ECE 598: Computational Power Systems **Winter 2025**

Lecture 2: Duality, optimality conditions, and electricity pricing

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2.1 Introduction to Optimization Problems

2.1.1 Objective Function

The objective function defines the goal of the optimization problem, such as minimizing cost or maximizing efficiency.

2.1.2 Constraints

Constraints define the feasible region, limiting the solution space to satisfy problem-specific conditions.

- Inequality Constraints: $g_i(\boldsymbol{x}) \leq 0, \quad \forall i = 1, \ldots, m$
- Equality Constraints: $h_i(\boldsymbol{x}) = 0, \quad \forall i = 1, \ldots, p$

2.2 Lagrangian and Duality

2.2.1 Lagrangian Function

The Lagrangian function enables optimization with constraints by turning them into terms in the objective function.

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}) = c(\boldsymbol{x}) + \sum_{i=1}^{m} \lambda_i g_i(\boldsymbol{x}) + \sum_{j=1}^{p} \nu_j h_j(\boldsymbol{x})$$

- $\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\nu})$ combines the objective function and its constraints.

2.2.2 Lagrange Multipliers

Lagrange multipliers show how sensitive the objective value is to changes in a constraint.

- $\boldsymbol{\lambda}$ represents the Lagrange multipliers for inequality constraints.

- $\boldsymbol{\lambda} \geq 0$ ensures that the constraints are properly penalized if violated.
- $\boldsymbol{\nu}$ represents the Lagrange multipliers for equality constraints.
- $\boldsymbol{\nu}$ can be positive or negative.

2.2.3 Dual Function

The dual function provides a lower bound for the primal problem, enabling duality-based optimization.

$$\phi(\boldsymbol{\lambda}, \boldsymbol{\nu}) = \min_{\boldsymbol{x}} \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}) = \min_{\boldsymbol{x}} \Big(c(\boldsymbol{x}) + \sum_{i=1}^{m} \lambda_i g_i(\boldsymbol{x}) + \sum_{j=1}^{p} \nu_j h_j(\boldsymbol{x}) \Big)$$

- The Dual Function **minimizes** the Lagrangian Function.
- Always concave.

2.2.4 Dual Problem

$$\underset{\boldsymbol{\lambda}, \boldsymbol{\nu}}{\text{maximize}} \ \phi(\boldsymbol{\lambda}, \boldsymbol{\nu}), \quad \text{subject to } \boldsymbol{\lambda} \geq 0$$

- The Dual Problem **maximizes** the Dual Function.
- The Dual Problem provides the best lower bound for the primal solution.
- Always concave.

Solving the dual problem often simplifies the computation of optimal solutions for the primal problem.

2.3 Duality Concepts

2.3.1 Weak Duality

Weak duality ensures the dual solution provides a valid lower bound for the primal solution.

- Non-zero duality gap: $\phi(\boldsymbol{\lambda}^*, \boldsymbol{\nu}^*) \leq c(\boldsymbol{x}^*)$
- Common for non-convex problems.

2.3.2 Strong Duality

Strong duality guarantees that the primal and dual solutions are equal and optimal for convex problems.

- Zero duality gap: $\phi(\boldsymbol{\lambda}^*, \boldsymbol{\nu}^*) = c(\boldsymbol{x}^*)$
- Holds for convex problems.
- We can use $\boldsymbol{\lambda}^*$ and $\boldsymbol{\nu}^*$ to certify optimality of $\boldsymbol{x}^*$.

2.3.3 Complementary Slackness Condition

Complementary slackness connects active constraints to their Lagrange multipliers.

$$\sum_{i=1}^{m} \lambda_i^* g_i(\boldsymbol{x}^*) = 0 \quad \Rightarrow \quad \lambda_i^* g_i(\boldsymbol{x}^*) = 0, \quad \forall i = 1, \ldots, m$$

- $\lambda_i^* = 0 \Leftrightarrow$ the i-th constraint is **inactive** at the optimum.
- $\lambda_i^* > 0 \Leftrightarrow$ the i-th constraint is **active** at the optimum.

2.4 Karush-Kuhn-Tucker (KKT) Conditions

The KKT conditions are necessary for optimality in constrained optimization problems. They include:

1. **Primal feasibility:**
$$g_i(\boldsymbol{x}^*) \leq 0, \quad h_j(\boldsymbol{x}^*) = 0, \quad \forall i, j$$
$$\text{for economic dispatch:} \quad \underline{\boldsymbol{p}} \leq \boldsymbol{p} \leq \overline{\boldsymbol{p}}, \quad \mathbf{1}^T \boldsymbol{p} = d$$

2. **Dual feasibility:**
$$\boldsymbol{\lambda}^* \geq 0$$
$$\text{for economic dispatch:} \quad \underline{\boldsymbol{\lambda}}, \overline{\boldsymbol{\lambda}} \geq 0$$

