ECE 598 Computational Power Systems

Online optimization for economic dispatch

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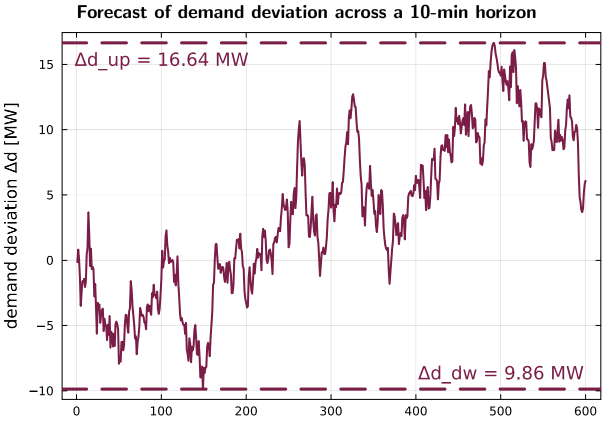
Last lecture recap

- Look around you and form teams of 2 people (1 min)
- Quickly review your notes or the slide deck (1 min)
- Share your **three** personal highlights with your partner (3 min)
- Get iClicker app ready

Real-time economic re-dispatch

- Re-dispatch generation w.r.t. any deviation from the day-ahead parameters
- Let Δd be the total demand deviation



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- Let Δd be the total demand deviation

- $\mathbf{p}_{\text{da}}$ is the scheduled, day-ahead dispatch for a particular period of time
- In real-time, generators deploy either upward ($\mathbf{r}_{\text{up}}$) and downward ($\mathbf{r}_{\text{dw}}$) regulation
- Typically, regulation costs are asymmetric, i.e., $\mathbf{c}_{\text{up}} \geqslant \mathbf{c}_{\text{da}} \geqslant \mathbf{c}_{\text{dw}} \geqslant \mathbf{0}$

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- For quartic generation cost function, the least-cost re-dispatch problem is

$$
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\underset{\mathbf{r}_{\text{up}},\mathbf{r}_{\text{dw}}}{\text{minimize}} \quad & \|\mathbf{p}_{\text{da}} + \mathbf{r}_{\text{up}} - \mathbf{r}_{\text{dw}}\|_{\mathbf{C}}^2 + \mathbf{c}_{\text{up}}^\top \mathbf{r}_{\text{up}} - \mathbf{c}_{\text{dw}}^\top \mathbf{r}_{\text{dw}} && \text{re-dispatch cost} \\
\text{subject to} \quad & \mathbf{1}^\top \mathbf{r}_{\text{up}} = \Delta d_{\text{up}} && \text{upward re-dispatch requirement} \\
& \mathbf{1}^\top \mathbf{r}_{\text{dw}} = \Delta d_{\text{dw}} && \text{downward re-dispatch requirement} \\
& \mathbf{0} \leqslant \mathbf{r}_{\text{up}} \leqslant \bar{\mathbf{r}}_{\text{up}} && \text{upward regulation limits} \\
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Q: What are the parameters and variables in this optimization?

Pricing real-time re-dispatch

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- System operators periodically (every 5 to 15 minutes) solve real-time re-dispatch.
- The power imbalances come from either measurements or short-term forecast
- The outcomes of real-time re-dipatch:
  - **Upward regulation:** generators maintain $r_{\text{up}}^\star$ MW in reserve for upward regulation; they are compensated with $\$\lambda_{\text{up}}^\star r_{\text{up}}^\star$
  - **Downward regulation:** generators maintain $r_{\text{dw}}^\star$ MW in reserve fow downward regulation; ; they are compensated with $\$\lambda_{\text{dw}}^\star r_{\text{dw}}^\star$

Q. What are the drawbacks of such market design?

Pricing real-time re-dispatch

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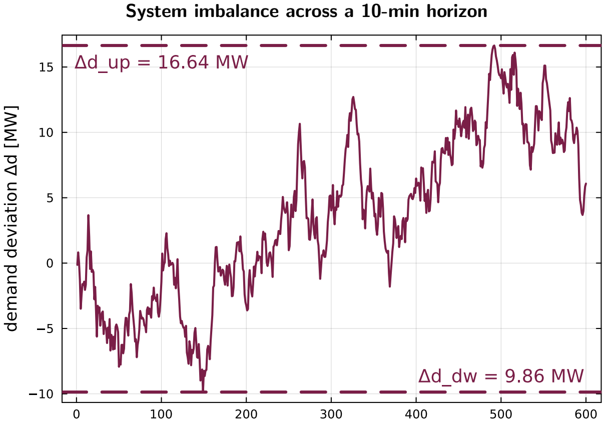
Q. What are the drawbacks of such market design?

