Chapter 26

Matrices and linear algebra

ap,mat

Contents

This appendix reviews some of the linear algebra and matrix analysis results that are useful for developing iterative algorithms and analyzing inverse problems.

In particular, the concept of the **adjoint** of a linear operator arises frequently when studying inverse problems, generalizing the familiar concept of the transpose of a matrix. This appendix also sketches the basic elements of functional analysis needed to describe an **adjoint**.

26.1 Matrix algebra

s,mat,det
26.1.1 Determinant (s,mat,det)

If $\boldsymbol{A} = a_{11} \in \mathbb{C}$ is a scalar, then the **determinant** of $\boldsymbol{A}$ is simply its value: $\det\{\boldsymbol{A}\} = a_{11}$. Using this definition as a starting point, the **determinant** of a square matrix $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ is defined recursively:

$$\det\{\boldsymbol{A}\} \triangleq \sum_{j=1}^{n} a_{ij}(-1)^{j+i} \det\{\boldsymbol{A}_{-i,-j}\}$$

for any $i \in \{1, \ldots, n\}$, where $\boldsymbol{A}_{-i,-j}$ denotes the $n-1 \times n-1$ matrix formed by removing the ith row and jth column from $\boldsymbol{A}$.

Properties of the determinant include the following.

- $\det\{\boldsymbol{AB}\} = \det\{\boldsymbol{BA}\}$ if $\boldsymbol{A} \in \mathbb{C}^{m\times n}$ and $\boldsymbol{B} \in \mathbb{C}^{n\times m}$.
- $\det\{\boldsymbol{A}\} = (\det\{\boldsymbol{A}'\})^*$, where "′" denotes the **Hermitian transpose** or **conjugate transpose**.
- $\boldsymbol{A}$ is **singular** (not **invertible**) if and only if $\det\{\boldsymbol{A}\} = 0$.

s,mat,eig
26.1.2 Eigenvalues and eigenvectors (s,mat,eig)

If $\boldsymbol{A} \in \mathbb{C}^{n\times n}$, *i.e.*, $\boldsymbol{A}$ is a $n \times n$ (square) matrix, then we call a nonzero vector $\boldsymbol{x} \in \mathbb{C}^n$ an **eigenvector** of $\boldsymbol{A}$ (or a **right eigenvector** of $\boldsymbol{A}$) and $\lambda \in \mathbb{C}$ the corresponding **eigenvalue** when

$$\boldsymbol{Ax} = \lambda \boldsymbol{x}.$$

Properties of eigenvalues include the following.

- The eigenvalues of $\boldsymbol{A}$ are the n roots of the **characteristic polynomial** $\det\{\boldsymbol{A} - \lambda \boldsymbol{I}\}$.
- The set of eigenvalues of a matrix $\boldsymbol{A}$ is called its **spectrum** and is denoted $\operatorname{eig}\{\boldsymbol{A}\}$ or $\lambda(\boldsymbol{A})$.
- If $\boldsymbol{A} \in \mathbb{C}^{m\times n}$ and $\boldsymbol{B} \in \mathbb{C}^{n\times m}$, then the eigenvalues satisfy the following **commutative property**:

e,eig(AB)=eig(BA)
$$\operatorname{eig}\{\boldsymbol{AB}\} - \{0\} = \operatorname{eig}\{\boldsymbol{BA}\} - \{0\}, \tag{26.1.1}$$

  i.e., the *nonzero* elements of each set of eigenvalues are the same.
  Proof. If $\lambda \in \operatorname{eig}\{\boldsymbol{AB}\} - \{0\}$ then $\exists \boldsymbol{x} \neq \boldsymbol{0}$ such that $\boldsymbol{ABx} = \lambda \boldsymbol{x} \neq \boldsymbol{0}$. Clearly $\boldsymbol{y} \triangleq \boldsymbol{Bx} \neq \boldsymbol{0}$ here. Multiplying both sides by $\boldsymbol{B}$ yields $\boldsymbol{BABx} = \lambda \boldsymbol{Bx} \Longrightarrow \boldsymbol{BAy} = \lambda \boldsymbol{y}$, so $\lambda \in \operatorname{eig}\{\boldsymbol{BA}\} - \{0\}$.
- If $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ then $\det\{\boldsymbol{A}\} = \prod_{i=1}^{n} \lambda_i$.
- $\operatorname{eig}\{\boldsymbol{A}'\} = \{\lambda^* : \lambda \in \operatorname{eig}\{\boldsymbol{A}\}\}$
- If $\boldsymbol{A}$ is invertible then $\lambda \in \operatorname{eig}\{\boldsymbol{A}\}$ iff $1/\lambda \in \operatorname{eig}\{\boldsymbol{A}^{-1}\}$.
- For $k \in \mathbb{N}$: $\operatorname{eig}\{\boldsymbol{A}^k\} = \{\lambda^k : \lambda \in \operatorname{eig}\{\boldsymbol{A}\}\}$.
- The **spectral radius** of a matrix $\boldsymbol{A}$ is defined as the largest eigenvalue magnitude:

e,mat,eig,rho
$$\rho(\boldsymbol{A}) \triangleq \max_{\lambda \in \operatorname{eig}\{\boldsymbol{A}\}} |\lambda|. \tag{26.1.2}$$

  See also (26.1.5) for Hermitian symmetric matrices.
- A corollary of (26.1.1) is the following symmetry property:

e,mat,rho,sym
$$\rho(\boldsymbol{AB}) = \rho(\boldsymbol{BA}). \tag{26.1.3}$$

- **Geršgorin Theorem**. For a $n \times n$ matrix $\boldsymbol{A}$, define $r_i(\boldsymbol{A}) = \sum_{j\neq i} |a_{ij}|$, $i = 1, \ldots, n$. Then all the eigenvalues of $\boldsymbol{A}$ are located in the following union of disks in the complex plane:

e,mat,eig,gersgorin
$$\lambda_i(\boldsymbol{A}) \in \bigcup_{i=1}^{n} B_2(a_{ii}, r_i(\boldsymbol{A})), \tag{26.1.4}$$

  where $B_2(c, r) \triangleq \{z \in \mathbb{C} : |z - c| \leq r\}$ is a disk of radius r centered at c in the complex plane $\mathbb{C}$.
- If $\rho(\boldsymbol{A}) < 1$, then $\boldsymbol{I} - \boldsymbol{A}$ is invertible and [1, p. 312]:

$$[\boldsymbol{I} - \boldsymbol{A}]^{-1} = \sum_{k=0}^{\infty} \boldsymbol{A}^k.$$

- **Weyl's inequality** [wiki] is useful for bounding the (real) eigenvalues of sums of Hermitian matrices. In particular it is useful for analyzing how **matrix perturbations** affect eigenvalues.

s,mat,hermitian

26.1.3 Hermitian and symmetric matrices

A (square) matrix $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ is called **Hermitian** or **Hermitian symmetric** iff $a_{ji} = a_{ij}^*$, $i, j = 1, \ldots, n$, *i.e.*, $\boldsymbol{A} = \boldsymbol{A}'$.

- A Hermitian matrix is diagonalizable by a unitary matrix:

$$\boldsymbol{A} = \boldsymbol{U} \operatorname{diag}\{\lambda_i\} \boldsymbol{U}',$$

  where $\boldsymbol{U}^{-1} = \boldsymbol{U}'$.
- The eigenvalues of a Hermitian matrix are all real [1, p. 170].
- If $\boldsymbol{A}$ is Hermitian, then $\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}$ is real for all $\boldsymbol{x} \in \mathbb{C}^n$ [1, p. 170].
- If $\boldsymbol{A}$ is Hermitian, then (see Problem 26.1 for consideration whether this condition is necessary):

e,mat,rho,ratio

$$\rho(\boldsymbol{A}) = \max_{\boldsymbol{x} \neq \boldsymbol{0}} \frac{|\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}|}{\boldsymbol{x}'\boldsymbol{x}}. \tag{26.1.5}$$

- If $\boldsymbol{A}$ is Hermitian, then it follows from (26.1.5) that

$$\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x} = \operatorname{real}\{\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}\} \leq |\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}| \leq \rho(\boldsymbol{A})\, \boldsymbol{x}'\boldsymbol{x}, \quad \forall \boldsymbol{x} \in \mathbb{C}^n.$$

- If $\boldsymbol{A}$ is *real* and symmetric, then we have upper bound that is sometimes tighter [1, p. 34]:

$$\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x} \leq \left(\max_{\lambda \in \operatorname{eig}\{\boldsymbol{A}\}} \lambda \right) \|\boldsymbol{x}\|^2, \quad \forall \boldsymbol{x} \in \mathbb{R}^n.$$

s,mat,trace

26.1.4 Matrix trace (s,mat,trace)

The **trace** of a matrix $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ is defined to be the sum of its diagonal elements:

e,mat,trace

$$\operatorname{trace}\{\boldsymbol{A}\} \triangleq \sum_{i=1}^n a_{ii}. \tag{26.1.6}$$

Properties of trace include the following.

- For $\boldsymbol{A} \in \mathbb{C}^{m \times n}$ and $\boldsymbol{B} \in \mathbb{C}^{n \times m}$ the trace operator has the following commutative property:

e,mat,trace,AB,BA

$$\operatorname{trace}\{\boldsymbol{A}\boldsymbol{B}\} = \operatorname{trace}\{\boldsymbol{B}\boldsymbol{A}\}. \tag{26.1.7}$$

- If $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$, then

e,mat,trace,eig

$$\operatorname{trace}\{\boldsymbol{A}\} = \sum_{i=1}^n \lambda_i. \tag{26.1.8}$$

s,mat,mil

26.1.5 Inversion formulas (s,mat,mil)

The following **matrix inversion lemma** is easily verified [2]:

e,mat,mil

$$[\boldsymbol{A} + \boldsymbol{B}\boldsymbol{C}\boldsymbol{D}]^{-1} = \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{B}\left[\boldsymbol{D}\boldsymbol{A}^{-1}\boldsymbol{B} + \boldsymbol{C}^{-1}\right]^{-1}\boldsymbol{D}\boldsymbol{A}^{-1}, \tag{26.1.9}$$

assuming that $\boldsymbol{A}$ and $\boldsymbol{C}$ are invertible. It is also known as the **Sherman-Morrison-Woodbury** formula [3–5]. (See [6] for the case where $\boldsymbol{A}$ is singular but positive semidefinite.)

