\right)$ whereas for a continuous rotation system with equally-spaced angular intervals we would have $\eta_{l}(\varphi)=$ $\mathbb{I}_{\{\pi l / L \leq \varphi<\pi(l+1) / L\}}$. Hint. Consider Problem 3.6 and Theorem 3.3.1.
(Need typed.)
Problem 4.11 Find an operator $\mathcal{C}$ such that the continuous-space penalty (2.4.1) can be written in the following form:

$$
\mathrm{R}(f)=\frac{1}{2}\|\mathcal{C} f\|^{2}
$$

for the usual $\mathcal{L}_{2}\left(\mathbb{R}^{2}\right)$ norm. Extend to the general case (4.3.12).
(Need typed.)
Problem 4.12 Repeat Example 4.3.1 for a tomographic system with a rectangular detector PSF: $b(r)=\operatorname{rect}(r)$. Comment on the sidelobe behavior.
(Need typed.)
Problem 4.13 Prove the equality (4.3.20).
(Need typed.)
Problem 4.14 Verify the following 3D X-ray transform pairs (of a 3D Dirac impulse and a $3 D$ unit sphere, respectively) using (4.6.3):

$$
\begin{aligned}
& \delta_{3}(x-a, y-b, z-c) \stackrel{3 D}{ } \text { Xray } \\
& \longleftrightarrow \delta_{2}(u-(a \cos \varphi+b \sin \varphi), v-(a \sin \varphi \sin \theta-b \cos \varphi \sin \theta+c \cos \varphi)) \\
& \mathbb{I}_{\left\{\sqrt{x^{2}+y^{2}+z^{2}} \leq 1\right\}} \stackrel{3 \mathrm{D} \text { Xray }}{\longleftrightarrow} 2 \sqrt{1-\left(u^{2}+v^{2}\right)} \mathbb{I}_{\left\{\sqrt{u^{2}+v^{2}} \leq 1\right\}} .
\end{aligned}
$$

(Need typed.)
Problem 4.15 Some $3 D$ functions, including an ellipsoid, can be written as $f(\overrightarrow{\mathrm{x}})=\mathbb{I}_{\{\|M \overrightarrow{\mathrm{x}}\| \leq 1\}}$, for some $3 \times 3$ matrix $\boldsymbol{M}$. For an ellipsoid, $\boldsymbol{M}=\operatorname{diag}\left\{1 / r_{x}, 1 / r_{y}, 1 / r_{z}\right\}$. Assuming that $\boldsymbol{M}^{\prime} \boldsymbol{M}$ is positive definite, verify that the $3 D$ $X$-ray transform of such an $f$ is given by

$$
\begin{aligned}
p(u, v ; \varphi, \theta) & =\frac{2}{A} \sqrt{B^{2}-A C} \mathbb{I}_{\left\{B^{2} \geq A C\right\}} \\
A & =\|\boldsymbol{M} \vec{e}\|^{2} \\
B & =(\boldsymbol{M} \vec{e})^{\prime} \boldsymbol{M} \vec{p} \\
C & =\|\boldsymbol{M} \vec{p}\|^{2}-1,
\end{aligned}
$$

where $\vec{e}=\vec{e}(\vec{\gamma})$ and $\vec{p}=\vec{p}(u, v ; \vec{\gamma})$ were defined in (4.6.1) and (4.6.2).
Problem 4.16 An elliptical cylinder of height $h$ can be written as $f(\overrightarrow{\mathrm{x}})=\mathbb{I}_{\{\|\boldsymbol{\rightharpoonup}\| \leq 1\}} \mathbb{I}_{\{|z| \leq h / 2\}}$, where $\boldsymbol{M}=$ $\operatorname{diag}\left\{1 / r_{x}, 1 / r_{y}, 0\right\}$. Verify that the corresponding $3 D X$-ray transform is given by

$$
\begin{aligned}
p(u, v ; \varphi, \theta) & =\left[\ell_{2}-\ell_{1}\right]_{+} \mathbb{I}_{\left\{B^{2} \geq A C\right\}} \\
\ell_{2} & =\min \left\{\ell_{+}, \tilde{\ell}_{+}\right\} \\
\ell_{1} & =\max \left\{\ell_{-}, \tilde{\ell}_{-}\right\} \\
\ell_{ \pm} & =\frac{-B \pm \sqrt{B^{2}-A C}}{A} \\
\tilde{\ell}_{ \pm} & = \pm(h / 2-v \cos \theta) / \sin \theta \\
A & =\|\boldsymbol{M} \vec{e}\|^{2} \\
B & =(\boldsymbol{M} \vec{e})^{\prime} \boldsymbol{M} \vec{p} \\
C & =\|\boldsymbol{M} \vec{p}\|^{2}-1,
\end{aligned}
$$

where $\vec{e}=\vec{e}(\vec{\gamma})$ and $\vec{p}=\vec{p}(u, v ; \vec{\gamma})$ were defined in (4.6.1) and (4.6.2).

