Online Electricity Pricing from Frequency Measurements

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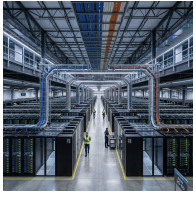
The problem: markets are slow, physics is fast

- Power imbalance shows up instantly as **frequency deviation**.
- Electricity is priced *offline*: day-ahead (every hour) and real-time (every 5–15 min).
- Between market clearings, fast automatic control (not the market) restores frequency.

$\Rightarrow$ Prices fixed for minutes cannot see sub-minute renewable and load variability.



Renewable power generation



Datacenter loads

Contribution

Derive an *online* electricity price that updates on frequency timescales, and show it recovers the theoretical properties of offline markets in real time.

Real-time economic dispatch

Day-ahead dispatch $\mathbf{p}^\star$, demand deviates by δ. Find optimal regulation $\mathbf{r}$:

$$\begin{aligned} \underset{\mathbf{r}\in\mathcal{P}}{\text{minimize}} \quad & c(\mathbf{r}) = \tfrac{1}{2}\|\mathbf{p}^\star + \mathbf{r}\|_{\mathbf{C}}^2 + \mathbf{c}^\top(\mathbf{p}^\star + \mathbf{r}) \\ \text{subject to} \quad & \mathbf{1}^\top(\mathbf{p}^\star + \mathbf{r}) - d - \delta = 0 \quad : \lambda \end{aligned}$$

- λ = dual variable = marginal electricity price
- Solved every 5–15 min in practice — we want it much faster

Primal-dual dynamics on the Lagrangian recover this equilibrium:

$$\dot{\mathbf{r}} = \Pi_{\mathcal{P}}[\lambda\mathbf{1} - \nabla_{\mathbf{r}} c(\mathbf{r})]\,, \qquad \dot{\lambda} = \delta - \mathbf{1}^\top\mathbf{r}$$

a classic Walrasian tâtonnement price-adjustment process.

Frequency dynamics

Copper-plate swing equation:

$$\dot{\omega} = \tfrac{1}{M}\left(\mathbf{1}^\top \mathbf{r} - D\omega - \delta\right)$$

- M: inertia, D: damping
- This is a gradient flow on

$$\mathcal{L}^{\mathsf{FD}}(\omega) = \tfrac{1}{2DM}(D\omega - \mathbf{1}^\top \mathbf{r} + \delta)^2$$

Idea

Frequency dynamics and market dynamics are both gradient/primal-dual flows related via active power balance $\Longrightarrow$ combine them into one system.

Integrated market–frequency dynamics

Composite Lagrangian, with auxiliary dual variable γ:

$$\mathcal{L}(\mathbf{r}, \omega, \lambda, \gamma) = \mathcal{L}^{\mathrm{ED}}(\mathbf{r}, \lambda) + \gamma\, \mathcal{L}^{\mathrm{FD}}(\omega)$$

Primal-dual (saddle-point) flow on $\mathcal{L}$:

$$\begin{aligned}
\dot{\mathbf{r}} &= \Pi_{\mathcal{P}}\left[\lambda \mathbf{1} - \tfrac{\gamma}{DM}(\mathbf{1}^\top \mathbf{r} - D\omega - \delta)\mathbf{1} - \nabla_{\mathbf{r}} c(\mathbf{r})\right] \\
\dot{\omega} &= \tfrac{\gamma}{M}(\mathbf{1}^\top \mathbf{r} - D\omega - \delta) \\
\dot{\lambda} &= \delta - \mathbf{1}^\top \mathbf{r} \\
\dot{\gamma} &= \tfrac{1}{2DM}(D\omega - \mathbf{1}^\top \mathbf{r} + \delta)^2
\end{aligned}$$

Generators now respond to a price informed by **both** power imbalance (λ) and frequency deviation (γ), on the same timescale.