Multiplying on the right by $\boldsymbol{B}$ and simplifying yields the following useful related equality, sometimes called the **push-through identity**:

e,mat,push,through

$$[\boldsymbol{A} + \boldsymbol{B}\boldsymbol{C}\boldsymbol{D}]^{-1}\boldsymbol{B} = \boldsymbol{A}^{-1}\boldsymbol{B}\left[\boldsymbol{D}\boldsymbol{A}^{-1}\boldsymbol{B} + \boldsymbol{C}^{-1}\right]^{-1}\boldsymbol{C}^{-1}. \tag{26.1.10}$$

The following inverse of 2×2 block matrices holds if $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible:

e,mat,block,2x2,inv

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{D} \\ \boldsymbol{C} & \boldsymbol{B} \end{bmatrix}^{-1} = \begin{bmatrix} [\boldsymbol{A} - \boldsymbol{D}\boldsymbol{B}^{-1}\boldsymbol{C}]^{-1} & -\boldsymbol{A}^{-1}\boldsymbol{D}\boldsymbol{\Delta}^{-1} \\ -\boldsymbol{\Delta}^{-1}\boldsymbol{C}\boldsymbol{A}^{-1} & \boldsymbol{\Delta}^{-1} \end{bmatrix}, \tag{26.1.11}$$

where $\boldsymbol{\Delta} = \boldsymbol{B} - \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{D}$ is the **Schur complement** of $\boldsymbol{A}$. A generalization is available even when $\boldsymbol{B}$ is not invertible [7, p. 656].

s,mat,kron

26.1.6 Kronecker products (s,mat,kron)

The **Kronecker product** of a $L \times M$ matrix $\boldsymbol{A}$ with a $K \times N$ matrix $\boldsymbol{B}$ is the $KL \times MN$ matrix defined as follows:

$$\boldsymbol{A} \otimes \boldsymbol{B} = \begin{bmatrix} a_{11}\boldsymbol{B} & \dots & a_{1M}\boldsymbol{B} \\ \vdots & \vdots & \vdots \\ a_{L1}\boldsymbol{B} & \dots & a_{LM}\boldsymbol{B} \end{bmatrix}. \qquad (26.1.12)$$ e,mat,kronecker

Properties of the Kronecker product include the following (among many others [8]):

- $(\boldsymbol{A} \otimes \boldsymbol{B})(\boldsymbol{C} \otimes \boldsymbol{D}) = (\boldsymbol{AC} \otimes \boldsymbol{BD})$ if the dimensions are compatible.
- In particular $(\boldsymbol{A} \otimes \boldsymbol{B})(\boldsymbol{u} \otimes \boldsymbol{v}) = (\boldsymbol{A}\,\boldsymbol{u}) \otimes (\boldsymbol{B}\,\boldsymbol{v})$.
  If $\boldsymbol{A}$ and $\boldsymbol{B}$ are Toeplitz or circulant matrices, then this property is the matrix analog of the separability property of 2D convolution.
- $(\boldsymbol{A} \otimes \boldsymbol{B})^{-1} = \boldsymbol{A}^{-1} \otimes \boldsymbol{B}^{-1}$ if $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible.
- $(\boldsymbol{A} \otimes \boldsymbol{B})' = \boldsymbol{A}' \otimes \boldsymbol{B}'$
- $[\boldsymbol{A} \otimes \boldsymbol{B}]_{(l-1)K+k,\,(m-1)N+n} = a_{lm} b_{kn}, \quad l = 1, \dots, L,\ k = 1, \dots, K,\ n = 1, \dots, N,\ m = 1, \dots, M.$
- $\det\{\boldsymbol{A} \otimes \boldsymbol{B}\} = (\det\{\boldsymbol{A}\})^m (\det\{B\})^n$ if $\boldsymbol{A}$ is $n \times n$ and $\boldsymbol{B}$ is $m \times m$
- If $\boldsymbol{A}\,\boldsymbol{u} = \lambda\,\boldsymbol{u}$ and $\boldsymbol{B}\,\boldsymbol{v} = \eta\,\boldsymbol{v}$ then $\boldsymbol{u} \otimes \boldsymbol{v}$ is an eigenvector of $\boldsymbol{A} \otimes \boldsymbol{B}$ with eigenvalue $\lambda\eta$.
- If $\boldsymbol{A}$ has singular values $\{\sigma_i\}$ and $\boldsymbol{B}$ has singular values $\{\eta_j\}$, then $\boldsymbol{A} \otimes \boldsymbol{B}$ has singular values $\{\sigma_i \eta_j\}$.

In the context of imaging problems, Kronecker products are useful for representing **separable operations** such as convolution with a **separable kernel** and the 2D DFT. To see this, consider that the linear operation

$$v[k] = \sum_{n=0}^{N-1} b[k,n] u[n], \qquad k = 0, \dots, K-1,$$

can be represented by the matrix-vector product $\boldsymbol{v} = \boldsymbol{B}\,\boldsymbol{u}$, where $\boldsymbol{B}$ is the $K \times N$ matrix with elements $b[k,n]$. Similarly, the **separable 2D operation**

$$v[k,l] = \sum_{m=0}^{M-1} a[l,m] \left(\sum_{n=0}^{N-1} b[k,n] u[n,m] \right), \qquad k = 0, \dots, K-1, \qquad l = 0, \dots, L-1,$$

can be represented by the matrix-vector product $\boldsymbol{v} = \boldsymbol{C}\,\boldsymbol{u}$, where $\boldsymbol{C} = \boldsymbol{A} \otimes \boldsymbol{B}$ and $\boldsymbol{A}$ is the $L \times M$ matrix with elements $a[l,m]$. Choosing $b[k,n] = \mathrm{e}^{-\imath \frac{2\pi}{N} kn}$ and $a[l,m] = \mathrm{e}^{-\imath \frac{2\pi}{M} lm}$ shows that the matrix representation of the (N, M)-point 2D DFT is $\boldsymbol{Q}_{2\mathrm{D}} = \boldsymbol{Q}_M \otimes \boldsymbol{Q}_N$, where $\boldsymbol{Q}_N$ denotes the N-point 1D DFT matrix.

For a $M \times N$ matrix $\boldsymbol{G}$, let $\mathrm{lex}\{\boldsymbol{G}\}$ denote the column vector formed by **lexicographic** ordering of its elements, *i.e.*, $\mathrm{lex}\{\boldsymbol{G}\} = (g_{11}, \dots, g_{M1}, g_{12}, \dots g_{M2}, \dots, g_{1N}, \dots, g_{MN})$, sometimes denoted $\mathrm{vec}(\boldsymbol{G})$. Then one can show that

$$(\boldsymbol{A} \otimes \boldsymbol{B})\,\mathrm{lex}\{\boldsymbol{G}\} = \mathrm{lex}\left\{\boldsymbol{A}\boldsymbol{G}\boldsymbol{B}^T\right\}. \qquad (26.1.13)$$ e,mat,kron,lex

The **Kronecker sum** [wiki] of $n \times n$ square matrix $\boldsymbol{A}$ with $m \times m$ square matrix $\boldsymbol{B}$ is defined as

$$\boldsymbol{A} \oplus \boldsymbol{B} = \boldsymbol{A} \otimes \boldsymbol{I}_m + \boldsymbol{I}_n \otimes \boldsymbol{B}.$$

s,mat,pd

26.2 Positive-definite matrices (s,mat,pd)

There is no standard definition for a **positive definite** matrix that is not **Hermitian symmetric**. Therefore we restrict attention to matrices that are **Hermitian symmetric**, which suffices for imaging applications. Matrices that are **positive definite** or **positive semidefinite** often arise as covariance matrices for random vectors and as Hessian matrices for convex cost functions.

d,mat,pd

Definition 26.2.1 *For a $n \times n$ matrix $\boldsymbol{M}$ that is **Hermitian symmetric**, we say $\boldsymbol{M}$ is positive definite [1, p. 396] iff $\boldsymbol{x}'\boldsymbol{M}\boldsymbol{x} > 0$ for all $\boldsymbol{x} \neq \boldsymbol{0} \in \mathbb{C}^n$.*

t,mat,pd,equiv

Theorem 26.2.2 *The following conditions are equivalent [9].*

- *$\boldsymbol{M}$ is positive definite.*
- $\boldsymbol{M} \succ \boldsymbol{0}$
- *All eigenvalues of $\boldsymbol{M}$ are positive (and real).*
- *For all $i = 1, \dots, n$ $\boldsymbol{M}_{1:i,1:i} \succ \boldsymbol{0}$ where $\boldsymbol{M}_{1:i,1:i}$ denotes the ith **principal minor**—the upper left $i \times i$ corner of $\boldsymbol{M}$.*
- *There exists a Hermitian matrix $\boldsymbol{S} \succ \boldsymbol{0}$, called a **matrix square root** of $\boldsymbol{M}$, such that $\boldsymbol{M} = \boldsymbol{S}^2$. Often we write $\boldsymbol{S} = \boldsymbol{M}^{1/2}$.*
- *There exists a unique lower triangular matrix $\boldsymbol{L}$ with positive diagonal entries such that $\boldsymbol{M} = \boldsymbol{L}\boldsymbol{L}'$. This factorization is called the **Cholesky decomposition**.*

One can similarly define **positive semidefinite** matrices (also known as **nonnegative definite**), using $\geq$ and $\succeq$ instead of $>$ and $\succ$.