Problem 4.17 Prove (4.3.7). For simplicity, ignore the effects of sampling.
Problem 4.18 Prove the shift property (4.6.6).
Problem 4.19 If $f(\overrightarrow{\mathrm{x}}) \stackrel{3 \mathrm{D} \text { Xray }}{\longleftrightarrow} p(u, v ; \varphi, \theta)$, then prove the following affine scaling property:

$$
\begin{align*}
f(x / a, y / b, z / c) & \stackrel{3 \mathrm{D} \text { Xray }}{\longleftrightarrow} p\left(u^{\prime}, v^{\prime} ; \varphi^{\prime}, \theta^{\prime}\right)  \tag{4.8.1}\\
\tan \varphi^{\prime} & =\frac{b}{a} \tan \varphi  \tag{4.8.2}\\
\tan \theta^{\prime} & =\frac{a b}{c} \frac{1}{\sqrt{(a \cos \varphi)^{2}+(b \sin \varphi)^{2}}} \tan \theta  \tag{4.8.3}\\
u^{\prime} & =\frac{1}{\sqrt{(a \cos \varphi)^{2}+(b \sin \varphi)^{2}}} u  \tag{4.8.4}\\
v^{\prime} & =\left[\left(\frac{b}{a}-\frac{a}{b}\right) \sin \varphi \cos \varphi \sin \theta^{\prime}\right] u+\frac{1}{c} \frac{\cos \theta^{\prime}}{\cos \theta} v . \tag{4.8.5}
\end{align*}
$$

Problem 4.20 For an $X$-ray point source at location $\overrightarrow{\mathrm{x}}_{0} \in \mathbb{R}^{3}$ and a detector element at location $\overrightarrow{\mathrm{x}}_{1} \in \mathbb{R}^{3}$, the integral of $f$ along the ray between those points is

$$
\int f\left(\overrightarrow{\mathrm{x}}_{0}+\alpha \vec{e}_{0}\right) \mathrm{d} \alpha
$$

where $\vec{e}_{0} \triangleq\left(\vec{x}_{1}-\vec{x}_{0}\right) /\left\|\vec{x}_{1}-\vec{x}_{0}\right\|$. Relate this line integral to a corresponding parallel-beam projection given in §4.6.1. If $\vec{e}_{0}=(a, b, c)$, show that the correspondence is

$$
\begin{aligned}
\varphi & =-\arctan (a / b) \\
\theta & =-\arcsin c \\
u & =\vec{e}_{1} \cdot \overrightarrow{\mathrm{x}}_{0} \\
v & =\vec{e}_{2} \cdot \overrightarrow{\mathrm{x}}_{0}
\end{aligned}
$$

See ir_coord_cb_flat_to_par.m.
Using (4.6.3), clearly $u=\vec{e}_{1} \cdot \overrightarrow{\mathrm{x}}_{0}$ and $v=\vec{e}_{2} \cdot \overrightarrow{\mathrm{x}}_{0}$. Furthermore, $\vec{e}= \pm \vec{e}_{0}$, so

$$
\begin{aligned}
a & =\mp \sin \varphi \cos \theta \\
b & = \pm \cos \varphi \cos \theta \\
c & = \pm \sin \theta
\end{aligned}
$$

Thus $\theta= \pm \arcsin c$ and $\cos \theta=1 / \sqrt{1-c^{2}}$ and $a / b=-\tan \varphi$.
Problem 4.21 It can be useful to relate cone-beam geometry coordinates, such as described in (3.10.4), to the corresponding parallel-beam coordinates given in §4.6.1. If $(s, t)$ denote the detector coordinates, then define

$$
\begin{aligned}
& x_{s}= \begin{cases}s, & D_{\mathrm{fd}}=\infty \\
D_{\mathrm{fd}} \sin \left(s / D_{\mathrm{fd}}\right), & D_{0 \mathrm{~d}} \leq D_{\mathrm{fd}}<\infty\end{cases} \\
& y_{s}= \begin{cases}-D_{0 \mathrm{~d}}, & D_{\mathrm{fd}}=\infty \\
D_{\mathrm{fs}}+D_{\mathrm{s} 0}-D_{\mathrm{fd}} \cos \left(s / D_{\mathrm{fd}}\right), & D_{0 \mathrm{~d}} \leq D_{\mathrm{fd}}<\infty\end{cases}
\end{aligned}
$$

