ECE 598: Computational Power Systems **Winter 2025**

Lecture 3: Optimal Power Flow & Locational Pricing

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3.1 Power Flow Models

The main objective of power flow is to calculate active power, reactive power, voltage magnitude, and voltage angle at each bus in a network.

3.1.1 Power Transmission Networks

Power transmission networks can be modeled as electric circuits with:

- N nodes (buses) and E edges (lines and transformers)
- AC voltages and currents as phasors: $\mathcal{V} = Ve^{j\theta} = \Re[\mathcal{V}] + j\Im[\mathcal{V}]$
- Ohm's law: $\mathcal{V} = \mathcal{Z}\mathcal{I}$

π-model can be utilized to model transmission lines with the below elements:

- Voltages $\mathcal{V}_n$ and $\mathcal{V}_m$ at line ends
- Line series impedance $z_{nm} = r_{nm} + jx_{nm}$
- Line series admittance $y_{nm} = \frac{1}{z_{nm}} = g_{nm} + jb_{nm}$
- Line series conductance $g_{nm} = \frac{r_{nm}}{r_{nm}^2 + x_{nm}^2}$
- Line series susceptance $b_{nm} = \frac{x_{nm}}{r_{nm}^2 + x_{nm}^2}$
- Line charging susceptance b_{nm}^c

Line currents entering nodes (buses): $\mathcal{I}_{nm} = \left(y_{nm} + j\frac{b_{nm}^c}{2}\right)\mathcal{V}_n - y_{nm}\mathcal{V}_m$

Kitchoff's current law can be used to sum all currents between node n and all other connected nodes:

$$\mathcal{I}_n = \sum_{m=1}^{N} \left(y_{nm} + j\frac{b_{nm}^c}{2}\right)\mathcal{V}_n - \sum_{m=1}^{N} y_{nm}\mathcal{V}_m, \quad \forall n = 1, ..., N, \text{such that } n \neq m \tag{3.1}$$

Multivariate Ohm's law considers multiple voltages, currents, and admittances in a matrix/vector form.

- $\mathbf{i} = \mathbf{Y}\mathbf{v}$, where $\mathbf{i}$ and $\mathbf{v}$ are vectors, and $\mathbf{Y}$ is a matrix.
- Bus admittance matrix is a descriptor of the network and it encodes line connections.

$$Y_{nm} = \begin{cases} \sum\limits_{k\neq n} y_{nk} + j\frac{b^c_{nk}}{2}, & n = m \\ -y_{nm}, & \exists \text{ line } (n,m) \\ 0, & \text{otherwise} \end{cases} \tag{3.2}$$

  Y_{nm} is symmetric, non-Hermitian, sparse and invertible (if $b^c_{nm} \neq 0$ for at least one line; otherwise $\mathbf{Y}\mathbf{1} = 0$)

- Bus impedance matrix $\mathbf{Z}$ is non-sparse and not the matrix of line impedances (i.e. $Z_{nm} \neq z_{nm} = \frac{1}{y_{nm}}$)

Complex power expression for power flow from node n to m: $S_{nm} = \mathcal{V}_m \mathcal{I}^*_{nm}$

3.1.2 AC Power Flow Model

- Polar Coordinates

$$P_n = V_n \sum_{m=1}^{N} V_m(G_{nm}cos(\theta_{nm}) + B_{nm}sin(\theta_{nm})) \tag{3.3}$$

$$Q_n = V_n \sum_{m=1}^{N} V_m(G_{nm}sin(\theta_{nm}) - B_{nm}cos(\theta_{nm})) \tag{3.4}$$

  Note that: $S_n = P_n + jQ_n$, $\theta_{nm} = \theta_n - \theta_m$

- Rectangular Coordinates

$$P_n = \Re[V_n] \sum_{m=1}^{N} (\Re[V_m]G_nm - \Im[V_m]B_nm) + \Im[V_n] \sum_{m=1}^{N} (\Im[V_m]G_nm - \Re[V_m]B_nm) \tag{3.5}$$

$$Q_n = \Im[V_n] \sum_{m=1}^{N} (\Re[V_m]G_nm - \Im[V_m]B_nm) - \Re[V_n] \sum_{m=1}^{N} (\Im[V_m]G_nm - \Re[V_m]B_nm) \tag{3.6}$$