Theorem 1: convergence to the social optimum

Theorem 1 (Social optimum)

The dynamical system above converges to the solution of

$$\begin{aligned} \underset{\mathbf{r}\in\mathcal{P},\,\omega}{\text{minimize}} \quad & c(\mathbf{r}) \\ \text{subject to} \quad & \mathbf{1}^\top \mathbf{r} - \delta = 0, \\ & D\omega - \mathbf{1}^\top \mathbf{r} + \delta = 0, \end{aligned}$$

i.e. minimum-cost generation, power balance, **and** zero frequency deviation, simultaneously.

Grid physics forces the hard constraint $\gamma = 1$ (the swing equation is a law, not a soft constraint) — this recovers the original swing equation and simplifies the dynamics.

Turning frequency measurements into a price

Power imbalance drives both $\dot{\lambda}$ and $\dot{\omega}$. Eliminating common $\mathbf{1}^\top \mathbf{r}$ between them:

$$\dot{\lambda} = -D\omega - M\dot{\omega}$$

Integrating from the day-ahead state $\lambda(0) = \lambda^{\mathsf{da}}$, $\omega(0) = 0$:

$$\lambda(t) = \lambda^{\mathsf{da}} - D\int_0^t \omega(\tau)\, d\tau - M\omega(t)$$

Result

The real-time price is the day-ahead price plus a correction computable **locally, from frequency measurements alone.**

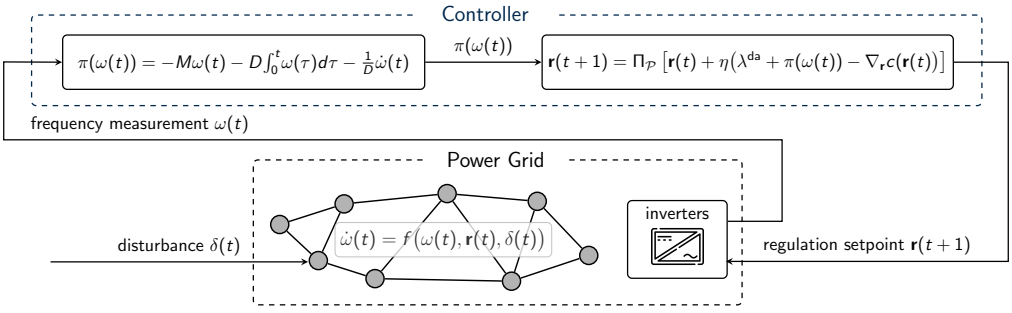
PID-structure of online electricity pricing

$$\pi(\omega(t)) = \underbrace{-M\omega(t)}_{\text{P}} \; \underbrace{-D\int_0^t \omega(\tau)\,d\tau}_{\text{I}} \; \underbrace{-\tfrac{1}{D}\dot{\omega}(t)}_{\text{D}}$$

- $\omega = 0$: $\pi = 0$, generators sit at the day-ahead schedule
- $\omega < 0$ (under-frequency): $\pi > 0$, price rises, generators ramp up
- $\omega > 0$ (over-frequency): $\pi < 0$, price falls, generators ramp down

Gains M, D are system inertia/damping, estimable and broadcast by the system operator; λ^{da} is public.

Block diagram



Theorem 2: cost recovery

Discrete-time update, step size η_i per generator i:

$$\hat{\mathbf{r}}_i(t{+}1) = \mathbf{r}_i(t) + \eta_i\big(\lambda^{\mathrm{da}} + \pi(\omega(t)) - \nabla_{\mathbf{r}} c_i(\mathbf{r}_i(t))\big), \quad \mathbf{r}_i(t{+}1) = \Pi_{\mathcal{P}_i}\big[\hat{\mathbf{r}}_i(t{+}1)\big]$$

- too small η_i: generator can't respond fast enough to price
- too large η_i: cost grows faster than price adjusts

Theorem 2 (Cost recovery)

Choosing $\eta_i = \mathbf{C}_{ii}^{-1}$ (inverse of the quadratic cost coefficient) guarantees **nonnegative profit** for generator i at *every* time step.