Show that the correspondence is

$$
\begin{aligned}
& \varphi=\beta+\arctan \left(s / D_{\mathrm{sd}}\right) \\
& \theta=-\arctan \left(\frac{t}{\sqrt{x_{s}^{2}+\left(y_{s}-D_{\mathrm{s} 0}\right)^{2}}}\right) \\
& u=D_{\mathrm{s} 0} \sin (\varphi-\beta)=D_{\mathrm{s} 0} \frac{s}{\sqrt{D_{\mathrm{sd}}^{2}+s^{2}}} \\
& v=-D_{\mathrm{s} 0} \sin \theta \cos (\varphi-\beta)=D_{\mathrm{s} 0} \frac{t}{\sqrt{x_{s}^{2}+\left(y_{s}-D_{\mathrm{s} 0}\right)^{2}+t^{2}}} \frac{D_{\mathrm{sd}}}{\sqrt{D_{\mathrm{sd}}^{2}+s^{2}}}
\end{aligned}
$$

(Note that when $D_{\mathrm{fd}}=\infty$, we have $y_{s}-D_{\mathrm{s} 0}=D_{\mathrm{sd}}$.)
See ir_coord_cb_flat_to_par.m.
Problem 4.22 Generalize the fan-beam BPF analysis in §3.9.4 to include the effects of detector blur, e.g., $p^{\prime}(s, \beta)=$ $p(s, \beta) * h(s)$.
(Solve?)