3.1.3 Solving Power Flow Equations

- 2N equations and 4N variables $\{(P_m, Q_m, V_m, \theta_m)\}_{m=1}^N$
- **Problem Statement**: using the known 2N variables, find the values of the rest of the variables (2N unknowns) that satisfy the nonlinear power flow equations.
- Where do the known variables come from? The known variables at each bus depend on the type of bus.
  1. N_d Load buses (PQ buses): (P_n, Q_n)
  2. N_g Generator buses (PV buses): (P_n, V_n)
  3. Reference bus: $(V_n, \theta_n = 0)$
- Total number of buses: $N = 1 + N_g + N_d$

- Resultant 2N power flow equations:

$$P_n = V_n \sum_{m=1}^{N} V_m(G_{nm}cos(\theta_{nm}) + B_{nm}sin(\theta_{nm})), \quad \forall n = 1, ..., N_d + N_g = N - 1 \tag{3.7}$$

$$Q_n = V_n \sum_{m=1}^{N} V_m(G_{nm}sin(\theta_{nm}) - B_{nm}cos(\theta_{nm})), \quad \forall n = 1, ..., N_d \tag{3.8}$$

- Recursive methods such as (Gauss-Seidel, Newton, FDPF) can be used to solve for $\{(V_n, \theta_n)\}_{n=1}^N$, then other quantities (injections, flows, currents, losses) can be calculated.

3.2 DC Power Flow Model

3.2.1 Formulation

DC power flow approximates AC power flow through the following assumptions:

1. Low r/x ratio in transmission lines

   $r_{nm} << x_{nm} \quad \rightarrow \quad g_{nm} << b_{nm} \quad \rightarrow \quad \mathbf{G} \approx \mathbf{0} \quad and \quad b_{nm} = \frac{x_{nm}}{r_{nm}^2 + x_{nm}^2}$

2. Small angle difference: $sin(\theta_n - \theta_m) \approx \theta_n - \theta_m$
3. Voltage magnitudes: $V_n \approx 1$

DC power flow model becomes:

$$P_n \approx \sum_{m:n\sim m} P_{nm} = \sum_{m:n\sim m} b_{nm}(\theta_n - \theta_m) \tag{3.9}$$

3.2.2 B Matrix

Equation (3.7) shows that power injections relate linearly to phase differences. The corresponding multivariate power flow model:

$$\mathbf{p} = \mathbf{B}\boldsymbol{\theta} \tag{3.10}$$

DC Bus Admittance Matrix: (different from matrix $\mathbf{B}$ in $\mathbf{Y} = \mathbf{G} + \text{j}\ \mathbf{B}$)

$$B_{nm} = \begin{cases} \sum_{k \neq n} b_{nk}, & n = m \\ -b_{nm}, & \exists \text{ line } (n, m) \\ 0, & \text{otherwise} \end{cases} \tag{3.11}$$

- $\mathbf{B}$ matrix is real, symmetric, sparse, and positive semidefinite
- Lossless lines: $\mathbf{B1} = \mathbf{0} \Rightarrow \mathbf{1}^T\mathbf{p} = 0 \quad (\mathbf{1}^T(\mathbf{p}^g - \mathbf{p}^d) = 0)$
- b_{nm} is often further simplified to: $b_{nm} = \frac{x_{nm}}{r_{nm}^2 + x_{nm}^2} \approx \frac{1}{x_{nm}}$ (based on the assumption $x_{nm} >> r_{nm}$)

3.3 Optimal Power Flow

3.3.1 Formulation in Rectangular Coordinates

- Collect nodal voltages in rectangular coordinates in $\boldsymbol{v} \in \mathbb{C}^N$

$$\boldsymbol{v} = \begin{bmatrix} \Re[\mathcal{V}_1] + j\Im[\mathcal{V}_1] & \dots & \Re[\mathcal{V}_N] + j\Im[\mathcal{V}_N] \end{bmatrix}^T$$

- Power injections and squared voltage magnitudes are quadratic functions of $\boldsymbol{v}$:

$$P_n(v) = \boldsymbol{v}^H \boldsymbol{M}_{P_n} \boldsymbol{v}$$
$$Q_n(v) = \boldsymbol{v}^H \boldsymbol{M}_{Q_n} \boldsymbol{v}$$
$$V_n^2(v) = \boldsymbol{v}^H \boldsymbol{M}_{V_n} \boldsymbol{v}$$

  where matrices $\boldsymbol{M}$ are Hermitian symmetric ($\boldsymbol{M} = \boldsymbol{M}^H$)