26.2.1 Properties of positive-definite matrices

- If $\boldsymbol{M} \succ \mathbf{0}$, then $\boldsymbol{M}$ is invertible and $\boldsymbol{M}^{-1} \succ \mathbf{0}$.
- If $\boldsymbol{M} \succ \mathbf{0}$ and $\alpha > 0$ is real, then $\alpha\boldsymbol{M} \succ \mathbf{0}$.
- If $\boldsymbol{A} \succ \mathbf{0}$ and $\boldsymbol{B} \succ \mathbf{0}$, then $\boldsymbol{A} + \boldsymbol{B} \succ \mathbf{0}$.
- If $\boldsymbol{A} \succ \mathbf{0}$ and $\boldsymbol{B} \succ \mathbf{0}$, then $\boldsymbol{A} \otimes \boldsymbol{B} \succ \mathbf{0}$.
- If $\boldsymbol{A} \succ \mathbf{0}$ then $a_{ii} > 0$ and is real.

s,mat,pd,order

26.2.2 Partial order

The notation $\boldsymbol{B} \succ \boldsymbol{A}$ is shorthand for saying $\boldsymbol{B} - \boldsymbol{A}$ is positive definite. This is a strict **partial order**, particularly because it satisfies **transitivity**: $\boldsymbol{C} \succ \boldsymbol{B}$ and $\boldsymbol{B} \succ \boldsymbol{A}$ implies $\boldsymbol{C} \succ \boldsymbol{A}$. Likewise, $\boldsymbol{B} \succeq \boldsymbol{A}$ is shorthand for saying $\boldsymbol{B} - \boldsymbol{A}$ is positive semidefinite, and $\succeq$ is also **transitive**. This **partial order** of matrices is called **Loewner order** [wiki]. These inequalities are important for designing **majorizers**. The following results are useful properties of these inequalities.

lemma,mat,ba

Lemma 26.2.3 *If $\boldsymbol{B} \succeq \boldsymbol{A}$, then $\boldsymbol{C}'\boldsymbol{B}\boldsymbol{C} \succeq \boldsymbol{C}'\boldsymbol{A}\boldsymbol{C}$ for any matrix $\boldsymbol{C}$ of suitable dimensions. (See Problem 26.5.)*

t,mat,spd,inv

Theorem 26.2.4 *If $\boldsymbol{B} \succeq \boldsymbol{A} \succ \mathbf{0}$, then $\boldsymbol{A}^{-1} \succeq \boldsymbol{B}^{-1}$. (See Problem 26.6.) In words: matrix inversion preserves the natural (partial) ordering of symmetric positive definite matrices.*

26.2.3 Diagonal dominance

- A $n \times n$ matrix $\boldsymbol{A}$ is called (weakly) **diagonally dominant** iff $|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$, $i = 1, \ldots, n$. It is called **strictly diagonally dominant** $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$, $i = 1, \ldots, n$.
- For Hermitian $\boldsymbol{A}$, if $\boldsymbol{A}$ is strictly diagonally dominant and $a_{ii} > 0$, $i = 1, \ldots, n$, then $\boldsymbol{A} \succ \mathbf{0}$ and in particular $\boldsymbol{A}$ is invertible [1, Cor. 7.2.3].
- If $\boldsymbol{A}$ is **strictly diagonally dominant**, then $\boldsymbol{A}$ is **invertible** [1, Cor. 5.6.17].
- If $\boldsymbol{A}$ is strictly diagonally dominant and $\boldsymbol{D} = \operatorname{diag}\{a_{ii}\}$, then $\rho(\boldsymbol{I} - \boldsymbol{D}^{-1}\boldsymbol{A}) < 1$ [1, p. 352, Ex. 6.1.9].

lemma,mat,dd

Lemma 26.2.5 *If $\boldsymbol{H} \in \mathbb{C}^{n \times n}$ is Hermitian and diagonally dominant and $h_{ii} \geq 0$, $i = 1, \ldots, n$, then $\boldsymbol{H} \succeq \mathbf{0}$.*
Proof:
By §26.1.3, $\boldsymbol{H}$ has real eigenvalues that, by the **Geršgorin Theorem** (26.1.4), satisfy $\lambda(\boldsymbol{H}) \geq h_{ii} - \sum_{j \neq i} |h_{ij}|$, and that latter quantitity is nonnegative by the assumed diagonal dominance. Thus by Theorem 26.2.2, $\boldsymbol{H}$ is positive semidefinite. □

26.2.4 Diagonal majorizers

We now use Lemma 26.2.5 to establish some **diagonal majorizers**.

c,mat,diag,B1

Corollary 26.2.6 *If $\boldsymbol{B}$ is a Hermitian matrix, then $\boldsymbol{B} \preceq \boldsymbol{D} \triangleq \operatorname{diag}\{|\boldsymbol{B}|\,\mathbf{1}\}$ where $|\boldsymbol{B}|$ denotes the matrix consisting of the absolute values of the elements of $\boldsymbol{B}$.*
Proof:
Let $\boldsymbol{H} \triangleq \boldsymbol{D} - \boldsymbol{B} = \operatorname{diag}\{|\boldsymbol{B}|\,\mathbf{1}\} - \boldsymbol{B}$. Then $h_{ii} = \sum_j |b_{ij}| - b_{ii} = \left(\sum_{j \neq i} |b_{ij}|\right) + (|b_{ii}| - b_{ii}) \geq \sum_{j \neq i} |b_{ij}|$ because $|b| - b \geq 0$. Also for $j \neq i$: $h_{ij} = -b_{ij}$ so $\sum_{j \neq i} |h_{ij}| = \sum_{j \neq i} |b_{ij}| \leq h_{ii}$. Thus $\boldsymbol{H}$ is diagonally dominant so $\boldsymbol{D} - \boldsymbol{B} \succeq \mathbf{0}$. □

c,mat,dd

Corollary 26.2.7 *If $\mathsf{F} = \boldsymbol{A}'\boldsymbol{W}\boldsymbol{A}$ where $\boldsymbol{W} = \operatorname{diag}\{w_i\}$ with $w_i \geq 0$, then $\mathsf{F} \preceq \boldsymbol{D} = \operatorname{diag}\{d_j\}$ where $d_j \triangleq \sum_{i=1}^{n_{\mathrm{d}}} w_i |a_{ij}|^2 / \pi_{ij}$ and $\pi_{ij} = |a_{ij}| / \sum_k |a_{ik}|$ (cf. (12.5.10)), i.e., $d_j = \sum_{i=1}^{n_{\mathrm{d}}} |a_{ij}| \, w_i \left(\sum_{k=1}^{n_{\mathrm{p}}} |a_{ik}|\right)$.*
Proof:
Define the Hermitian matrix $\boldsymbol{H} = \boldsymbol{D} - \mathsf{F}$ for which $h_{jj} = d_j - f_{jj} = \sum_{i=1}^{n_{\mathrm{d}}} w_i |a_{ij}|^2 / \pi_{ij} - \sum_{i=1}^{n_{\mathrm{d}}} w_i |a_{ij}|^2 \geq 0$. So by Lemma 26.2.5, it suffices to show that $\boldsymbol{H}$ is diagonally dominant:

$$\begin{aligned}
h_{jj} - \sum_{k \neq j} |h_{jk}| = d_j - f_{jj} - \sum_{k \neq j} |f_{jk}| &= \sum_{i=1}^{n_{\mathrm{d}}} w_i |a_{ij}|^2 / \pi_{ij} - \sum_{i=1}^{n_{\mathrm{d}}} w_i |a_{ij}|^2 - \sum_{k \neq j} \left| \sum_{i=1}^{n_{\mathrm{d}}} w_i a_{ik}^* a_{ij} \right| \\
&\geq \sum_{i=1}^{n_{\mathrm{d}}} w_i |a_{ij}|^2 / \pi_{ij} - \sum_{i=1}^{n_{\mathrm{d}}} w_i |a_{ij}|^2 - \sum_{k \neq j} \sum_{i=1}^{n_{\mathrm{d}}} w_i |a_{ik}| \, |a_{ij}| \\
&= \sum_{i=1}^{n_{\mathrm{d}}} w_i |a_{ij}|^2 / \pi_{ij} - \sum_{i=1}^{n_{\mathrm{d}}} w_i |a_{ij}| \sum_k |a_{ik}| = 0.
\end{aligned}$$

□

Another way of writing the diagonal majorizer in Corollary 26.2.7 is

e,mat,pd,AWA,D

$$\boldsymbol{A}'\boldsymbol{W}\boldsymbol{A} \preceq \boldsymbol{D} \triangleq \mathsf{diag}\{|\boldsymbol{A}'|\,\boldsymbol{W}\,|\boldsymbol{A}|\,\mathbf{1}\}\,. \tag{26.2.1}$$

When $\boldsymbol{A}$ (and $\boldsymbol{W}$) have nonnegative elements (*e.g.*, in CT, PET, SPECT), an alternative simpler proof is to use Corollary 26.2.6 directly with $\boldsymbol{B} = \boldsymbol{A}'\boldsymbol{W}\boldsymbol{A}$.

The following theorem generalizes Corollary 26.2.7 (*cf.* (12.5.14)). See Problem 26.12.

t,mat,dd

Theorem 26.2.8 *For $\boldsymbol{B} \in \mathbb{C}^{n_\mathrm{d} \times n_\mathrm{p}}$ and any $\pi_{ij} \geq 0$ and $\sum_{j=1}^{n_\mathrm{p}} \pi_{ij} = 1$ for which $\pi_{ij} = 0$ only if $b_{ij} = 0$:*

e,mat,dd

$$\boldsymbol{B}'\boldsymbol{B} \preceq \boldsymbol{D} \triangleq \mathsf{diag}\{d_j\}, \quad d_j \triangleq \sum_{i=1}^{n_\mathrm{d}} |b_{ij}|^2 / \pi_{ij}. \tag{26.2.2}$$

Proof:
$\boldsymbol{x}'\boldsymbol{B}'\boldsymbol{B}\boldsymbol{x} = \sum_{i=1}^{n_\mathrm{d}} \left|\sum_{j=1}^{n_\mathrm{p}} b_{ij} x_j\right|^2 = \sum_{i=1}^{n_\mathrm{d}} \left|\sum_{j=1}^{n_\mathrm{p}} \pi_{ij} \left(\frac{b_{ij}}{\pi_{ij}} x_j\right)\right|^2 \leq \sum_{i=1}^{n_\mathrm{d}} \sum_{j=1}^{n_\mathrm{p}} \pi_{ij} \left|\frac{b_{ij}}{\pi_{ij}} x_j\right|^2 = \sum_{j=1}^{n_\mathrm{p}} |x_j|^2 d_j = \boldsymbol{x}'\boldsymbol{D}\boldsymbol{x}$, using the convexity of $|\cdot|^2$. □

c,mat,dd,equal

Corollary 26.2.9 *For $\boldsymbol{B} \in \mathbb{C}^{n_\mathrm{d} \times n_\mathrm{p}}$:*

$$\boldsymbol{B}'\boldsymbol{B} \preceq \boldsymbol{D} = \alpha \boldsymbol{I}, \quad \alpha = \sum_{i=1}^{n_\mathrm{d}} \sum_{j=1}^{n_\mathrm{p}} |b_{ij}|^2 = \|\boldsymbol{B}\|_{\mathrm{Frob}}^2.$$