### 4.9 Bibliography

[1] D. G. Luenberger. Optimization by vector space methods. New York: Wiley, 1969. URL: http: / /books.google.com/books?id=lZU0CAH4RccC (cit. on pp. 4.2, 4.3, 4.6).
[10] A. Caponnetto and M. Bertero. "Tomography with a finite set of projections: singular value decomposition and re solution." In: Inverse Prob. 13.5 (Oct. 1997), 1191-205. DOI: $10.1088 / 0266$-5611/13/5/006 (cit. on p. 4.5).
[11] R. N. Bracewell. "Strip integration in radio astronomy." In: Aust. J. Phys. 9 (1956), 198-217. URL: http: //adsabs.harvard.edu/cgi-bin/nph-bib_query?bibcode=1956AuJPh...9..198B (cit. on p. 4.5).
[12] G. Chu and K. C. Tam. "Three-dimensional imaging in the positron camera using Fourier techniques." In: Phys. Med. Biol. 22.2 (Mar. 1977), 245-65. DoI: $10.1088 / 0031-9155 / 22 / 2 / 005$ (cit. on p. 4.7).
[13] A. Macovski. Medical imaging systems. New Jersey: Prentice-Hall, 1983 (cit. on p. 4.8).
[14] S. M. Kay. Fundamentals of statistical signal processing: Estimation theory. New York: Prentice-Hall, 1993 (cit. on pp. 4.10, 4.16).
[15] J. A. Fessler. "Tomographic reconstruction using information weighted smoothing splines." In: Information Processing in Medical Im. Ed. by H H Barrett and A F Gmitro. Vol. 687. Lecture Notes in Computer Science. Berlin: Springer-Verlag, 1993, pp. 372-86. DOI: 10.1007 /BF.b 0013800 (cit. on p. 4.11).
[16] J. W. Stayman and J. A. Fessler. "Compensation for nonuniform resolution using penalized-likelihood reconstruction in space-variant imaging systems." In: IEEE Trans. Med. Imag. 23.3 (Mar. 2004), 269-84. DOI: $10.1109 /$ TMI .2003 .823063 (cit. on pp. 4.12, 4.16).
[17] J. Nuyts and J. A. Fessler. "A penalized-likelihood image reconstruction method for emission tomography, compared to post-smoothed maximum-likelihood with matched spatial resolution." In: IEEE Trans. Med. Imag. 22.9 (Sept. 2003), 1042-52. DOI: $10.1109 /$ TMI. 2003.816960 (cit. on p. 4.12).
[18] R. Lecomte, D. Schmitt, and G. Lamoureux. "Geometry study of a high resolution PET detection system using small detectors." In: IEEE Trans. Nuc. Sci. 31.1 (Feb. 1984), 556-61. DOI: $10.1109 /$ TNS . 1984.4333318 (cit. on p. 4.14).
[19] J. G. Rogers. "A method for correcting the depth-of-interaction blurring in PET cameras." In: IEEE Trans. Med. Imag. 14.1 (Mar. 1995), 146-50. DOI: $10.1109 / 42.370411$ (cit. on p. 4.14).
[20] J. A. Fessler and W. L. Rogers. "Spatial resolution properties of penalized-likelihood image reconstruction methods: Space-invariant tomographs." In: IEEE Trans. Im. Proc. 5.9 (Sept. 1996), 1346-58. DOI: $10.1109 / 83.535846$ (cit. on pp. 4.14, 4.16).
[21] J. W. Stayman and J. A. Fessler. "Regularization for uniform spatial resolution properties in penalized-likelihood image reconstruction." In: IEEE Trans. Med. Imag. 19.6 (June 2000), 601-15. DOI: 10.1109/42.870666 (cit. on pp. 4.14, 4.15, 4.16).
[22] J. W. Stayman and J. A. Fessler. "Efficient calculation of resolution and covariance for fully-3D SPECT." In: IEEE Trans. Med. Imag. 23.12 (Dec. 2004), 1543-56. DOI: $10.1109 /$ TMI. 2004.837790 (cit. on p. 4.16).
[23] J. Qi and R. M. Leahy. "A theoretical study of the contrast recovery and variance of MAP reconstructions from PET data." In: IEEE Trans. Med. Imag. 18.4 (Apr. 1999), 293-305. DOI: $10.1109 / 42.768839$ (cit. on p. 4.16).
[24] J. Qi and R. M. Leahy. "Resolution and noise properties of MAP reconstruction for fully 3D PET." In: IEEE Trans. Med. Imag. 19.5 (May 2000), 493-506. DOI: $10.1109 / 42.870259$ (cit. on p. 4.16).
[25] H. R. Shi and J. A. Fessler. "Quadratic regularization design for 2D CT." In: IEEE Trans. Med. Imag. 28.5 (May 2009), 645-56. DOI: $10.1109 /$ TMI .2008 .2007366 (cit. on pp. 4.16, 4.18).
[26] A. C. Kak and M. Slaney. Principles of computerized tomographic imaging. New York: IEEE Press, 1988. DOI: $10.1137 / 1.9780898719277$. URL: http: / /www. slaney.org/pct (cit. on p. 4.19).
[27] J. G. Colsher. "Fully three dimensional positron emission tomography." In: Phys. Med. Biol. 25.1 (Jan. 1980), 103-15. DOI: $10.1088 / 0031-9155 / 25 / 1 / 010$ (cit. on pp. 4.23, 4.26).
[28] S. M. Schmitt, M. M. Goodsitt, and J. A. Fessler. "Fast variance prediction for iteratively reconstructed CT images with locally quadratic regularization." In: IEEE Trans. Med. Imag. 36.1 (Jan. 2017), 17-26. DOI: $10.1109 /$ TMI. 2016.2593259 (cit. on pp. 4.23, 4.27).
[29] S. Schmitt and J. A. Fessler. "Fast variance computation for quadratically penalized iterative reconstruction of 3D axial CT images." In: Proc. IEEE Nuc. Sci. Symp. Med. Im. Conf. 2012, 3287-92. DOI:
10.1109/NSSMIC. 2012.6551749 (cit. on p. 4.23).
[30] S. M. Schmitt and J. A. Fessler. "Fast variance prediction for iterative reconstruction of 3D helical CT images." In: Proc. Intl. Mtg. on Fully 3D Image Recon. in Rad. and Nuc. Med. 2013, 162-5. URL: proc/13/web/schmitt-13-fvp.pdf (cit. on p. 4.23).