- Every bus contributes two quadratic constraints (active and reactive power) on $\boldsymbol{v}$

3.3.2 Finding $\boldsymbol{M}$ Matrices

- Voltage magnitude ($\boldsymbol{e}_n$ is the n-th canonical vector)

$$V_n^2(\boldsymbol{v}) = \mathcal{V}_n^* \mathcal{V}_n = \boldsymbol{v}^H \boldsymbol{e}_n \boldsymbol{e}_n^T \boldsymbol{v} \quad \Rightarrow \quad \boldsymbol{M}_{V_n} = \boldsymbol{e}_n \boldsymbol{e}_n^T$$

- Complex power injection

$$S_n = P_n + jQ_n = \mathcal{V}_n \mathcal{I}_n^* = (\boldsymbol{v}^T \boldsymbol{e}_n)(\boldsymbol{e}_n^T \boldsymbol{i}^*) = v^T \boldsymbol{e}_n \boldsymbol{e}_n^T \boldsymbol{Y}^* \boldsymbol{v}^* = \boldsymbol{v}^H \boldsymbol{Y}^* \boldsymbol{e}_n \boldsymbol{e}_n^T \boldsymbol{v}$$

- Active and reactive power then take the form

$$P_n = \frac{1}{2}(S_n + S_n^*) = \boldsymbol{v}^H \boldsymbol{M}_{P_n} \boldsymbol{v}, \quad \boldsymbol{M}_{P_n} = \frac{1}{2}(\boldsymbol{Y}^* \boldsymbol{e}_n \boldsymbol{e}_n^T + \boldsymbol{e}_n \boldsymbol{e}_n^T \boldsymbol{Y}^*)$$
$$Q_n = \frac{1}{2}(S_n - S_n^*) = \boldsymbol{v}^H \boldsymbol{M}_{Q_n} \boldsymbol{v}, \quad \boldsymbol{M}_{Q_n} = \frac{1}{2}(\boldsymbol{Y}^* \boldsymbol{e}_n \boldsymbol{e}_n^T - \boldsymbol{e}_n \boldsymbol{e}_n^T \boldsymbol{Y}^*)$$

3.3.3 Power Flow as a Feasibility Problem

- System state as solution of feasibility problem:

$$\begin{aligned} &\text{find } v \\ &\text{s.t. } \boldsymbol{v}^H \boldsymbol{M}_k \boldsymbol{v} = s_k, \quad \forall k = 1, \dots, 2N \quad [\text{note: } \boldsymbol{v}^H \boldsymbol{M}_k \boldsymbol{v} = \mathrm{Tr}(\boldsymbol{M}_k \boldsymbol{v} \boldsymbol{v}^H)] \end{aligned}$$

- Introduce matrix variable $\boldsymbol{V} = \boldsymbol{v}\boldsymbol{v}^H$:

$$\begin{aligned} &\text{find } (\boldsymbol{v}, \boldsymbol{V}) \\ &\text{s.t. } \mathrm{Tr}(\boldsymbol{M}_k \boldsymbol{V}) = s_k, \quad \forall k = 1, \dots, 2N \\ &\quad \boldsymbol{V} = \boldsymbol{v}\boldsymbol{v}^H \end{aligned}$$

- Eliminate variable $\boldsymbol{v}$; non-convex problem due to rank constraint:

$$\begin{aligned}
&\text{find } (\boldsymbol{V}) \\
&\text{s.t. } \operatorname{Tr}(\boldsymbol{M}_k\boldsymbol{V}) = s_k, \quad \forall k = 1, \ldots, 2N \\
&\quad \boldsymbol{V} \succeq \boldsymbol{0}, \quad \operatorname{rank}(\boldsymbol{V}) = 1
\end{aligned}$$

  Note: The rank constraint ensures finding a unique set of voltages.

3.3.4 Semidefinite Program Relaxation

- Drop rank constraint to get semidefinite program (SDP):

$$\begin{aligned}
&\text{find } (\boldsymbol{V}) \\
&\text{s.t. } \operatorname{Tr}(\boldsymbol{M}_k\boldsymbol{V}) = s_k, \quad \forall k = 1, \ldots, 2N \\
&\quad \boldsymbol{V} \succeq \boldsymbol{0}
\end{aligned}$$

  which is a convex problem.

- If the solution $\boldsymbol{V}^\star$ is rank-1, the relaxation is said to be exact.
- If exact, find $\boldsymbol{v}^\star$ from $\boldsymbol{V}^\star = \boldsymbol{v}^\star\boldsymbol{v}^{\star H}$
- Relaxation is oftentimes exact under practical system conditions.

