ECE 598 Computational Power Systems

ADMM – Applications to OPF

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Last lecture recap

- Look around you and form teams of 2 people (1 min)
- Quickly review your notes or the slide deck (1 min)
- Share your **three** personal highlights with your partner (3 min)
- Get iClicker app ready

Recap of DC-OPF

Consensus ADMM for DC-OPF

Exchange ADMM for DC-OPF

ADMM for distribution AC-OPF

Recap of DC-OPF

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DC-OPF Recap - I

Only active power flow
$$P_n = V_n \sum_{m=1}^{N} V_m \left(G_{nm} \cos\theta_{nm} + B_{nm} \sin\theta_{nm} \right)$$

Assumptions

A1 Low r/x ratios in transmission lines (1/5-1/10 for 220-400kV)

$$r_{nm} \ll x_{nm} \quad \rightarrow \quad g_{nm} \ll b_{nm} \quad \rightarrow \quad \mathbf{G} \approx \mathbf{0} \quad \text{and} \quad b_{nm} = \frac{x_{nm}}{r_{nm}^2 + x_{nm}^2}$$

A2 Small angle difference $\sin(\theta_n - \theta_m) \approx \theta_n - \theta_m$

A3 Voltage magnitudes $V_n \approx 1$

DC power flow model

$$P_n \approx \sum_{m:n\sim m} P_{nm} = \sum_{m:n\sim m} b_{nm}(\theta_n - \theta_m)$$

DC-OPF Recap - II

$b\theta-$formulation

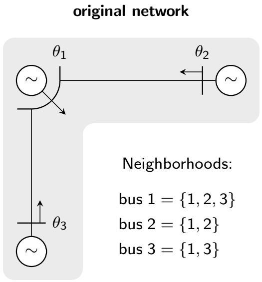
$$\begin{aligned}\underset{\mathbf{p},\boldsymbol{\theta}}{\text{minimize}} \quad & c(\mathbf{p}) \\ \text{subject to} \quad & \mathbf{B}\boldsymbol{\theta} = \mathbf{p} - \mathbf{d} \\ & \underline{\mathbf{p}} \leqslant \mathbf{p} \leqslant \overline{\mathbf{p}} \\ & |f(\boldsymbol{\theta})| \leqslant \bar{\mathbf{f}}\end{aligned}$$

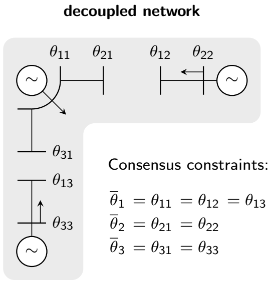
PTDF-based formulation

$$\begin{aligned}\underset{\mathbf{p}}{\text{minimize}} \quad & c(\mathbf{p}) \\ \text{subject to} \quad & \mathbf{1}^\top(\mathbf{p} - \mathbf{d}) = 0 \\ & |\mathbf{F}(\mathbf{p} - \mathbf{d})| \leqslant \bar{\mathbf{f}} \\ & \underline{\mathbf{p}} \leqslant \mathbf{p} \leqslant \overline{\mathbf{p}}\end{aligned}$$

- $f : \mathbb{R}^n \mapsto \mathbb{R}^e$ maps the vector of n voltage angles to e power flows
- The $b\theta-$formulation suits consensus ADMM (voltage decoupling)
- PTDF-based formulation suits exchange ADMM (LMP exchange)

Voltage variable decoupling and consensus





- Each node creates copies of the own and neighboring voltage angles
- Consensus constraints force local copies to agree on the same values
- Relaxing consensus constraints leads to a set of smaller OPF sub-problems
- No OPF data from the sub-problem needs to be shared (local optimization)

Decoupled DC-OPF and consensus

- The DC-OPF takes the following form:

$$\begin{aligned} \underset{\mathbf{p},\boldsymbol{\theta}}{\text{minimize}} \quad & c(\mathbf{p}) \\ \text{subject to} \quad & \mathbf{B}\boldsymbol{\theta} = \mathbf{p} - \mathbf{d} \\ & \underline{\mathbf{p}} \leqslant \mathbf{p} \leqslant \overline{\mathbf{p}} \\ & |f(\boldsymbol{\theta})| \leqslant \overline{\mathbf{f}} \end{aligned}$$

- The DC-OPF with decoupled voltage variables:

$$\begin{aligned} \underset{\mathbf{p},\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_n,\overline{\boldsymbol{\theta}}}{\text{minimize}} \quad & \sum_{i=1}^{n} c_i(p_i) && \\ \text{subject to} \quad & \mathbf{B}_i^\top \boldsymbol{\theta}_i = p_i - d_i && \textit{power balance for each bus } i \\ & \underline{p}_i \leqslant p_i \leqslant \overline{p}_i && \textit{gen limits for each bus } i \\ & |f_i(\boldsymbol{\theta}_i)| \leqslant \overline{\mathbf{f}}_i && \textit{flow limits for adjacent to bus } i \textit{ lines} \\ & \boldsymbol{\theta}_i = \overline{\boldsymbol{\theta}} \quad : \boldsymbol{\mu}_i && \textit{voltage consensus for each bus } i \end{aligned}$$