[31] J. H. Cho and J. A. Fessler. "Quadratic regularization design for 3D axial CT." In: Proc. Intl. Mtg. on Fully 3D Image Recon. in Rad. and Nuc. Med. 2013, 78-81. URL: proc / 13/web/cho-13-qrd-f3d.pdf (cit. on p. 4.23).
[32] J. H. Cho and J. A. Fessler. "Quadratic regularization design for 3D axial CT: Towards isotropic noise." In: Proc. IEEE Nuc. Sci. Symp. Med. Im. Conf. 2013, 1-5. DOI: $10.1109 /$ NSSMIC. 2013.6829355 (cit. on p. 4.23).
[33] M. Defrise et al. "Exact and approximate rebinning algorithms for 3-D PET data." In: IEEE Trans. Med. Imag. 16.2 (Apr. 1997), 145-58. DOI: $10.1109 / 42.563660$ (cit. on pp. 4.24, 4.27).
[34] B. Schorr, D. Townsend, and R. Clack. "A general method for three-dimensional filter computation." In: Phys. Med. Biol. 28.9 (Sept. 1983), 1009-19. DOI: $10.1088 / 0031-9155 / 28 / 9 / 001$ (cit. on p. 4.26).
[35] N. J. Pelc. "A generalized filtered backprojection algorithm for three-dimensional reconstruction." PhD thesis. Boston: Harvard, 1979 (cit. on p. 4.26).
[36] J. G. Rogers, R. Harrop, and P. E. Kinahan. "The theory of three-dimensional image reconstruction for PET." In: IEEE Trans. Med. Imag. 6.3 (Sept. 1987), 239-43. DOI: $10.1109 /$ TMI .1987 .4307832 (cit. on p. 4.26).
[37] P. E. Kinahan et al. "Three-dimensional image reconstruction in object space." In: IEEE Trans. Nuc. Sci. 35.1 (Feb. 1988), 635-8. DOI: $10.1109 / 23.12802$ (cit. on p. 4.26).
[38] P. E. Kinahan and J. G. Rogers. "Analytic 3D image reconstruction using all detected events." In: IEEE Trans. Nuc. Sci. 36.1 (Feb. 1989), 964-8. DOI: $10.1109 / 23.34585$ (cit. on p. 4.26).
[39] M. Magnusson, P-E. Danielsson, and J. Sunnegardh. "Handling of long objects in iterative improvement of nonexact reconstruction in helical cone-beam CT." In: IEEE Trans. Med. Imag. 25.7 (July 2006), 935-40. DOI: 10.1109/TMI. 2006.876156 (cit. on p. 4.26).
[40] R. M. Lewitt, G. Muehllehner, and J. S. Karp. "Three-dimensional image reconstruction for PET by multi-slice rebinning and axial image filtering." In: Phys. Med. Biol. 39.3 (Mar. 1994), 321-9. DOI: 10.1088/0031-9155/39/3/002 (cit. on p. 4.27).
[41] M. Defrise and X. Liu. "A fast rebinning algorithm for 3D positron emission tomography using John's equation." In: Inverse Prob. 15.4 (Aug. 1999), 1047-65. DOI: $10.1088 / 0266-5611 / 15 / 4 / 314$ (cit. on p. 4.27).
[42] X. Liu et al. "Exact rebinning methods for three-dimensional PET." In: IEEE Trans. Med. Imag. 18.8 (Aug. 1999), 657-664. DOI: 10.1109 / 42.796279 (cit. on p. 4.27).
[43] F. Noo, M. Defrise, and R. Clackdoyle. "Single-slice rebinning method for helical cone-beam CT." In: Phys. Med. Biol. 44.2 (Feb. 1999), 561-70. DOI: $10.1088 / 0031-9155 / 44 / 2 / 019$ (cit. on p. 4.27).
[44] M. Kachelrieß, S. Schaller, and W. A. Kalender. "Advanced single-slice rebinning in cone-beam spiral CT." In: Med. Phys. 27.4 (Apr. 2000), 754-72. DOI: $10.1118 / 1.598938$ (cit. on p. 4.27).
[45] M. Kachelrieß et al. "Advanced single-slice rebinning for tilted spiral cone-beam CT." In: Med. Phys. 28.6 (June 2001), 1033-41. DOI: $10.1118 / 1.1373675$ (cit. on p. 4.27).
[46] S. Schaller et al. "Exact Radon rebinning algorithm for the long object problem in helical cone-beam CT." In: IEEE Trans. Med. Imag. 19.5 (May 2000), 361-75. DOI: $10.1109 / 42.870247$ (cit. on p. 4.27).
[47] M. Defrise, Frédéric Noo, and H. Kudo. "Rebinning-based algorithms for helical cone-beam CT." In: Phys. Med. Biol. 46.11 (Nov. 2001), 2911-28. DOI: $10.1088 / 0031-9155 / 46 / 11 / 311$ (cit. on p. 4.27).
[48] M. Defrise, Frédéric Noo, and H. Kudo. "Improved two-dimensional rebinning of helical cone-beam computerized tomography data using John's equation." In: Inverse Prob. 19.6 (Dec. 2003), S41-54. DOI: 10.1088/0266-5611/19/6/053 (cit. on p. 4.27).
[49] S. Matej et al. "Performance of the Fourier rebinning algorithm for PET with large acceptance angles." In: Phys. Med. Biol. 43.4 (Apr. 1998), 787-96. DOI: $10.1088 / 0031-9155 / 43 / 4 / 008$ (cit. on p. 4.27).
[50] Y. Ye, J. Zhu, and G. Wang. "Minimum detection windows, PI-line existence and uniqueness for helical cone-beam scanning of variable pitch." In: Med. Phys. 31.3 (Mar. 2004), 566-72. DOI: 10.1118/1.1646041 (cit. on p. 4.27).
[51] J. A. Fessler. Resolution properties of regularized image reconstruction methods. Tech. rep. 297. Univ. of Michigan, Ann Arbor, MI, 48109-2122: Comm. and Sign. Proc. Lab., Dept. of EECS, Aug. 1995. URL: http://web.eecs.umich.edu/~fessler/papers/lists/files/tr/95, 297, rpo.pdf (cit. on p. 4.27).
[52] J. A. Fessler. Spatial resolution properties of penalized weighted least-squares image reconstruction with model mismatch. Tech. rep. 308. Available from web.eecs.umich.edu/~fessler. Univ. of Michigan, Ann Arbor, MI, 48109-2122: Comm. and Sign. Proc. Lab., Dept. of EECS, Mar. 1997. URL:
http://web.eecs.umich.edu/~fessler/papers/lists/files/tr/97,308, srp.pdf.