3.3.5 Optimal Power Flow Using Semidefinite Relaxation

- OPF problem:

$$\begin{aligned}
&\min_{\boldsymbol{V}\succeq\boldsymbol{0}} \quad \operatorname{Tr}(\boldsymbol{M}\boldsymbol{V}) \\
&\text{s.t. } \operatorname{Tr}(\boldsymbol{M}_k\boldsymbol{V}) = s_k, \quad \forall k = 1, \ldots, 2N
\end{aligned}$$

- Design matrix $\boldsymbol{M}$ to strengthen the relaxation (favor rank-1 solutions)[1]:
  - Selecting $\boldsymbol{M} = \boldsymbol{Y}^H\boldsymbol{Y}$ minimizes $\|\boldsymbol{i}\|_2^2$
  - Selecting $\boldsymbol{M} = \boldsymbol{B}$ minimizes losses
  - Both yield the "high-voltage solution" of the power flow equations
- Use $\sum_{n=1}^{N} c_n \operatorname{Tr}(\boldsymbol{M}_{P_n}\boldsymbol{V})$ as an objective to minimize the dispatch cost[2]
- Incorporate squared voltage bounds as: $\underline{\boldsymbol{v}}^2 \leq \operatorname{Tr}(\boldsymbol{M}_{V_n}\boldsymbol{V}) \leq \overline{\boldsymbol{v}}^2$

[1] R. Madani, J. Lavaei, and R. Baldick. *Convexification of power flow problem over arbitrary networks.* IEEE CDC 2015.
[2] J. Lavaei and S. Low. *Zero duality gap in optimal power flow problem.* IEEE Trans. on Power Systems. 2012.

3.3.6 Formulation in Polar Coordinates

- AC power flow model:

$$P_n(v,\theta) = V_n \sum_{m=1}^{N} V_m (G_{nm}\cos\theta_{nm} + B_{nm}\sin\theta_{nm}), \quad \forall n = 1,\dots,N$$
$$Q_n(v,\theta) = V_n \sum_{m=1}^{N} V_m (G_{nm}\sin\theta_{nm} - B_{nm}\cos\theta_{nm}), \quad \forall n = 1,\dots,N$$

- Classic AC-OPF problem formulation:

$$\begin{aligned}
\min_{\boldsymbol{p}^g,\boldsymbol{q}^g,\boldsymbol{v},\boldsymbol{\theta}} \quad & c(\boldsymbol{p}^g) && \text{generation cost}\\
\text{s.t.} \quad & \boldsymbol{p}(\boldsymbol{v},\boldsymbol{\theta}) = \boldsymbol{p}^g - \boldsymbol{p}^d && \text{active power flow}\\
& \boldsymbol{q}(\boldsymbol{v},\boldsymbol{\theta}) = \boldsymbol{q}^g - \boldsymbol{q}^d && \text{reactive power flow}\\
& \boldsymbol{p}^{g_{\min}} \le \boldsymbol{p}^g \le \boldsymbol{p}^{g_{\max}} && \text{min/max gen p-limits}\\
& \boldsymbol{q}^{g_{\min}} \le \boldsymbol{q}^g \le \boldsymbol{q}^{g_{\max}} && \text{min/max gen q-limits}\\
& \boldsymbol{v}^{\min} \le \boldsymbol{v} \le \boldsymbol{v}^{\max} && \text{min/max voltage mag limits}\\
& \boldsymbol{\theta}^{\min} \le \boldsymbol{\theta} \le \boldsymbol{\theta}^{\max} && \text{min/max voltage angle limits}
\end{aligned}$$

- Minimize generation cost subject to power flow equations and variable limits.

3.3.7 Security-Constrained AC-OPF (SC-AC-OPF)

- Optimization problem:

$$\begin{aligned}
\min_{\boldsymbol{p}^g,\boldsymbol{q}^g,\boldsymbol{v}_0,\boldsymbol{\theta}_0,\boldsymbol{v}_c,\boldsymbol{\theta}_c} \quad & c(\boldsymbol{p}^g) && \text{generation cost}\\
\text{s.t.} \quad & \boldsymbol{p}_0(\boldsymbol{v}_0,\boldsymbol{\theta}_0) = \boldsymbol{p}^g - \boldsymbol{p}^d && \text{nominal active power flow}\\
& \boldsymbol{q}_0(\boldsymbol{v}_0,\boldsymbol{\theta}_0) = \boldsymbol{q}^g - \boldsymbol{q}^d && \text{nominal reactive power flow}\\
& \boldsymbol{p}_c(\boldsymbol{v}_c,\boldsymbol{\theta}_c) = \boldsymbol{p}^g - \boldsymbol{p}^d, \quad \forall c = 1,\dots,N_c && \text{post-contingency active power flow}\\
& \boldsymbol{q}_c(\boldsymbol{v}_c,\boldsymbol{\theta}_c) = \boldsymbol{q}^g - \boldsymbol{q}^d, \quad \forall c = 1,\dots,N_c && \text{post-contingency reactive power flow}\\
& + \text{ limits on optimization variables}
\end{aligned}$$

- Constraints for the nominal and all contingency scenarios.
- Line outage: $\boldsymbol{p}_c()$ and $\boldsymbol{q}_c()$ include new admittances $\boldsymbol{Y}_c$.
- One dispatch $(\boldsymbol{p}^g, \boldsymbol{q}^g)$ is computed for the nominal and all contingency scenarios.

- SC-AC-OPF costs $\geq$ classic AC-OPF since SC-AC-OPF has more constraints which shrinks the feasible solution region.
- If single-line outages are considered: N_c = number of lines
- If double-line outages are considered: $N_c = \frac{\text{number of lines (number of lines}-1)}{2}$
- Similarly, generator outage security constraints are added to SC-AC-OPF.

3.3.8 DC-OPF ($b\theta$-Formulation)

$$
\begin{aligned}
\min_{\boldsymbol{p},\boldsymbol{\theta}} \quad & c(\boldsymbol{p}) && \text{generation cost} \\
\text{s.t.} \quad & \boldsymbol{B\theta} = \boldsymbol{p} - \boldsymbol{d} && \text{active power balance} \\
& \underline{\boldsymbol{p}} \leq \boldsymbol{p} \leq \overline{\boldsymbol{p}} && \text{min/max gen p-limits} \\
& -\overline{f}_{nm} \leq b_{nm}(\theta_n - \theta_m) \leq \overline{f}_{nm}, && \text{power flow limits}
\end{aligned}
$$

- New notation: $\boldsymbol{p}^g \to \boldsymbol{p}$ and $\boldsymbol{p}^d \to \boldsymbol{d}$.
- Acts on the DC power flow approximation.
- Active power only; reactive power disregarded.
- Double-sided power flow constraints ensure that power does not exceed rated line capacity in either direction.

3.3.9 DC-OPF (PTDF Formulation)

- Formulate the DC-OPF problem in one variable $\boldsymbol{p}^g$ only.
- Use matrix $\boldsymbol{F} \in \mathbb{R}^{E \times N}$ of power transfer distribution factors (PTDF):
  - Defines how power flow in line e changes with a power injection at node n.
  - Obtained by manipulating the DC bus admittance matrix $\boldsymbol{B}$.
- Power flow is given by: $f = \boldsymbol{F}(\boldsymbol{p} - \boldsymbol{d})$ (distribution of net injections across power lines)
- The new DC-OPF formulation:

$$
\begin{aligned}
\min_{\boldsymbol{p}} \quad & c(\boldsymbol{p}) && \text{generation cost} \\
\text{s.t.} \quad & \boldsymbol{1}^\top(\boldsymbol{p} - \boldsymbol{d}) = \boldsymbol{0} && \text{active power balance} \\
& |\boldsymbol{F}(\boldsymbol{p} - \boldsymbol{d})| \leq \overline{f} && \text{power flow limits} \\
& \underline{\boldsymbol{p}} \leq \boldsymbol{p} \leq \overline{\boldsymbol{p}} && \text{min/max gen p-limits}
\end{aligned}
$$

- Less variables than in $b\theta$-formulation but requires more memory to store and operate with matrix $\boldsymbol{F}$.
- Often used for locational marginal pricing in high-voltage grids.