- Diff. timescales of grid (in millisec to sec) and market (5-15 minutes) dynamics
- Prices $\lambda_{\text{up}}^\star$ and $\lambda_{\text{dw}}^\star$ do not reflect changes within the market-clearing period

Towards **online** economic re-dispatch

- Starting from $t = 1$, we take measurements $\Delta d(t)$ every Δt
- We continuously observe the imbalance sequence $\Delta d(1), \dots, \Delta d(t), \dots$



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Peer discussion:

- Look around you and form teams of 2 people (1 min)
- Select one person to represent your group, who will share your response (30 sec)
- Quickly review your notes or the slide deck from previous lecture (2 min)
- Answer the following questions (3 min):
  - How do you solve this problem in an offline fashion?
  - How do you solve this problem in an online fashion?
  - Which online algorithm suits well to solve this problem?

Application of mixed-saddle flow to economic re-dispatch

$$
\begin{aligned}
\underset{\mathbf{r}_{\mathrm{up}},\mathbf{r}_{\mathrm{dw}}}{\text{minimize}} \quad & \|\mathbf{p}_{\mathrm{da}} + \mathbf{r}_{\mathrm{up}} - \mathbf{r}_{\mathrm{dw}}\|_{\mathbf{C}}^2 + \mathbf{c}_{\mathrm{up}}^\top \mathbf{r}_{\mathrm{up}} - \mathbf{c}_{\mathrm{dw}}^\top \mathbf{r}_{\mathrm{dw}} & \\
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\end{aligned}
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Partial Lagrangian function:

$$
\mathcal{L}(\mathbf{r}_{\text{up}}, \mathbf{r}_{\text{dw}}, \lambda \mid \Delta d(t)) = \|\mathbf{p}_{\text{da}} + \mathbf{r}_{\text{up}} - \mathbf{r}_{\text{dw}}\|_{\mathbf{C}}^2 + \mathbf{c}_{\text{up}}^\top \mathbf{r}_{\text{up}} - \mathbf{c}_{\text{dw}}^\top \mathbf{r}_{\text{dw}} - \lambda(\mathbf{1}^\top(\mathbf{r}_{\text{up}} - \mathbf{r}_{\text{dw}}) - \Delta d(t))
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Q. Why do we dualize the constraint with "minus"? (there is economic interpretation)

Application of mixed-saddle flow to economic re-dispatch

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Q. Why do we dualize the constraint with “minus”? (there is economic interpretation)

Mixed-saddle flow:

$$\begin{aligned}
\dot{\mathbf{r}}_{\mathrm{up}} &= P_{[0,\bar{\mathbf{r}}_{\mathrm{up}}]}\left[-\nabla_{\mathbf{r}_{\mathrm{up}}} \mathcal{L}(\mathbf{r}_{\mathrm{up}}, \mathbf{r}_{\mathrm{dw}}, \lambda \mid \Delta d(t))\right] & \text{projected descent} \\
\dot{\mathbf{r}}_{\mathrm{dw}} &= P_{[0,\bar{\mathbf{r}}_{\mathrm{dw}}]}\left[-\nabla_{\mathbf{r}_{\mathrm{dw}}} \mathcal{L}(\mathbf{r}_{\mathrm{up}}, \mathbf{r}_{\mathrm{dw}}, \lambda \mid \Delta d(t))\right] & \text{projected descent} \\
\dot{\lambda} &= \nabla_{\lambda} \mathcal{L}(\mathbf{r}_{\mathrm{up}}, \mathbf{r}_{\mathrm{dw}}, \lambda \mid \Delta d(t)) = \underbrace{\mathbf{1}^\top(\mathbf{r}_{\mathrm{up}} - \mathbf{r}_{\mathrm{dw}}) - \Delta d(t)}_{\text{total imbalance } \delta(t)} & \text{ascent}
\end{aligned}$$

To update the dual price, we only need the measurement of the total imbalance $\delta(t)$!