Proof:
In Theorem 26.2.8 take $\pi_{ij} = |b_{ij}|^2 / \sum_{k=1}^{n_\mathrm{p}} |b_{ik}|^2$. □

26.2.5 Simultaneous diagonalization

If $\boldsymbol{S}$ is symmetric and $\boldsymbol{A}$ is symmetric positive definite and of the same size, then there exists an invertible matrix $\boldsymbol{B}$ that diagonalizes both $\boldsymbol{S}$ and $\boldsymbol{A}$, *i.e.*, $\boldsymbol{B}'\boldsymbol{S}\boldsymbol{B} = \boldsymbol{D}$ and $\boldsymbol{B}'\boldsymbol{A}\boldsymbol{B} = \boldsymbol{I}$ where $\boldsymbol{D}$ is diagonal [wiki] [1, p. 218]. However, $\boldsymbol{B}$ is not orthogonal in general.

s,mat,vnorm

26.3 Vector norms (s,mat,vnorm)

The material in this section is derived largely from [1, Ch. 5] [10].

vnorm

Definition 26.3.1 *Let $\mathcal{V}$ be a vector space over a field such as $\mathbb{R}$ or $\mathbb{C}$. A function $\|\cdot\| : \mathcal{V} \to \mathbb{R}$ is a* ***vector norm*** *iff for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$:*

- $\|\boldsymbol{x}\| \geq 0$ *(nonnegative)*
- $\|\boldsymbol{x}\| = 0$ *iff* $\boldsymbol{x} = \mathbf{0}$ *(positive)*
- $\|c\boldsymbol{x}\| = |c|\,\|\boldsymbol{x}\|$ *for all scalars c in the field (homogeneous)*
- $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$ ***(triangle inequality)***

26.3.1 Examples of vector norms

- For $1 \leq p < \infty$, the **ℓ_p norm** is

e,mat,vnorm,lp

$$\|\boldsymbol{x}\|_p \triangleq \left(\sum_i |x_i|^p\right)^{1/p}. \tag{26.3.1}$$

- The **max norm** or **infinity norm** or ℓ_∞ norm is

e,mat,vnorm,linf

$$\|\boldsymbol{x}\|_\infty \triangleq \sup\{|x_1|, |x_2|, \ldots\}, \tag{26.3.2}$$

where sup denotes the **supremum** (least upper bound) of a set. One can show [10, Prob. 2.12] that

e,mat,vnorm,linf,limit

$$\|\boldsymbol{x}\|_\infty = \lim_{p\to\infty} \|\boldsymbol{x}\|_p\,. \tag{26.3.3}$$

- For quantifying sparsity, it is useful to note that

e,mat,vnorm,l0

$$\lim_{p\to 0} \|\boldsymbol{x}\|_p^p = \sum_i \mathbb{I}_{\{x_i \neq 0\}} \triangleq \|\boldsymbol{x}\|_0\,. \tag{26.3.4}$$

However, the "0-norm" $\|\boldsymbol{x}\|_0$ is *not* a vector norm because it does not satisfy all the conditions of Definition 26.3.1. The proper name for $\|\boldsymbol{x}\|_0$ is **counting measure**.

26.3.2 Inequalities

To establish that (26.3.1) and (26.3.2) are vector norms, that hardest part is proving the **triangle inequality**. The proofs use the following inequalities.
The **Hölder inequality** [10, p. 29]
If $p \in [1, \infty]$ and $q \in [1, \infty]$ satisfy $1/p + 1/q = 1$, and if $\boldsymbol{x} = (x_1, x_2, \ldots) \in \ell_p$ and $\boldsymbol{y} = (y_1, y_2, \ldots) \in \ell_q$, then

e,mat,holder

$$\sum_i |x_i y_i| \leq \|\boldsymbol{x}\|_p \|\boldsymbol{y}\|_q . \tag{26.3.5}$$

Equality holds iff either $\boldsymbol{x}$ or $\boldsymbol{y}$ equal $\mathbf{0}$, or both $\boldsymbol{x}$ and $\boldsymbol{y}$ are nonzero and $\left(|x_i|/\|\boldsymbol{x}\|_p\right)^{1/q} = \left(|y_i|/\|\boldsymbol{y}\|_q\right)^{1/p}$, $\forall i$.
The **Minkowski inequality** [10, p. 31]
If $\boldsymbol{x}$ and $\boldsymbol{y}$ are in ℓ_p, for $1 \leq p \leq \infty$,

e,mat,minkoswki

$$\left(\sum_i |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_i |x_i|^p\right)^{1/p} + \left(\sum_i |y_i|^p\right)^{1/p} . \tag{26.3.6}$$

For $1 \leq p < \infty$, equality holds iff $\boldsymbol{x}$ and $\boldsymbol{y}$ are linearly dependent.

26.3.3 Properties

e,mat,vnorm,wtd

- If $\|\cdot\|$ is a vector norm then
$$\|\boldsymbol{x}\|_{\boldsymbol{T}} \triangleq \|\boldsymbol{T}\boldsymbol{x}\| \tag{26.3.7}$$
is also a vector norm for any *nonsingular*[1] matrix $\boldsymbol{T}$ (with appropriate dimensions).
- Let $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ be any two vector norms on a *finite-dimensional* space. Then there exist finite positive constants C_m and C_M such that (see Problem 26.3):

e,mat,vnorm,equiv

$$C_m \|\cdot\|_\alpha \leq \|\cdot\|_\beta \leq C_M \|\cdot\|_\alpha . \tag{26.3.8}$$
Thus, convergence of $\{\boldsymbol{x}^{(n)}\}$ to a limit $\boldsymbol{x}$ with respect to some vector norm implies convergence of $\{\boldsymbol{x}^{(n)}\}$ to that limit with respect to any vector norm.
- For any vector norm:
$$\left|\|\boldsymbol{x}\| - \|\boldsymbol{y}\|\right| \leq \|\boldsymbol{x} - \boldsymbol{y}\| .$$
- All vector norms are **convex** functions:
$$\|\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{z}\| \leq \alpha \|\boldsymbol{x}\| + (1-\alpha) \|\boldsymbol{z}\| , \ \forall \alpha \in [0, 1].$$
This is easy to prove using the **triangle inequality** and the homogeneity property in Definition 26.3.1.
- The quadratic function $f(\boldsymbol{x}) \triangleq \|\boldsymbol{x}\|_2^2$ is **strictly convex** because its Hessian is positive definite. However, $f(\boldsymbol{x}) \triangleq \|\boldsymbol{x}\|_2$ is not strictly convex.
- For $p > 1$, $f(\boldsymbol{x}) \triangleq \|\boldsymbol{x}\|_p^p$ is strictly convex on $\mathbb{C}^n$ and ℓ_p. See Problem 26.11 and Example 27.9.10.

s,mat,inprod

26.4 Inner products (s,mat,inprod)

For a vector space $\mathcal{V}$ over the field $\mathbb{C}$, an **inner product** operation $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$, must satisfy the following axioms $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$, $\alpha \in \mathbb{C}$.

- $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{y}, \boldsymbol{x} \rangle^*$ (**Hermitian symmetry**), where * denotes complex conjugate.
- $\langle \boldsymbol{x} + \boldsymbol{y}, \boldsymbol{z} \rangle = \langle \boldsymbol{x}, \boldsymbol{z} \rangle + \langle \boldsymbol{y}, \boldsymbol{z} \rangle$ (**additivity**)
- $\langle \alpha \boldsymbol{x}, \boldsymbol{y} \rangle = \alpha \langle \boldsymbol{x}, \boldsymbol{y} \rangle$ (**scaling**)
- $\langle \boldsymbol{x}, \boldsymbol{x} \rangle \geq 0$ and $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ iff $\boldsymbol{x} = \mathbf{0}$. (**positive definite**)

26.4.1 Examples

x,mat,inprod,w,l11

Example 26.4.1 *For the space of (suitably regular) functions on $[a, b]$, a valid inner product is*

$$\langle f, g \rangle = \int_a^b w(t) f(t) g^*(t) \, \mathrm{d}t,$$

where $w(t) > 0$, $\forall t$ is some (real) weighting function. The usual choice is $w = 1$.

x,mat,inprod,2

Example 26.4.2 *In Euclidean space, $\mathbb{C}^n$, the usual inner product (aka "**dot product**") is*

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i=1}^n x_i y_i^*, \text{ where } \boldsymbol{x} = (x_1, \ldots, x_n) \text{ and } \boldsymbol{y} = (y_1, \ldots, y_n).$$

[1] We also use the notation $\|\boldsymbol{x}\|_{\boldsymbol{T}}$ even when $\boldsymbol{T}$ might be singular, in which case the resulting functional is a **semi-norm** rather than a norm, because the positivity condition in Definition 26.3.1 no longer holds.