3. **Lagrangian optimality:**
$$\nabla_{\boldsymbol{x}} c(\boldsymbol{x}^*) + \sum_i \lambda_i^* \nabla_{\boldsymbol{x}} g_i(\boldsymbol{x}^*) + \sum_j \nu_j^* \nabla_{\boldsymbol{x}} h_j(\boldsymbol{x}^*) = 0$$
$$\text{for economic dispatch:} \quad \nabla_{\boldsymbol{p}} c(\boldsymbol{p}) - \underline{\boldsymbol{\lambda}} + \overline{\boldsymbol{\lambda}} + \mathbf{1}\nu = 0$$

4. **Complementary slackness:**
$$\lambda_i^* g_i(\boldsymbol{x}^*) = 0, \quad \forall i$$
$$\text{for economic dispatch:} \quad \underline{\boldsymbol{\lambda}} \circ (\boldsymbol{p} - \underline{\boldsymbol{p}}) = 0, \quad \overline{\boldsymbol{\lambda}} \circ (\overline{\boldsymbol{p}} - \boldsymbol{p}) = 0$$

Note: For convex problems (minimization) or concave problems (maximization), if $\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\nu}^*$ satisfy the KKT conditions:

- There is zero duality gap: $c(\boldsymbol{x}^*) = \phi(\boldsymbol{\lambda}^*, \boldsymbol{\nu}^*)$
- The primal and dual solutions are globally optimal.

The KKT conditions combine feasibility, optimality, and slackness, providing a unified framework for solving constrained optimization problems.

2.5 Steps for Solving Optimization Problems

2.5.1 Linear Programs (LP)

LPs are optimization problems where duality and constraints are applied to determine optimal solutions.

1. **Construct the Lagrangian:** combine the linear objective function with the constraints
$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}) = \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{\lambda}^T(\boldsymbol{G}\boldsymbol{x} - \boldsymbol{h}) + \boldsymbol{\nu}^T(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}).$$
2. **Find the Dual Function:** minimize the Lagrangian with respect to x.
$$\phi(\boldsymbol{\lambda}, \boldsymbol{\nu}) = \min_{\boldsymbol{x}} \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}).$$
3. **Formulate the Dual Problem:** maximize the dual function over feasible $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$.
$$\text{maximize } \phi(\boldsymbol{\lambda}, \boldsymbol{\nu}) \quad \text{subject to } \boldsymbol{\lambda} \geq 0.$$
4. **Solve for the primal solution:** use the dual solution $(\boldsymbol{\lambda}^*, \boldsymbol{\nu}^*)$ and constraints to determine $\boldsymbol{x}^*$.
$$\boldsymbol{G}\boldsymbol{x} \leq \boldsymbol{h}, \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}.$$
5. **Code the solution and verify strong duality.**
$$c(\boldsymbol{x}^*) = \phi(\boldsymbol{\lambda}^*, \boldsymbol{\nu}^*)$$

2.5.2 Quadratic Programs (QP)

QPs extend LPs with quadratic terms, making them ideal for modeling energy systems and financial optimization problems.

1. **Construct the Lagrangian:** combine the quadratic objective function with the constraints.
$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}) = \boldsymbol{c}^T\boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{\lambda}^T(\boldsymbol{G}\boldsymbol{x} - \boldsymbol{h}) + \boldsymbol{\nu}^T(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}).$$
2. **Solve for $\boldsymbol{x}^*$:** take the derivative of the Lagrangian with respect to $\boldsymbol{x}$ and set it to zero.
$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{x}} = \boldsymbol{c} + \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{G}^T\boldsymbol{\lambda} + \boldsymbol{A}^T\boldsymbol{\nu} = 0.$$
3. **Find the Dual Function:** substitute $\boldsymbol{x}^*$ back into the Lagrangian.
$$\phi(\boldsymbol{\lambda}, \boldsymbol{\nu}) = \mathcal{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}, \boldsymbol{\nu}).$$
4. **Formulate the Dual Problem:** maximize the dual function over feasible $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$.
$$\text{maximize } \phi(\boldsymbol{\lambda}, \boldsymbol{\nu}) \quad \text{subject to } \boldsymbol{\lambda} \geq 0.$$
5. **Solve for the primal solution:** use the dual solution $(\boldsymbol{\lambda}^*, \boldsymbol{\nu}^*)$ and constraints to determine $\boldsymbol{x}^*$.
$$\boldsymbol{G}\boldsymbol{x} \leq \boldsymbol{h}, \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}.$$
6. **Code the solution and verify strong duality.**
$$c(\boldsymbol{x}^*) + \frac{1}{2}(\boldsymbol{x}^*)^T\boldsymbol{Q}\boldsymbol{x}^* = \phi(\boldsymbol{\lambda}^*, \boldsymbol{\nu}^*)$$

2.6 Economic Dispatch

Economic dispatch is the process of allocating generation among available units to minimize the total generation cost while satisfying system constraints. It ensures that power generation meets demand efficiently while adhering to operational limits.