Q. What feasibility guarantees can we establish? What are the consequences?

Implementation of the mixed-saddle flow

- Forward Euler discretization with a step size η:

for $t = 1, \ldots, +\infty$ **do**

$$\lambda(t+1) \leftarrow \lambda(t) + \eta\delta(t)$$

$$\mathbf{r}_{\mathrm{up}}(t+1) \leftarrow P_{[0,\bar{\mathbf{r}}_{\mathrm{up}}]}\left[\mathbf{r}_{\mathrm{up}}(t) - \eta\Big(2\mathbf{C}(\mathbf{p}_{\mathrm{da}} + \mathbf{r}_{\mathrm{up}}(t) - \mathbf{r}_{\mathrm{dw}}(t)) + \mathbf{c}_{\mathrm{up}} - \lambda(t+1)\mathbf{1}\Big)\right]$$

$$\mathbf{r}_{\mathrm{dw}}(t+1) \leftarrow P_{[0,\bar{\mathbf{r}}_{\mathrm{dw}}]}\left[\mathbf{r}_{\mathrm{dw}}(t) + \eta\Big(2\mathbf{C}(\mathbf{p}_{\mathrm{da}} + \mathbf{r}_{\mathrm{up}}(t) - \mathbf{r}_{\mathrm{dw}}(t)) + \mathbf{c}_{\mathrm{dw}} - \lambda(t+1)\mathbf{1}\Big)\right]$$

end for

- The step size η controls the rate of change of generator's response

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end for

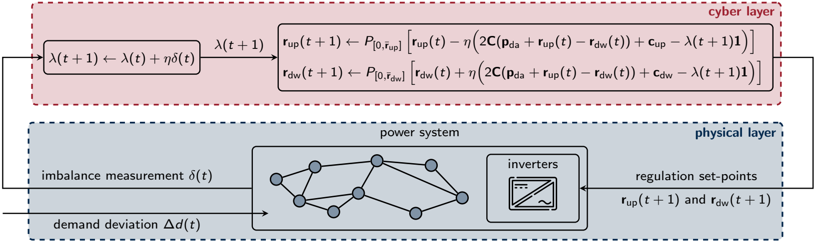
- The step size η controls the rate of change of generator's response

Peer discussion:

Q1. What are the practical considerations in selecting the step size η?

Q2. What information needs to be communicated to implement the algorithm?

Implementation of the mixed-saddle flow (cont'd)



- Online re-dispatch takes one measurement and broadcasts one signal
- Almost no communication burden (one public price signal $\lambda(t)$)
- This way, it is similar to regulation signals in the PJM market

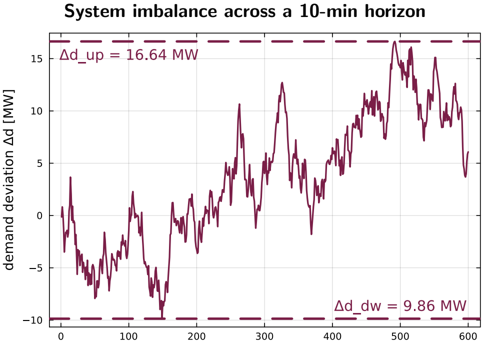
Q. Which optimization is easier to implement: offline or online?

Example: two-unit economic re-dispatch

A simplified economic re-dispatch (with no upper regulation limits):

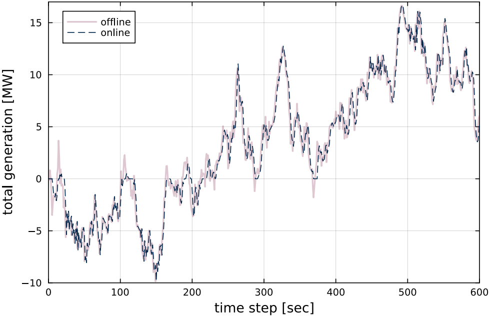
$$\begin{aligned} \underset{\mathbf{r}_{\text{up}},\mathbf{r}_{\text{dw}}}{\text{minimize}} \quad & \|\mathbf{p}_{\text{da}} + \mathbf{r}_{\text{up}} - \mathbf{r}_{\text{dw}}\|_{\mathbf{C}}^2 + \mathbf{c}_{\text{up}}^\top \mathbf{r}_{\text{up}} - \mathbf{c}_{\text{dw}}^\top \mathbf{r}_{\text{dw}} \\ \text{subject to} \quad & \mathbf{1}^\top(\mathbf{r}_{\text{up}} - \mathbf{r}_{\text{dw}}) = \Delta d(t) \\ & \mathbf{r}_{\text{up}} \geqslant 0, \quad \mathbf{r}_{\text{dw}} \geqslant 0 \end{aligned}$$

with $\mathbf{p}_{\text{da}} = \begin{bmatrix} 100 \\ 50 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 0.5 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$, $\mathbf{c}_{\text{up}} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$, $\mathbf{c}_{\text{dw}} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$, and imbalance dynamics:



Example: two-unit economic re-dispatch (cont'd)