[^0]:    ${ }^{1}$ The domain of $\mathcal{P}$ really should be something like the subspace of $\mathcal{L}_{2}\left(\mathbb{R}^{2}\right)$ on which the integrals (3.2.4) are well defined, but I am not intending to try to make this very rigorous.
    ${ }^{2} \mathcal{P}$ is a bounded linear operator for the usual $\mathcal{L}_{2}$ inner products [2]:

    $$
    \begin{aligned}
    \|\mathcal{P} f\|^{2} & =\int_{0}^{\pi} \int_{-\infty}^{\infty}\left|p_{\varphi}(r)\right|^{2} \mathrm{~d} r \mathrm{~d} \varphi \\
    & =\int_{0}^{\pi} \int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} f(r \cos \varphi-\ell \sin \varphi, r \sin \varphi+\ell \cos \varphi) \mathrm{d} \ell\right|^{2} \mathrm{~d} r \mathrm{~d} \varphi \\
    & \leq \int_{0}^{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty}|f(r \cos \varphi-\ell \sin \varphi, r \sin \varphi+\ell \cos \varphi)|^{2} \mathrm{~d} \ell \mathrm{~d} r \mathrm{~d} \varphi \\
    & =\int_{0}^{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty}|f(x, y)|^{2} \mathrm{~d} x \mathrm{~d} y \mathrm{~d} \varphi=\pi\|f\|^{2}
    \end{aligned}
    $$

[^1]:    ${ }^{3}$ We reuse $R$ here because it should always be evident whether any given expression involves the penalty $\mathrm{R}(f)$ or the spectrum $R(\rho, \Phi)$.

[^2]:    ${ }^{4}$ An ideal tomograph would have $a(r, \varphi ; x, y)=\delta(r-[x \cos \varphi+y \sin \varphi])$.
    ${ }^{5}$ This approximation is reasonable for a parallel-beam geometry but would require modification for fan-beam cases.

[^3]:    ${ }^{6}$ In all of the expressions involving $\int_{\mathcal{T}}$ in this section, the integral could be replaced by a sum if $\mathcal{T}$ is a discrete set as it is in practice.

[^4]:    ${ }^{7}$ In this expression, $\operatorname{sgn}(x)=\left\{\begin{array}{ll}1, & x \geq 0, \\ -1, & x<0 .\end{array}\right.$ Also, $\angle(b, a)=\pi / 2-\angle(a, b)$ and $\angle(a,-b)=-\angle(a, b)$.