3.4 Locational Electricity Pricing

3.4.1 Duality of DC-OPF

- Focus on the coupling constraints (i.e., linking generators and loads):

$$\begin{aligned} \min_{\underline{\boldsymbol{p}} \leq \boldsymbol{p} \leq \overline{\boldsymbol{p}}} \quad & c(\boldsymbol{p}) \\ \text{s.t.} \quad & \mathbf{1}^\top(\boldsymbol{p}-\boldsymbol{d}) &= 0 \quad & : \lambda \\ & \boldsymbol{F}(\boldsymbol{p}-\boldsymbol{d}) &\leq \overline{f} \quad & : \overline{\boldsymbol{\mu}} \\ & -\boldsymbol{F}(\boldsymbol{p}-\boldsymbol{d}) &\leq \overline{f} \quad & : \underline{\boldsymbol{\mu}} \end{aligned}$$

- Partial Lagrangian function (dualizing only the coupling constraints):

$$\max_{\lambda, \overline{\boldsymbol{\mu}}, \underline{\boldsymbol{\mu}}} \quad \min_{\underline{\boldsymbol{p}} \leq \boldsymbol{p} \leq \overline{\boldsymbol{p}}} \quad \mathcal{L}(\boldsymbol{p}, \lambda, \overline{\boldsymbol{\mu}}, \underline{\boldsymbol{\mu}}) = c(\boldsymbol{p}) - \lambda \mathbf{1}^\top(\boldsymbol{p}-\boldsymbol{d}) + \overline{\boldsymbol{\mu}}^\top(\boldsymbol{F}(\boldsymbol{p}-\boldsymbol{d}) - \overline{\boldsymbol{f}}) + \underline{\boldsymbol{\mu}}^\top(-\boldsymbol{F}(\boldsymbol{p}-\boldsymbol{d}) - \overline{\boldsymbol{f}})$$

- Group terms corresponding to dispatch $\boldsymbol{p}$, demand $\boldsymbol{d}$, and line limits $\overline{\boldsymbol{f}}$:

$$\begin{aligned} \mathcal{L} &= \mathcal{L}^{\boldsymbol{p}} + \mathcal{L}^{\boldsymbol{d}} + \mathcal{L}^{\boldsymbol{f}}, \quad \text{where} \\ \mathcal{L}^{\boldsymbol{p}}(\boldsymbol{p}, \lambda, \overline{\boldsymbol{\mu}}, \underline{\boldsymbol{\mu}}) &= c(\boldsymbol{p}) - (\mathbf{1}\lambda - \boldsymbol{F}^\top \overline{\boldsymbol{\mu}} + \boldsymbol{F}^\top \underline{\boldsymbol{\mu}})^\top \boldsymbol{p} \\ \mathcal{L}^{\boldsymbol{d}}(\lambda, \overline{\boldsymbol{\mu}}, \underline{\boldsymbol{\mu}}) &= (\mathbf{1}\lambda - \boldsymbol{F}^\top \overline{\boldsymbol{\mu}} + \boldsymbol{F}^\top \underline{\boldsymbol{\mu}})^\top \boldsymbol{d} \\ \mathcal{L}^{\boldsymbol{f}}(\overline{\boldsymbol{\mu}}, \underline{\boldsymbol{\mu}}) &= -(\overline{\boldsymbol{\mu}} + \underline{\boldsymbol{\mu}})^\top \boldsymbol{f} \end{aligned}$$

Note: Power dispatch $\boldsymbol{p}$ and demand $\boldsymbol{d}$ share the same multiplier but with opposite signs.

3.4.2 Locational Marginal Prices (LMPs)

$$\boldsymbol{\pi}^\star(\lambda^\star, \overline{\boldsymbol{\mu}}^\star, \underline{\boldsymbol{\mu}}^\star) = \underbrace{\mathbf{1}\lambda^\star}_{\text{uniform}} \underbrace{- \boldsymbol{F}^\top(\overline{\boldsymbol{\mu}}^\star - \underline{\boldsymbol{\mu}}^\star)}_{\text{congestion}} \in \mathbb{R}^N$$

- $\boldsymbol{\pi}_n^\star$ is the cost of supplying the next unit of demand at node n.
- If congestion occurs ($\overline{\boldsymbol{\mu}}^\star > 0$ or $\underline{\boldsymbol{\mu}}^\star > 0$), electricity price varies across the grid.
- The price at the reference bus is $\boldsymbol{\lambda}^\star$ (since the reference column of $\boldsymbol{F}$ is zero).