Proof idea: the update is the KKT solution of a proximal problem; the optimal step size makes the dual objective — hence the primal profit bound — exactly balance to zero.

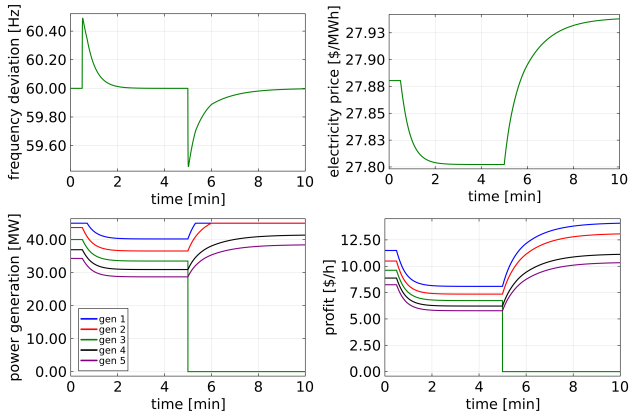
Numerical setup

Single-area system, five generators.

- Nominal demand $d = 200$ MW, generator capacity up to 50 MW each
- Linear cost \$27.4/MWh; quadratic cost \$0.01–0.015/MWh^2
- Inertia $M = 12$ s, damping $D = 35$ MW/Hz
- Step sizes η_i set via Theorem 2
- Frequency simulated at 0.05 s, sampled every 0.25 s

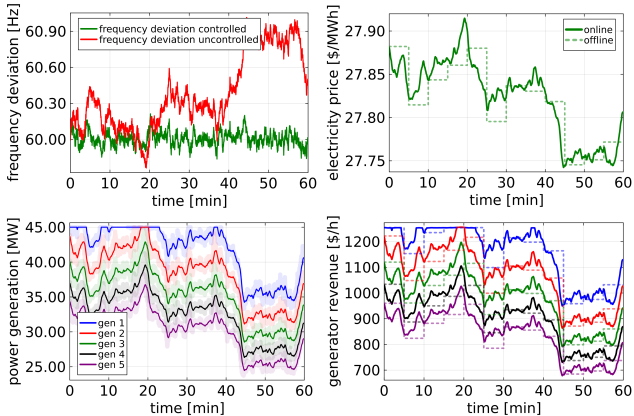
Two scenarios: **constant** step disturbances, and **time-varying** stochastic demand.

Results: constant disturbance



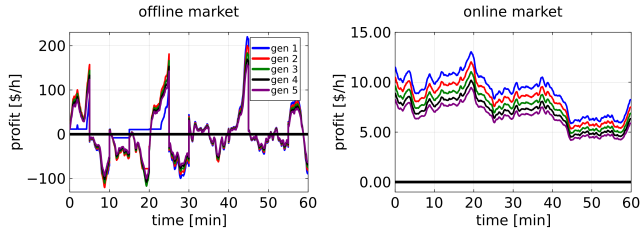
Step demand drop at $t = 30$s, generator outage at $t = 5$min. Frequency is restored both times; price falls after the first event, rises after the second; generation and profit track accordingly.

Results: time-varying disturbance



Stochastic demand (Wiener process). Online pricing tracks intra-interval frequency changes that 5-minute offline pricing simply misses.

Results: profit, offline vs. online



Offline prices ignore the real-time cost of frequency response: profits are volatile and often negative. Online prices keep profit stable and nonnegative, confirming Theorem 2.

Conclusion

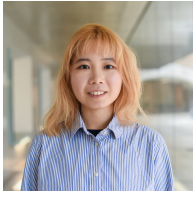
- Integrated market–frequency dynamics give *online* electricity pricing directly from local frequency measurements.
- The resulting controller is a local PID: day-ahead price plus a frequency-derived correction.
- Provable properties: convergence to the social optimum, cost recovery at every step.

Future work

Extend to transmission networks and locational marginal prices from bus-level frequency; establish budget balance in the networked online market.

Thank you

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Xinwei Liu (2nd-year PhD student)



Vladimir Dvorkin