26.4.2 Properties

- Bilinearity:

$$\left\langle \sum_i \alpha_i \boldsymbol{x}_i,\, \sum_j \beta_j \boldsymbol{y}_j \right\rangle = \sum_i \sum_j \alpha_i \beta_j^* \langle \boldsymbol{x}_i,\, \boldsymbol{y}_j \rangle .$$

- The following **induced norm** is a valid vector norm:

e,mat,vnorm,induced
$$\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x},\, \boldsymbol{x} \rangle}. \tag{26.4.1}$$

- A vector norm satisfies the **parallelogram identity**:

$$\frac{1}{2}\left(\|\boldsymbol{x}+\boldsymbol{y}\|^2 + \|\boldsymbol{x}-\boldsymbol{y}\|^2\right) = \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2$$

iff it is induced by an inner product via (26.4.1). The required inner product is

$$\begin{aligned}\langle \boldsymbol{x},\, \boldsymbol{y} \rangle &\triangleq \frac{1}{4}\left(\|\boldsymbol{x}+\boldsymbol{y}\|^2 - \|\boldsymbol{x}-\boldsymbol{y}\|^2 + \imath\,\|\boldsymbol{x}+\imath\boldsymbol{y}\|^2 - \imath\,\|\boldsymbol{x}-\imath\boldsymbol{y}\|^2\right)\\ &= \frac{\|\boldsymbol{x}+\boldsymbol{y}\|^2 - \|\boldsymbol{x}\|^2 - \|\boldsymbol{y}\|^2}{2} + \imath\,\frac{\|\boldsymbol{x}+\imath\boldsymbol{y}\|^2 - \|\boldsymbol{x}\|^2 - \|\boldsymbol{y}\|^2}{2}.\end{aligned}$$

- The **Schwarz inequality** or **Cauchy-Schwarz** inequality states:

e,mat,schwarz
$$|\langle \boldsymbol{x},\, \boldsymbol{y} \rangle| \le \|\boldsymbol{x}\|\,\|\boldsymbol{y}\| = \sqrt{\langle \boldsymbol{x},\, \boldsymbol{x} \rangle}\sqrt{\langle \boldsymbol{y},\, \boldsymbol{y} \rangle}, \tag{26.4.2}$$

for a norm $\|\cdot\|$ induced by an inner product $\langle \cdot,\, \cdot \rangle$ via (26.4.1), with equality iff $\boldsymbol{x}$ and $\boldsymbol{y}$ are linearly dependent.

s,mat,mnorm
26.5 Matrix norms (s,mat,mnorm)

The set $\mathbb{C}^{m\times n}$ of $m \times n$ matrices over $\mathbb{C}$ is a **vector space** and one can define norms on this space that satisfy the properties in Definition 26.3.1, as follows [1, Ch. 5.6].

mat,norm
Definition 26.5.1 *A function $\|\cdot\| : \mathbb{C}^{m\times n} \to \mathbb{R}$ is a (vector)* ***norm*** *for $\mathbb{C}^{m\times n}$ iff it satisfies the following properties for all $\boldsymbol{A} \in \mathbb{C}^{m\times n}$ and $\boldsymbol{B} \in \mathbb{C}^{m\times n}$.*
- $\|\boldsymbol{A}\| \ge 0$ *(nonnegative)*
- $\|\boldsymbol{A}\| = 0$ *iff* $\boldsymbol{A} = \boldsymbol{0}$ *(positive)*
- $\|c\boldsymbol{A}\| = |c|\,\|\boldsymbol{A}\|$ *for all* $c \in \mathbb{C}$ *(homogeneous)*
- $\|\boldsymbol{A}+\boldsymbol{B}\| \le \|\boldsymbol{A}\| + \|\boldsymbol{B}\|$ *(**triangle inequality**)*

In addition, many, but not all, norms for the space $\mathbb{C}^{n\times n}$ of **square matrices** are **submultiplicative**, meaning that they satisfy the following inequality:

e,mat,mnorm,submult
$$|||\boldsymbol{A}\boldsymbol{B}||| \le |||\boldsymbol{A}|||\,|||\boldsymbol{B}|||, \qquad \forall \boldsymbol{A}, \boldsymbol{B} \in \mathbb{C}^{n\times n}. \tag{26.5.1}$$

We use the notation $|||\cdot|||$ to distinguish such **matrix norms** on $\mathbb{C}^{n\times n}$ from the ordinary vector norms $\|\cdot\|$ on $\mathbb{C}^{m\times n}$ that need not satisfy this extra condition.

For example, the **max norm** on $\mathbb{C}^{m\times n}$ is the element-wise maximum: $\|\boldsymbol{A}\|_{\max} = \max_{i,j} |a_{ij}|$. This is a (vector) norm on $\mathbb{C}^{m\times n}$ but does not satisfy the **submultiplicative** condition (26.5.1). Most of the norms of interest in imaging problems are submultiplicative, so these matrix norms are our primary focus hereafter.

26.5.1 Induced norms

If $\|\cdot\|$ is a vector norm that is suitable for both $\mathbb{C}^n$ and $\mathbb{C}^m$, then

e,mat,mnorm,from,vnorm
$$|||\boldsymbol{A}||| \triangleq \max_{\|\boldsymbol{x}\|=1} \|\boldsymbol{A}\boldsymbol{x}\| = \max_{\boldsymbol{x}\neq\boldsymbol{0}} \frac{\|\boldsymbol{A}\boldsymbol{x}\|}{\|\boldsymbol{x}\|} \tag{26.5.2}$$

is a matrix norm for $\mathbb{C}^{m\times n}$, and

e,mat,norm,Ax
$$\|\boldsymbol{A}\boldsymbol{x}\| \le |||\boldsymbol{A}|||\,\|\boldsymbol{x}\|, \qquad \forall \boldsymbol{x} \in \mathbb{C}^n. \tag{26.5.3}$$

In such cases, we say the matrix norm $|||\cdot|||$ is **induced** by the vector norm $\|\cdot\|$. Furthermore, the **submultiplicative** property (26.5.1) holds not only for square matrices, but also whenever the number of columns of $\boldsymbol{A}$ matches the number of rows of $\boldsymbol{B}$.

x,mat,norm,induce
Example 26.5.2 *The most important matrix norms are induced by the vector norm $\|\cdot\|_p$.*

- *The* ***spectral norm*** $\|\cdot\|_2$*, often denoted simply* $\|\cdot\|$*, is defined on* $\mathbb{C}^{m\times n}$ *by*

$$\|A\|_2 \triangleq \max\left\{\sqrt{\lambda} : \lambda \in \mathsf{eig}\{A'A\}\right\},$$

  which is real and nonnegative. This is the matrix norm induced by the Euclidean vector norm $\|\cdot\|_2$, i.e.,

$$\|A\|_2 = \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}.$$

- *The* ***maximum row sum matrix norm*** *is defined on* $\mathbb{C}^{m\times n}$ *by*

$$\|A\|_\infty \triangleq \max_{1\leq i\leq m} \sum_{j=1}^n |a_{ij}|\,. \tag{26.5.4}$$ e,mnorm,max

  It is induced by the ℓ_∞ *vector norm.*

- *The* ***maximum column sum matrix norm*** *is defined on* $\mathbb{C}^{m\times n}$ *by*

$$\|A\|_1 \triangleq \max_{x\neq 0} \frac{\|Ax\|_1}{\|x\|_1} = \max_{1\leq j\leq n} \sum_{i=1}^m |a_{ij}|\,. \tag{26.5.5}$$ e,mnorm,1

  It is induced by the ℓ_1 *vector norm.*

26.5.2 Other examples

- The **Frobenius norm** is defined on $\mathbb{C}^{m\times n}$ by

$$\|A\|_{\text{Frob}} \triangleq \sqrt{\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2} = \sqrt{\mathsf{trace}\{A'A\}}, \tag{26.5.6}$$ e,mnorm,frobenius

  and is also called **Schur norm** and **Hilbert-Schmidt norm**. It is often the easiest norm to compute.
  This norm is invariant to unitary transformations [11, p. 442], because of the trace property (26.1.7).
  This is not an induced norm [12], but nevertheless it is **compatible** with the Euclidean vector norm because

e,mat,mnorm,frob,ineq
$$\|Ax\|_2 \leq \|A\|_{\text{Frob}} \|x\|_2\,. \tag{26.5.7}$$

  However, this is not a tight upper bound in general. By combining (26.5.7) with the definition of matrix multiplication, one can show easily that the Frobenius norm is **submultiplicative** [1, p. 291].

26.5.3 Properties

- All matrix norms are **equivalent** in the sense given for vectors in (26.3.8).
- Two vector norms can induce the same matrix norm if and only if one of the vector norms is a constant scalar multiple of the other.
- No induced matrix norm can be uniformly dominated by another induced matrix norm:

$$\|A\|_\alpha \leq \|A\|_\beta, \qquad \forall A \in \mathbb{C}^{m\times n}$$

  if and only if

$$\|A\|_\alpha = \|A\|_\beta.$$

- A **unitarily invariant matrix norm** satisfies $\|A\| = \|UAV\|$ for all $A \in \mathbb{C}^{m\times n}$ and all unitary matrices $U \in \mathbb{C}^{m\times m}$, $V \in \mathbb{C}^{n\times n}$.
  The spectral norm $\|\cdot\|_2$ is the only matrix norm that is both induced and unitarily invariant.
- A **self adjoint matrix norm** satisfies $\|A'\| = \|A\|$.
  The spectral norm $\|\cdot\|_2$ is the only matrix norm that is both induced and self adjoint.
- If $A \in \mathbb{C}^{m\times n}$ has **rank** $r \leq \min(m,n)$, then [13, p. 57]:

$$\|A\|_2 \leq \|A\|_{\text{Frob}} \leq \sqrt{r}\|A\|_2. \tag{26.5.8}$$ e,mat,mnorm,2,F

- By [13, p. 58],

$$\|A\|_2 \leq \sqrt{\|A\|_1 \|A\|_\infty}.$$

- Using the **spectral radius** $\rho(\cdot)$ defined in (26.1.2):

$$\|A\|_2 = \sqrt{\rho(A'A)}. \tag{26.5.9}$$ e,mat,mnorm,2,rho

26.5.4 Properties for square matrices

- For $k \in \mathbb{N}$
$$\|\boldsymbol{A}^k\| \leq \|\boldsymbol{A}\|^k.$$
- If $\|\cdot\|$ is a matrix norm on $\mathbb{C}^{n\times n}$, and if $\boldsymbol{T} \in \mathbb{C}^{n\times n}$ is invertible, then the following is a matrix norm:
$$\|\boldsymbol{A}\|_{\boldsymbol{T}} \triangleq \|\boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}\|.$$
- If $\|\boldsymbol{A}\| < 1$ for some matrix norm, then $\lim_{k\to\infty} \boldsymbol{A}^k = \boldsymbol{0}$.