- Step size $\eta = 0.7$, initial values for $\mathbf{r}_{\text{up}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{r}_{\text{dw}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\lambda = 110$
- Tracking the optimal solution with mixed-saddle flow

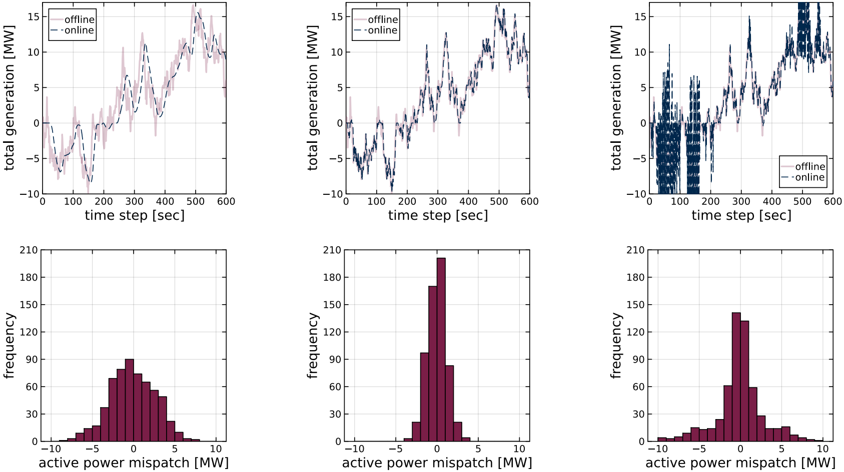


How good is the online solution?

Example: two-unit economic re-dispatch (cont'd)

Impact of the step size η on the total power mismatch

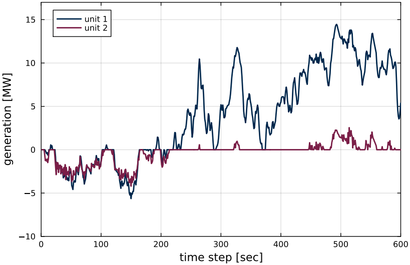
step size $\eta = 0.1$ | step size $\eta = 0.7$ | step size $\eta = 1.0$



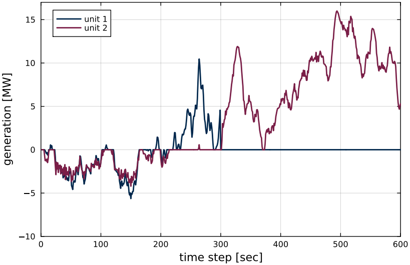
Example: two-unit economic re-dispatch (cont'd)

Impact of generator outage on dispatch dynamics

No generator outage



Unit 1 trips at $t = 300$



- Unit 1 is more economically efficient to provide upward regulation
- If unit 1 trips, unit 2 will pick up the load (as fast as the step size η allows)
- The mixed-saddle flow is robust to contingencies (e.g., generator or line outage)

Tutorial: Solve the following problem using safe gradient flow

$$\begin{aligned} \underset{\mathbf{x}\in\mathbb{R}^2}{\text{minimize}} \quad & 0.125\|\mathbf{x}\|_2^2 - 0.5x_1 + 0.25x_2 \\ \text{subject to} \quad & x_1 - x_2 \leqslant 0 \\ & x_2 \geqslant 0 \end{aligned}$$

- Control-affine dynamical system

$$\dot{x} = -\nabla f(\mathbf{x}) - \frac{\partial g(\mathbf{x})}{\partial \mathbf{x}}^\top \boldsymbol{\mu}$$

- At every step, select dual $\boldsymbol{\mu}$ by solving an optimization

$$\boldsymbol{\mu}(\mathbf{x}) \in \underset{\boldsymbol{\mu}\in K_\alpha(\mathbf{x})}{\text{argmin}} \left\| \frac{\partial g(\mathbf{x})}{\partial \mathbf{x}}^\top \boldsymbol{\mu} \right\|_2^2$$

where $K_\alpha(\mathbf{x})$ is the admissible control set:

$$K_\alpha(\mathbf{x}) = \left\{ \boldsymbol{\mu} \in \mathbb{R}_+^m \;\middle|\; -\frac{\partial g(\mathbf{x})}{\partial \mathbf{x}} \frac{\partial g(\mathbf{x})}{\partial \mathbf{x}}^\top \boldsymbol{\mu} \leqslant \frac{\partial g(\mathbf{x})}{\partial \mathbf{x}} \nabla f(\mathbf{x}) - \alpha g(\mathbf{x}) \right\}$$