3.4.3 Equilibrium Interpretation

- Equilibrium problem:

$$\boldsymbol{p}(\boldsymbol{\pi}^\star) = \underset{\underline{\boldsymbol{p}} \leq \boldsymbol{p} \leq \overline{\boldsymbol{p}}}{\text{argmin}} \quad \mathcal{L}^{\boldsymbol{p}}(\boldsymbol{p}, \boldsymbol{\pi}^\star) = c(\boldsymbol{p}) - \boldsymbol{\pi}^{\star\top} \boldsymbol{p}$$

The market operator finds equilibrium prices $\boldsymbol{\pi}^\star(\boldsymbol{\lambda}^\star, \overline{\boldsymbol{\mu}}^\star, \underline{\boldsymbol{\mu}}^\star)$ such that:

$$\begin{aligned} \mathbf{1}^\top(\boldsymbol{p}(\boldsymbol{\pi}^\star) - \boldsymbol{d}) &= 0 \\ |\boldsymbol{F}(\boldsymbol{p}(\boldsymbol{\pi}^\star) - \boldsymbol{d})| &\leq \overline{f} \end{aligned}$$

- Once equilibrium is found:
  - Inelastic demands are charged at $\boldsymbol{\pi}^\star \circ \boldsymbol{d}$.
  - Transmission operator collects congestion rent $(\overline{\boldsymbol{\mu}}^\star + \underline{\boldsymbol{\mu}}^\star)^\top \overline{\boldsymbol{f}}$.
  - If no congestion ($\overline{\boldsymbol{\mu}}^\star = \underline{\boldsymbol{\mu}}^\star = 0$), the problem reduces to economic dispatch.
  - Elastic demand $\rightarrow$ utility-maximization problem for each demand.

3.4.4 Some Desirable Market Properties

- **Market efficiency**: Equilibrium LMPs yield the least-cost dispatch.

  Proof: Extrapolate the solution to Assignment 1 (Problem 2) to the network-constrained case.

- **Cost recovery**: For fully dispatchable generators (i.e., $\underline{\boldsymbol{p}} = \boldsymbol{0}$).

  Proof:

Primal Problem	Dual Problem
$\max\limits_{\boldsymbol{p}} \quad {\boldsymbol{\pi}^\star}^\top \boldsymbol{p} - \boldsymbol{c}^\top \boldsymbol{p}$	$\min\limits_{\overline{\boldsymbol{\vartheta}},\underline{\boldsymbol{\vartheta}} \geq \boldsymbol{0}} \quad \overline{\boldsymbol{\vartheta}}^\top \overline{\boldsymbol{p}} - \underline{\boldsymbol{\vartheta}}^\top \underline{\boldsymbol{p}}$
subject to $\quad \underline{\boldsymbol{p}} \leq \boldsymbol{p} \leq \overline{\boldsymbol{p}}$	subject to $\quad \boldsymbol{c} - \boldsymbol{\pi}^\star + \overline{\boldsymbol{\vartheta}} - \underline{\boldsymbol{\vartheta}} = 0$

Strong duality: ${\boldsymbol{\pi}^\star}^\top \boldsymbol{p}^\star - \boldsymbol{c}^\top \boldsymbol{p}^\star = {\overline{\boldsymbol{\vartheta}}^\star}^\top \overline{\boldsymbol{p}} - {\underline{\boldsymbol{\vartheta}}^\star}^\top \underline{\boldsymbol{p}}$.

Since $\overline{\boldsymbol{\vartheta}}^\star, \underline{\boldsymbol{\vartheta}}^\star \geq \boldsymbol{0}$ and $\underline{\boldsymbol{p}} = \boldsymbol{0}$, the profit is non-negative and thus, generators that are paid LMP's are able to recover their costs.

- What holds for aggregate also holds for individual generators.
- **Revenue adequacy** in markets with fully dispatchable units (i.e., $\underline{\boldsymbol{p}} = \boldsymbol{0}$):

  Market operator does not run into deficit (demand charges $\geq$ generator revenues).

  Proof: The Lagrangian function in optimality decomposes into:

$$\underbrace{\mathcal{L}(\boldsymbol{p}^\star, \lambda^\star, \overline{\boldsymbol{\mu}}^\star, \underline{\boldsymbol{\mu}}^\star)}_{=c(\boldsymbol{p}^\star)} = \underbrace{c(\boldsymbol{p}^\star) - {\boldsymbol{\pi}^\star}^\top \boldsymbol{p}^\star}_{\text{minus profit}} + \underbrace{{\boldsymbol{\pi}^\star}^\top \boldsymbol{d}}_{\text{charges}} - \underbrace{\overline{\boldsymbol{f}}^\top (\overline{\boldsymbol{\mu}}^\star + \underline{\boldsymbol{\mu}}^\star)}_{\text{congestion rent}} \Longleftrightarrow {\boldsymbol{\pi}^\star}^\top \boldsymbol{d} - {\boldsymbol{\pi}^\star}^\top \boldsymbol{p}^\star = (\overline{\boldsymbol{\mu}}^\star + \underline{\boldsymbol{\mu}}^\star)^\top \boldsymbol{f}$$