26.5.4.1 Invertibility

- If $\boldsymbol{A}$ is invertible then
$$\|\boldsymbol{A}^{-1}\| \geq \|\boldsymbol{I}\| / \|\boldsymbol{A}\|.$$
- If $\|\boldsymbol{A}\| < 1$ for any matrix norm, then $\boldsymbol{I} - \boldsymbol{A}$ is invertible and
$$[\boldsymbol{I} - \boldsymbol{A}]^{-1} = \sum_{k=0}^{\infty} \boldsymbol{A}^k.$$

26.5.4.2 Relationship with spectral radius

- If $\boldsymbol{A}$ is Hermitian symmetric, then the relation (26.5.9) specializes to
$$\|\boldsymbol{A}\|_2 = \rho(\boldsymbol{A}).$$
- If $\|\cdot\|$ is any matrix norm on $\mathbb{C}^{n\times n}$ and if $\boldsymbol{A} \in \mathbb{C}^{n\times n}$, then

e,rho<norm
$$\rho(\boldsymbol{A}) \leq \|\boldsymbol{A}\|. \tag{26.5.10}$$
- Given $\boldsymbol{A} \in \mathbb{C}^{n\times n}$, the spectral radius is the smallest matrix norm:
$$\rho(\boldsymbol{A}) = \inf\left\{\|\boldsymbol{A}\| : \|\cdot\| \text{ is a matrix norm}\right\}.$$
- If $\boldsymbol{A} \in \mathbb{C}^{n\times n}$, then $\lim_{k\to\infty} \boldsymbol{A}^k = \boldsymbol{0}$ if and only if $\rho(\boldsymbol{A}) < 1$.
- For any matrix norm $\|\cdot\|$:

e,rho,lim,mnorm
$$\rho(\boldsymbol{A}) = \lim_{k\to\infty} \|\boldsymbol{A}^k\|^{1/k}. \tag{26.5.11}$$
- If $\boldsymbol{A} \in \mathbb{C}^{n\times n}$, then the series $\sum_{k=0}^{\infty} \alpha_k \boldsymbol{A}^k$ converges if there is a matrix norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$ such that the numerical series $\sum_{k=0}^{\infty} |\alpha_k| \, \|\boldsymbol{A}\|^k$ converges.

t,mat,mnorm2,mleq
- **Theorem 26.5.3** *If $\boldsymbol{A}$ is symmetric* ***positive semidefinite****, i.e., $\boldsymbol{A} \succeq \boldsymbol{0}$, then (Problem 26.4)*
$$\|\boldsymbol{A}\|_2 \leq 1 \iff \boldsymbol{A} \preceq \boldsymbol{I}.$$

s,mat,svd

26.6 Singular values (s,mat,svd)

Eigenvalues are defined only for square matrices. For any rectangular matrix $\boldsymbol{A} \in \mathbb{C}^{n\times m}$, the **singular values**, denoted $\sigma_1, \ldots, \sigma_n$ are the square roots of the eigenvalues of the $n \times n$ square matrix $\boldsymbol{A}'\boldsymbol{A}$. (Because $\boldsymbol{A}'\boldsymbol{A}$ is positive semidefinite, its eigenvalues are all real and nonnegative.) Written concisely:

e,mat,svd,sig
$$\sigma_i(\boldsymbol{A}) = \sqrt{\lambda_i(\boldsymbol{A}'\boldsymbol{A})}. \tag{26.6.1}$$

If $\boldsymbol{A}$ is Hermitian positive definite, then $\sigma_i = \lambda_i$.

Usually the singular values are ordered from largest to smallest, *i.e.*, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$. With this order, the rth singular value is related to a **low-rank approximation** to $\boldsymbol{A}$ as follows [wiki]:
$$\sigma_r(\boldsymbol{A}) = \inf\left\{\|\boldsymbol{A} - \boldsymbol{L}\|_2 : \boldsymbol{L} \in \mathbb{C}^{n\times m} \text{ has rank } < r\right\}.$$

s,mat,cond
26.7 Condition numbers and linear systems (s,mat,cond)

The **condition number** [wiki] for matrix inversion with respect to matrix norm $\|\cdot\|$ is defined:

e,mat,cond,mnorm
$$\kappa(\boldsymbol{A}) \triangleq \begin{cases} \|\boldsymbol{A}\| \|\boldsymbol{A}^{-1}\|, & \boldsymbol{A} \text{ invertible} \\ \infty, & \boldsymbol{A} \text{ singular.} \end{cases} \tag{26.7.1}$$

In particular, for the spectral norm $\|\cdot\|_2$ we have

e,mat,cond
$$\kappa(\boldsymbol{A}) = \frac{\sigma_{\max}(\boldsymbol{A})}{\sigma_{\min}(\boldsymbol{A})}, \tag{26.7.2}$$

where $\sigma_{\max}$ and $\sigma_{\min}$ denote the maximum and minimum **singular values** of $\boldsymbol{A}$. A concept of **condition number** has also been developed for problems with **constraints** [14]. Condition numbers are submultiplicative:

$$\kappa(\boldsymbol{A}\boldsymbol{B}) \leq \kappa(\boldsymbol{A})\kappa(\boldsymbol{B}).$$

Suppose we want to solve $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, but the right-hand side is perturbed (*e.g.*, by noise or numerical error) so instead we solve $\boldsymbol{A}\,\hat{\boldsymbol{x}} = \boldsymbol{b} + \boldsymbol{\varepsilon}$. Then the error propagation depends on the condition number [1, p. 338]:

$$\frac{\|\boldsymbol{x} - \hat{\boldsymbol{x}}\|}{\|\boldsymbol{x}\|} \leq \kappa(\boldsymbol{A}) \frac{\|\boldsymbol{\varepsilon}\|}{\|\boldsymbol{b}\|}.$$

See [15] [16, p. 89] for generalizations to nonlinear problems.

s,mat,adjoint
26.8 Adjoints (s,mat,adjoint)

Recall the following fact from linear algebra. If $\boldsymbol{A} \in \mathbb{C}^{m\times n}$, then

$$\langle \boldsymbol{A}\boldsymbol{x},\, \boldsymbol{y}\rangle_{\mathbb{C}^m} = \boldsymbol{y}'\boldsymbol{A}\boldsymbol{x} = (\boldsymbol{A}'\boldsymbol{y})'\boldsymbol{x} = \langle \boldsymbol{x},\, \boldsymbol{A}'\boldsymbol{y}\rangle_{\mathbb{C}^n},$$

where $\boldsymbol{A}'$ denotes the Hermitian transpose of $\boldsymbol{A}$. For analyzing some image reconstruction problems, we need to generalize the above relationship to operators $\boldsymbol{\mathcal{A}}$ in function spaces (specifically Hilbert spaces). The appropriate generalization of "**transpose**" is called the **adjoint** of $\boldsymbol{\mathcal{A}}$ and is denoted $\boldsymbol{\mathcal{A}}^*$ [17, p. 352].

Let $\mathcal{X}$ and $\mathcal{Y}$ denote vector spaces with inner products $\langle\cdot,\,\cdot\rangle_{\mathcal{X}}$ and $\langle\cdot,\,\cdot\rangle_{\mathcal{Y}}$ respectively. Let $\mathcal{B}(\mathcal{X},\mathcal{Y})$ denote the space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$, *i.e.*, if $\boldsymbol{\mathcal{A}} \in \mathcal{B}(\mathcal{X},\mathcal{Y})$ then the following supremum is finite:

$$\|\boldsymbol{\mathcal{A}}\| \triangleq \sup_{f\in\mathcal{X},\ f\neq 0} \frac{\|\boldsymbol{\mathcal{A}} f\|_{\mathcal{Y}}}{\|f\|_{\mathcal{X}}},$$

where $\|\cdot\|_{\mathcal{X}}$ is the norm on $\mathcal{X}$ corresponding to $\langle\cdot,\,\cdot\rangle_{\mathcal{X}}$, defined in (26.4.1), and likewise for $\|\cdot\|_{\mathcal{Y}}$.

If $\mathcal{X}$ and $\mathcal{Y}$ are **Hilbert spaces**, *i.e.*, **complete** vector spaces under their respective inner products, and if $\boldsymbol{\mathcal{A}} \in \mathcal{B}(\mathcal{X},\mathcal{Y})$, then one can show that there exists a unique bounded linear operator $\boldsymbol{\mathcal{A}}^* \in \mathcal{B}(\mathcal{Y},\mathcal{X})$, called the **adjoint** of $\boldsymbol{\mathcal{A}}$, that satisfies

e,mat,adjoint,inprod
$$\langle \boldsymbol{\mathcal{A}} f,\, g\rangle_{\mathcal{Y}} = \langle f,\, \boldsymbol{\mathcal{A}}^* g\rangle_{\mathcal{X}}, \quad \forall\, f \in \mathcal{X},\ g \in \mathcal{Y}. \tag{26.8.1}$$

26.8.1 Examples

If $\mathcal{X} = \mathbb{C}^n$ and $\mathcal{Y} = \mathbb{C}^m$ and $\boldsymbol{A} \in \mathbb{C}^{m\times n}$, then $\boldsymbol{A}^* = \boldsymbol{A}'$. So adjoint and transpose are the same in Euclidean space.

As another finite-dimensional example, consider $\mathcal{X} = \mathbb{C}^{n\times n}$ and $\mathcal{Y} = \mathbb{C}$ and the trace operator $\boldsymbol{\mathcal{A}} : \mathbb{C}^{n\times n} \to \mathbb{C}$ defined by $y = \boldsymbol{\mathcal{A}}\boldsymbol{X}$ iff $y = \mathrm{trace}\{\boldsymbol{X}\}$. To determine the adjoint we massage the inner products:

$$\langle \boldsymbol{\mathcal{A}}\boldsymbol{X},\, y\rangle_{\mathcal{Y}} = \left(\sum_{i=1}^n X_{ii}\right) y^* = \left(\sum_{i,j=1}^n \delta[i-j]\, X_{ij}\right) y^* = \sum_{i,j=1}^n X_{ij}\, (\delta[i-j]\, y)^* = \sum_{i,j=1}^n X_{ij} \left([\boldsymbol{I}_n y]_{ij}\right)^*.$$

Thus $\boldsymbol{\mathcal{A}}^* y = y\boldsymbol{I}_n$ is the adjoint of the trace operator.