2.6.1 Problem Formulation

$$\begin{aligned} \underset{\boldsymbol{p}=(p_1,\ldots,p_n)}{\text{minimize}} \quad & c(\boldsymbol{p}), \\ \text{subject to} \quad & \underline{\boldsymbol{p}} \leq \boldsymbol{p} \leq \overline{\boldsymbol{p}} \quad : \underline{\boldsymbol{\lambda}}, \overline{\boldsymbol{\lambda}} \\ & \mathbf{1}^T \boldsymbol{p} = d \qquad\quad : \nu \end{aligned}$$

- $c(\boldsymbol{p})$: **Dispatch cost.** Total cost of power generation as a function of power generation.
- $\overline{\boldsymbol{p}}$: **Dispatch limits.** Maximum generation limits for generators.
- $\underline{\boldsymbol{p}}$: **Dispatch limits.** Minimum generation limits for generators.
- d: **Power balance** Total power demand.
- $\underline{\boldsymbol{\lambda}}, \overline{\boldsymbol{\lambda}}, \nu$: Lagrange multipliers for constraints.

2.6.2 Steps to Solve Economic Dispatch

1. **Construct the Lagrangian:** combine the objective function and constraints.
$$\mathcal{L}(\boldsymbol{p}, \underline{\boldsymbol{\lambda}}, \overline{\boldsymbol{\lambda}}, \nu) = c(\boldsymbol{p}) + \underline{\boldsymbol{\lambda}}^T(\boldsymbol{p} - \underline{\boldsymbol{p}}) + \overline{\boldsymbol{\lambda}}^T(\overline{\boldsymbol{p}} - \boldsymbol{p}) + \nu(\mathbf{1}^T\boldsymbol{p} - d).$$
2. **Solve for the Dual Function:** minimize the Lagrangian with respect to p:
$$\phi(\underline{\boldsymbol{\lambda}}, \overline{\boldsymbol{\lambda}}, \nu) = \min_{\boldsymbol{p}} \mathcal{L}(\boldsymbol{p}, \underline{\boldsymbol{\lambda}}, \overline{\boldsymbol{\lambda}}, \nu).$$
For linear cost $c(\boldsymbol{p}) = \boldsymbol{c}^T\boldsymbol{p}$, the dual function becomes:
$$\phi(\underline{\boldsymbol{\lambda}}, \overline{\boldsymbol{\lambda}}, \nu) = \begin{cases} \overline{\boldsymbol{\lambda}}^T \overline{\boldsymbol{p}} - \underline{\boldsymbol{\lambda}}^T \underline{\boldsymbol{p}} - \nu d & \text{if } c - \underline{\boldsymbol{\lambda}} + \overline{\boldsymbol{\lambda}} + \mathbf{1}\nu = 0, \\ -\infty & \text{otherwise.} \end{cases}$$
3. **Formulate the Dual Problem:** maximize the dual function over feasible $\underline{\boldsymbol{\lambda}}, \overline{\boldsymbol{\lambda}}, \nu$:
$$\text{maximize } \phi(\underline{\boldsymbol{\lambda}}, \overline{\boldsymbol{\lambda}}, \nu), \quad \text{subject to } \underline{\boldsymbol{\lambda}}, \overline{\boldsymbol{\lambda}} \geq 0.$$
4. **Solve for the primal solution:** use the dual solution $(\underline{\boldsymbol{\lambda}}^*, \overline{\boldsymbol{\lambda}}^*, \nu^*)$ and constraints to determine $\boldsymbol{p}^*$.
$$\underline{\boldsymbol{p}} \leq \boldsymbol{p} \leq \overline{\boldsymbol{p}}, \quad \mathbf{1}^T\boldsymbol{p}^* = d.$$
5. **Interpret the electricity price:** the dual variable ν^* represents the marginal cost of electricity. It is calculated as follows $\boldsymbol{\zeta}$ is a marginal change in demand):
$$\nu^* = \frac{\partial \mathcal{L}(\boldsymbol{p}^*, \underline{\boldsymbol{\lambda}}^*, \overline{\boldsymbol{\lambda}}^*, \nu^*, \boldsymbol{\zeta})}{\partial \boldsymbol{\zeta}},$$
6. **Code the solution and verify strong duality.**
$$c(\boldsymbol{p}^*) = \phi(\underline{\boldsymbol{\lambda}}^*, \overline{\boldsymbol{\lambda}}^*, \nu^*),$$