We thus only need to show that congestion rent is non-negative:

The dual feasibility conditions $\overline{\boldsymbol{\mu}}^\star, \underline{\boldsymbol{\mu}}^\star \geq \boldsymbol{0}$ imply that ${\boldsymbol{\pi}^\star}^\top \boldsymbol{d} - {\boldsymbol{\pi}^\star}^\top \boldsymbol{p}^\star \geq 0$.

Since congestion rent is non-negative, revenue adequacy holds.

3.4.5 Unit Commitment (UC) Problem

$$\begin{aligned}
\min_{\boldsymbol{p}_t, \boldsymbol{u}_t} \quad & \sum_{t=1}^{T} c(\boldsymbol{p}_t, \boldsymbol{u}_t) \\
\text{subject to} \quad & \boldsymbol{1}^\top (\boldsymbol{p}_t - \boldsymbol{d}_t) = 0 \\
& \boldsymbol{F}(\boldsymbol{p}_t - \boldsymbol{d}_t) \leq \overline{f} \\
& -\boldsymbol{F}(\boldsymbol{p}_t - \boldsymbol{d}_t) \leq \overline{f} \\
& \boldsymbol{u}_t \circ \underline{\boldsymbol{p}} \leq \boldsymbol{p}_t \leq \boldsymbol{u}_t \circ \overline{\boldsymbol{p}} \\
& + \text{other constraints} \quad \forall t = 1, \ldots, T
\end{aligned}$$

- Binary unit commitment decisions $\boldsymbol{u}_t \in \{0, 1\}^N$.
- Discontinuous generator cost functions $\Rightarrow$ non-convex problem.
- Because of discontinuity, the dual variables do not exist.
- Mixed-integer (MI) linear (or quadratic) program.
- How to price electricity using unit commitment? Solve the UC problem to optimaly find the generators that need to be running, then solve an optimal power flow problem (such as DC-OPF).

3.4.6 Electricity Pricing with Discontinuous Costs

- Let $\boldsymbol{u}_1^\star, \ldots, \boldsymbol{u}_T^\star$ be the optimal UC decisions (e.g., after solving UC with an MI solver).
- Formulate the relaxed problem:

$$\begin{aligned}
\min_{\boldsymbol{p}_t, \boldsymbol{u}_t} \quad & \sum_{t=1}^{T} c(\boldsymbol{p}_t, \boldsymbol{u}_t) \\
\text{subject to} \quad & \begin{array}{rl}
\boldsymbol{1}^\top (\boldsymbol{p}_t - \boldsymbol{d}_t) = 0 & : \lambda_t \\
\boldsymbol{F}(\boldsymbol{p}_t - \boldsymbol{d}_t) \leq \overline{f} & : \boldsymbol{\mu}_t \\
-\boldsymbol{F}(\boldsymbol{p}_t - \boldsymbol{d}_t) \leq \overline{f} & : \underline{\boldsymbol{\mu}}_t \\
\boldsymbol{u}_t \circ \underline{\boldsymbol{p}} \leq \boldsymbol{p}_t \leq \boldsymbol{u}_t \circ \overline{\boldsymbol{p}} & \\
\boldsymbol{u}_t = \boldsymbol{u}_t^\star & : \boldsymbol{\vartheta}_t
\end{array}
\end{aligned}$$

  This is a convex optimization problem with the dual solution[3]:

  - Prices $\boldsymbol{\pi}_t^\star(\lambda_t^\star, \overline{\boldsymbol{\mu}}_t^\star, \underline{\boldsymbol{\mu}}_t^\star)$ and $\boldsymbol{\vartheta}_t^\star$ solve a competitive equilibrium with discontinuous costs.
  - Uplift payment $\boldsymbol{\vartheta}_t^\star \circ \boldsymbol{u}_t$ remunerates costs not covered by LMPs.
  - This applies to not fully dispatchable units ($\underline{\boldsymbol{p}} \neq 0$).

- UC is solved days ahead to compute uplifts, while OPF is solved later to price electricity.

[3] O'Neill et al. Efficient market-clearing prices in markets with nonconvexities. EJOR, 2025.