Now we turn to infinite-dimensional examples.

x,mat,adjoint,dtft
Example 26.8.1 *Consider $\mathcal{X} = \ell_2$, the space of square summable sequences, and $\mathcal{Y} = \mathcal{L}_2[-\pi,\pi]$, the space of square integrable functions on $[-\pi,\pi]$. The **discrete-time Fourier transform** (**DTFT**) operator $\boldsymbol{\mathcal{A}} : \mathcal{X} \to \mathcal{Y}$ is defined by*

$$F = \boldsymbol{\mathcal{A}} f \iff F(\omega) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{-\imath\omega n} f_n, \quad \forall\omega \in [-\pi,\pi].$$

The fact that this linear operator is bounded is equivalent to ***Parseval's theorem****:*

$$\|F\|_{\mathcal{Y}}^2 = \int_{-\pi}^{\pi} |F(\omega)|^2 \,\mathrm{d}\omega = 2\pi \sum_{n=-\infty}^{\infty} |f_n|^2 = 2\pi \|f\|_{\mathcal{X}}^2 .$$

Thus $\|\mathcal{A}\| = 2\pi$. *To determine the adjoint, manipulate the inner product:*

$$\begin{aligned}\langle \mathcal{A} f,\, G\rangle_{\mathcal{Y}} &= \int_{-\pi}^{\pi} (\mathcal{A} f)(\omega) G^*(\omega) \,\mathrm{d}\omega = \int_{-\pi}^{\pi} \left(\sum_{n=-\infty}^{\infty} \mathrm{e}^{-\imath\omega n} f_n \right) G^*(\omega) \,\mathrm{d}\omega \\ &= \sum_{n=-\infty}^{\infty} f_n \left(\int_{-\pi}^{\pi} \mathrm{e}^{\imath\omega n} G(\omega) \,\mathrm{d}\omega \right)^* = \langle f,\, \mathcal{A}^* G\rangle_{\mathcal{X}},\end{aligned}$$

where

$$[\mathcal{A}^* G]_n = \int_{-\pi}^{\pi} \mathrm{e}^{\imath\omega n} G(\omega) \,\mathrm{d}\omega .$$

In this particular example, $\mathcal{A}^{-1} = \frac{1}{2\pi}\mathcal{A}^*$, *but in general the adjoint is* not *related to the inverse of* $\mathcal{A}$.

x,mat,adjoint,conv

Example 26.8.2 *Consider* $\mathcal{X} = \mathcal{Y} = \ell_2$ *and the (linear) discrete-time convolution operator* $\mathcal{A} : \ell_2 \to \ell_2$ *defined by*

$$\boldsymbol{z} = \mathcal{A}\boldsymbol{x} \iff z_n = \sum_{k=-\infty}^{\infty} h_{n-k} x_k, \quad n \in \mathbb{Z},$$

where we assume that $h \in \ell_1$. *One can show that* $\|\mathcal{A}\boldsymbol{x}\|_2 \leq \|h\|_1 \|\boldsymbol{x}\|_2$, *so* $\mathcal{A}$ *is bounded with* $\|\mathcal{A}\| \leq \|h\|_1$. *Since* $\mathcal{A}$ *is bounded, it is legitimate to search for its adjoint:*

$$\langle \mathcal{A}\boldsymbol{x},\, \boldsymbol{y}\rangle = \sum_{n=-\infty}^{\infty} y_n^* \left[\sum_{k=-\infty}^{\infty} x_k h_{n-k} \right] = \sum_{k=-\infty}^{\infty} x_k \left[\sum_{n=-\infty}^{\infty} y_n h_{n-k}^* \right]^* = \sum_{k=-\infty}^{\infty} x_k [\mathcal{A}^* \boldsymbol{y}]_k^* = \langle \boldsymbol{x},\, \mathcal{A}^* \boldsymbol{y}\rangle,$$

where the adjoint is

$$[\mathcal{A}^* \boldsymbol{y}]_k = \sum_{n=-\infty}^{\infty} h_{n-k}^* y_n \Longrightarrow [\mathcal{A}^* \boldsymbol{y}]_n = \sum_{k=-\infty}^{\infty} h_{k-n}^* y_k,$$

which is convolution with h_{-n}^*.

26.8.2 Properties

The following properties of adjoints all concur with those of Hermitian transpose in Euclidean space.

- $\mathcal{I}^* = \mathcal{I}$, where $\mathcal{I}$ denotes the identity operator: $\mathcal{I} f = f$
- $(\mathcal{A}^*)^* = \mathcal{A}$
- $(\mathcal{A}\mathcal{B})^* = \mathcal{B}^* \mathcal{A}^*$
- $(\mathcal{A} + \mathcal{B})^* = \mathcal{A}^* + \mathcal{B}^*$
- $(\alpha\mathcal{A})^* = \alpha^* \mathcal{A}^*$
- $\|\mathcal{A}^*\| = \|\mathcal{A}\|$
- $\|\mathcal{A}^*\mathcal{A}\| = \|\mathcal{A}\mathcal{A}^*\| = \|\mathcal{A}\|^2 = \|\mathcal{A}^*\|^2$.
- If $\mathcal{A} \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ is invertible, then $\mathcal{A}^*$ is invertible and $(\mathcal{A}^*)^{-1} = (\mathcal{A}^{-1})^*$.

s,mat,pseudo

26.9 Pseudo inverse / generalized inverse (s,mat,pseudo)

The **Moore-Penrose generalized inverse** or **pseudo inverse** of a matrix $\boldsymbol{A} \in \mathbb{C}^{m\times n}$ is the unique matrix $\boldsymbol{A}^\dagger \in \mathbb{C}^{n\times m}$ that satisfies [1, p. 421]

e,mat,pseudo

$$\begin{gathered}\boldsymbol{A}^\dagger \boldsymbol{A} \text{ and } \boldsymbol{A}\boldsymbol{A}^\dagger \text{are Hermitian} \\ \boldsymbol{A}\boldsymbol{A}^\dagger\boldsymbol{A} = \boldsymbol{A} \\ \boldsymbol{A}^\dagger\boldsymbol{A}\boldsymbol{A}^\dagger = \boldsymbol{A}^\dagger.\end{gathered} \tag{26.9.1}$$

The pseudo inverse is related to **minimum-norm least-squares** (**MNLS**) problems as follows. Of all the vectors $\boldsymbol{x}$ that minimize $\|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|_2$, the unique vector having minimum (Euclidean) norm $\|\boldsymbol{x}\|_2$ is

e,mat,pseudo,mnls

$$\hat{\boldsymbol{x}} = \boldsymbol{A}^\dagger \boldsymbol{b}. \tag{26.9.2}$$

Properties of the pseudo inverse include the following.

- [18, p. 252]
$$\boldsymbol{A}^{\dagger} = \boldsymbol{A}'[\boldsymbol{A}\boldsymbol{A}']^{\dagger} = [\boldsymbol{A}'\boldsymbol{A}]^{\dagger}\boldsymbol{A}'.$$
- By [wiki] [13, p. 215], if $\boldsymbol{A} \in \mathbb{C}^{m\times n}$ then
$$\boldsymbol{A}^{\dagger} = \underset{\boldsymbol{B}\in\mathbb{C}^{n\times m}}{\arg\min} \, \|\boldsymbol{A}\boldsymbol{B} - \boldsymbol{I}_m\|_{\mathrm{Frob}}.$$
- $$\mathcal{P}_{\boldsymbol{A}} = \boldsymbol{A}\boldsymbol{A}^{\dagger} = \boldsymbol{A}[\boldsymbol{A}'\boldsymbol{A}]^{\dagger}\boldsymbol{A}', \quad \mathcal{P}_{\boldsymbol{A}'} = \boldsymbol{A}^{\dagger}\boldsymbol{A} = \boldsymbol{A}'[\boldsymbol{A}\boldsymbol{A}']^{\dagger}\boldsymbol{A}.$$
- If $\boldsymbol{U} \in \mathbb{C}^{m\times m}$ is unitary and $\boldsymbol{V} \in \mathbb{C}^{n\times n}$ is unitary and $\boldsymbol{A} \in \mathbb{C}^{m\times n}$, then (see Problem 26.9.3):
$$(\boldsymbol{U}\boldsymbol{A}\boldsymbol{V})^{\dagger} = \boldsymbol{V}'\boldsymbol{A}^{\dagger}\boldsymbol{U}'. \tag{26.9.3}$$
e,mat,pseudo,uav
- The pseudo inverse of a product is characterized by [19, Thm. 1.4.1, p. 20]:
$$(\boldsymbol{A}\boldsymbol{B})^{\dagger} = (\boldsymbol{A}^{\dagger}\boldsymbol{A}\boldsymbol{B})^{\dagger}(\boldsymbol{A}\boldsymbol{B}\boldsymbol{B}^{\dagger})^{\dagger} = (\mathcal{P}_{\boldsymbol{A}}\boldsymbol{B})^{\dagger}(\boldsymbol{A}\mathcal{P}_{\boldsymbol{B}})^{\dagger}. \tag{26.9.4}$$
e,mat,pseudo,ab

26.10 Matrices and derivatives (s,mat,grad)

s,mat,grad

Let $\boldsymbol{X}$ denote a $N \times M$ matrix and let $f(\boldsymbol{X})$ denote some functional $f : \mathbb{R}^{N\times M} \to \mathbb{R}$ of that matrix. Then the **gradient** of f with respect to $\boldsymbol{X}$ is defined as the $N \times M$ matrix having entries
$$[\nabla_{\boldsymbol{X}} f(\boldsymbol{X})]_{ij} \triangleq \lim_{\alpha\to\infty} \frac{1}{\alpha}\left[f\left(\boldsymbol{X} + \alpha \boldsymbol{e}_i \boldsymbol{e}_j'\right) - f(\boldsymbol{X})\right]. \tag{26.10.1}$$
e,mat,grad

Using this definition, one can show that
$$f(\boldsymbol{X}) = \mathsf{trace}\{\boldsymbol{A}\boldsymbol{X}\boldsymbol{B}\boldsymbol{X}'\boldsymbol{C}\} \Longrightarrow \nabla_{\boldsymbol{X}} f(\boldsymbol{X}) = \boldsymbol{A}'\boldsymbol{C}\boldsymbol{X}\boldsymbol{B}' + \boldsymbol{C}\boldsymbol{A}\boldsymbol{X}\boldsymbol{B}. \tag{26.10.2}$$
e,mat,grad,tr

The derivative of a matrix inverse with respect to a parameter also can be useful [wiki]:
$$\frac{\partial}{\partial t}[\boldsymbol{A}(t)]^{-1} = -\boldsymbol{A}^{-1}\left(\frac{\partial}{\partial t}\boldsymbol{A}(t)\right)\boldsymbol{A}^{-1}. \tag{26.10.3}$$
e,mat,grad,inv

26.11 The four spaces (s,mat,4space)

s,mat,4space

The range space and null space of a matrix $\boldsymbol{A} \in \mathbb{C}^{m\times n}$ and related quantities can be important.
$$\begin{aligned}
\mathcal{R}_{\boldsymbol{A}} &\triangleq \{\boldsymbol{y} : \boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}\} \\
\mathcal{N}_{\boldsymbol{A}} &\triangleq \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}\} \\
\mathcal{N}_{\boldsymbol{A}'}^{\perp} &\triangleq \{\boldsymbol{y} : \boldsymbol{A}'\boldsymbol{y}_0 = \boldsymbol{0} \Longrightarrow \boldsymbol{y}'\boldsymbol{y}_0 = 0\} \\
\mathcal{R}_{\boldsymbol{A}'}^{\perp} &\triangleq \{\boldsymbol{x} : \boldsymbol{x}'\boldsymbol{A}'\boldsymbol{y} = 0 \; \forall \boldsymbol{y}\}.
\end{aligned}$$

All four are linear spaces, so all include the zero vector. These spaces have the following relationships (see Problem 26.10):
$$\mathcal{R}_{\boldsymbol{A}} = \mathcal{N}_{\boldsymbol{A}'}^{\perp} \tag{26.11.1}$$
$$\mathcal{N}_{\boldsymbol{A}} = \mathcal{R}_{\boldsymbol{A}'}^{\perp} \tag{26.11.2}$$
$$\mathcal{R}_{\boldsymbol{A}} - \boldsymbol{0} \subseteq \mathcal{N}_{\boldsymbol{A}'}^{c} \tag{26.11.3}$$
$$\mathcal{N}_{\boldsymbol{A}} - \boldsymbol{0} \subseteq \mathcal{R}_{\boldsymbol{A}'}^{c} \tag{26.11.4}$$
$$\mathcal{N}_{\boldsymbol{A}'}^{\perp} \subseteq \mathcal{N}_{\boldsymbol{A}'}^{c}. \tag{26.11.5}$$
e,mat,4space

s,mat,pca

26.12 Principal components analysis (low-rank approximation) (s,mat,pca)

Given data y_{ij} for $i = 1, \ldots, N$ and $j = 1, \ldots, M$, often we wish to find a set of L orthonormal vectors $\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_L \in \mathbb{C}^N$ and corresponding coefficients $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_M \in \mathbb{C}^L$ to minimize the following WLS approximation error

$$\sum_{j=1}^{M} w_j \left\| \boldsymbol{y}_j - \sum_{l=1}^{L} \boldsymbol{\phi}_l x_{lj} \right\|^2 = \sum_{j=1}^{M} w_j \left\| \boldsymbol{y}_j - \boldsymbol{B} \boldsymbol{x}_j \right\|^2,$$

where $\boldsymbol{y}_j = (y_{1j}, \ldots, y_{Nj})$, $\boldsymbol{x}_j = (x_{1j}, \ldots, x_{LM})$, and $\boldsymbol{B} = [\boldsymbol{\phi}_1 \ \ldots \ \boldsymbol{\phi}_L]$. Defining $\tilde{\boldsymbol{y}}_j \triangleq \sqrt{w_j}\boldsymbol{y}_j$ and $\tilde{\boldsymbol{x}}_j \triangleq \sqrt{w_j}\boldsymbol{x}_j$, we can rewrite this low-rank matrix approximation problem as

$$\min_{\tilde{\boldsymbol{X}},\, \boldsymbol{B} : \boldsymbol{B}'\boldsymbol{B} = \boldsymbol{I}_L} \sum_{j=1}^{M} \|\tilde{\boldsymbol{y}}_j - \boldsymbol{B}\tilde{\boldsymbol{x}}_j\|_2^2 = \min_{\tilde{\boldsymbol{X}},\, \boldsymbol{B} : \boldsymbol{B}'\boldsymbol{B} = \boldsymbol{I}_L} \|\tilde{\boldsymbol{Y}} - \boldsymbol{B}\tilde{\boldsymbol{X}}\|_{\text{Frob}}^2,$$

where $\tilde{\boldsymbol{X}} \triangleq [\tilde{\boldsymbol{x}}_1, \ldots, \tilde{\boldsymbol{x}}_M]$ and $\tilde{\boldsymbol{Y}} \triangleq [\tilde{\boldsymbol{y}}_1, \ldots, \tilde{\boldsymbol{y}}_M]$.

Since $\boldsymbol{B}$ has orthonormal columns, minimizing over $\tilde{\boldsymbol{x}}_j$ yields $\tilde{\boldsymbol{x}}_j = \boldsymbol{B}'\tilde{\boldsymbol{y}}_j$ or equivalently $\boldsymbol{x}_j = \boldsymbol{B}'\boldsymbol{y}_j$ and $\tilde{\boldsymbol{X}} = \boldsymbol{B}'\tilde{\boldsymbol{Y}}$. In this form, $\boldsymbol{B}\tilde{\boldsymbol{X}}$ is a **low-rank approximation** of $\tilde{\boldsymbol{Y}}$. Thus to find $\boldsymbol{B}$ we must minimize

$$\begin{aligned} \|\tilde{\boldsymbol{Y}} - \boldsymbol{B}\boldsymbol{B}'\tilde{\boldsymbol{Y}}\|_{\text{Frob}}^2 &= \operatorname{trace}\left\{ \left(\tilde{\boldsymbol{Y}} - \boldsymbol{B}\boldsymbol{B}'\tilde{\boldsymbol{Y}}\right)' \left(\tilde{\boldsymbol{Y}} - \boldsymbol{B}\boldsymbol{B}'\tilde{\boldsymbol{Y}}\right) \right\} \\ &\equiv -\operatorname{trace}\left\{ \tilde{\boldsymbol{Y}}'\boldsymbol{B}\boldsymbol{B}'\tilde{\boldsymbol{Y}} \right\}. \end{aligned}$$

Thus we want to maximize $\operatorname{trace}\left\{\boldsymbol{B}'\tilde{\boldsymbol{Y}}\tilde{\boldsymbol{Y}}'\boldsymbol{B}\right\}$, where $\boldsymbol{K} \triangleq \tilde{\boldsymbol{Y}}\tilde{\boldsymbol{Y}}'$, subject to the constraint that the columns of $\boldsymbol{B}$ must be orthonormal. Taking the gradient with respect to $\boldsymbol{\phi}_l$ of the Lagrangian $\sum_{l=1}^{L} \boldsymbol{\phi}_l'\boldsymbol{K}\boldsymbol{\phi}_l - \lambda_l(\|\boldsymbol{\phi}_l\|^2 - 1)$ yields $\boldsymbol{K}\boldsymbol{\phi}_l = \lambda_l\boldsymbol{\phi}_l$. Thus each $\boldsymbol{\phi}_l$ is an eigenvector of $\boldsymbol{K}$. So the optimal $\boldsymbol{B}$ is the first L singular vectors of $\boldsymbol{K}$. This is called the **Eckart-Young theorem** [20].

Mat `svds`

s,mat,prob

26.13 Problems (s,mat,prob)

p,mat,rho,ratio

Problem 26.1 *Prove or disprove the ratio property for $\rho(\boldsymbol{A})$ in (26.1.5) in the general case where $\boldsymbol{A}$ is square but not necessarily symmetric.*

p,mat,rho,ab

Problem 26.2 *Equation (26.1.3) states that $\rho(\boldsymbol{A}\boldsymbol{B}) = \rho(\boldsymbol{B}\boldsymbol{A})$ when $\boldsymbol{A} \in \mathbb{C}^{m \times n}$ and $\boldsymbol{B} \in \mathbb{C}^{n \times m}$. Prove or disprove: $\rho(\boldsymbol{I}_m - \boldsymbol{A}\boldsymbol{B}) \stackrel{?}{=} \rho(\boldsymbol{I}_n - \boldsymbol{B}\boldsymbol{A})$.*

p,mat,vnorm,equiv

Problem 26.3 *Determine the constants in relating norms in (26.3.8) for the case $\alpha = 2$ and $\beta = 1$.*

p,mat,mnorm2,mleq

Problem 26.4 *Prove Theorem 26.5.3,* i.e., $\boldsymbol{A} \succeq \boldsymbol{0} \implies (\|\boldsymbol{A}\|_2 \leq 1 \iff \boldsymbol{A} \preceq \boldsymbol{I})$.

p,mat,lemma,ba

Problem 26.5 *Prove Lemma 26.2.3 relating to matrix partial orderings.*

p,mat,spd,inv

Problem 26.6 *Prove Theorem 26.2.4, relating to the inverse of partially Hermitian positive definite matrices.*

p,mat,frob

Problem 26.7 *Prove the Frobenius norm inequality (26.5.7) and show that it is not tight.*

Problem 26.8 *Following Example 26.8.2, determine the adjoint of the 2D convolution operator $g = \boldsymbol{\mathcal{A}} f \iff g(x, y) = \iint h(x - x', y - y')\, f(x', y')\, \mathrm{d}x'\, \mathrm{d}y'$.*

p,mat,pseudo,uav

Problem 26.9 *Prove the equality (26.9.3) for unitary transforms of pseudo inverses.*

p,mat,4space

Problem 26.10 *Prove the relationships between the four spaces in (26.11.5).*

mat,lp,p,strict,convex

Problem 26.11 *Prove that $f(\boldsymbol{x}) = \|\boldsymbol{x}\|_p^p$ is strictly convex for $p > 1$.*

p,mat,diag,gers

Problem 26.12 *Either prove the generalized diagonal dominance theorem Theorem 26.2.8 using the Geršgorin theorem, or construct a counter example showing that (26.2.2) truly is a generalization,* i.e., *a case where $\boldsymbol{D} - \boldsymbol{B}'\boldsymbol{B}$ is not diagonally dominant.* *(Solve?)*

26.14 Bibliography

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