- Solutions to original and decoupled DC-OPF problems are the same

Consensus ADMM for decoupled DC-OPF

$$
\begin{aligned}
\underset{\mathbf{p},\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_n,\overline{\boldsymbol{\theta}}}{\text{minimize}} \quad & \sum_{i=1}^{n} c_i(p_i) \\
\text{subject to} \quad & \mathbf{B}_i^\top \boldsymbol{\theta}_i = p_i - d_i, && \forall i = 1, \ldots, n \\
& \underline{p}_i \leqslant p_i \leqslant \overline{p}_i, && \forall i = 1, \ldots, n \\
& |f_i(\boldsymbol{\theta}_i)| \leqslant \overline{\mathbf{f}}_i, && \forall i = 1, \ldots, n \\
& \boldsymbol{\theta}_i = \overline{\boldsymbol{\theta}} \quad : \boldsymbol{\mu}_i, && \forall i = 1, \ldots, n
\end{aligned}
$$

- Dualize the consensus constraint $\rightarrow$ the problem is separable per bus
- Add the regularization term for consensus constraints with some $\rho > 0$

Consensus ADMM for decoupled DC-OPF

$$
\begin{aligned}
\underset{\mathbf{p},\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_n,\overline{\boldsymbol{\theta}}}{\text{minimize}} \quad & \sum_{i=1}^{n} c_i(p_i) + \sum_{i=1}^{n} \boldsymbol{\mu}_i^\top (\boldsymbol{\theta}_i - \overline{\boldsymbol{\theta}}) \\
\text{subject to} \quad & \mathbf{B}_i^\top \boldsymbol{\theta}_i = p_i - d_i, \quad \forall i = 1, \ldots, n \\
& \underline{p}_i \leqslant p_i \leqslant \overline{p}_i, \qquad \forall i = 1, \ldots, n \\
& |f_i(\boldsymbol{\theta}_i)| \leqslant \overline{\mathbf{f}}_i, \qquad \forall i = 1, \ldots, n
\end{aligned}
$$

- Dualize the consensus constraint $\rightarrow$ the problem is separable per bus
- Add the regularization term for consensus constraints with some $\rho > 0$

Consensus ADMM for decoupled DC-OPF

$$
\begin{array}{ll}
\underset{\mathbf{p},\boldsymbol{\theta}_1,\dots,\boldsymbol{\theta}_n,\overline{\boldsymbol{\theta}}}{\text{minimize}} & \sum_{i=1}^{n} c_i(p_i) + \sum_{i=1}^{n} \boldsymbol{\mu}_i^\top (\boldsymbol{\theta}_i - \overline{\boldsymbol{\theta}}) + \sum_{i=1}^{n} \frac{\rho}{2} \left\| \boldsymbol{\theta}_i - \overline{\boldsymbol{\theta}} \right\|_2^2 \\
\text{subject to} & \mathbf{B}_i^\top \boldsymbol{\theta}_i = p_i - d_i, \quad \forall i = 1,\dots,n \\
& \underline{p}_i \leqslant p_i \leqslant \overline{p}_i, \quad \forall i = 1,\dots,n \\
& |f_i(\boldsymbol{\theta}_i)| \leqslant \overline{\mathbf{f}}_i, \quad \forall i = 1,\dots,n
\end{array}
$$

- Dualize the consensus constraint $\rightarrow$ the problem is separable per bus
- Add the regularization term for consensus constraints with some $\rho > 0$

Consensus ADMM algorithm

for $k = 1, \ldots, K$ **do**

update local copies of voltage angles

for $i = 1, \ldots, n$ **do**

$$\boldsymbol{\theta}_i^k = \underset{p_i, \boldsymbol{\theta}_i}{\operatorname{argmin}}\ c_i(p_i) + \boldsymbol{\mu}_i^{k-1\top} \boldsymbol{\theta}_i + \frac{\rho}{2} \left\| \boldsymbol{\theta}_i - \overline{\boldsymbol{\theta}}^{k-1} \right\|_2^2$$

subject to local OPF constraints

end for

update consensus variable

$$\overline{\boldsymbol{\theta}}^k = \underset{\overline{\boldsymbol{\theta}}}{\operatorname{argmin}}\ -\sum_{i=1}^{n} \boldsymbol{\mu}_i^{k-1\top} \overline{\boldsymbol{\theta}} + \sum_{i=1}^{n} \frac{\rho}{2} \left\| \boldsymbol{\theta}_i^{k-1} - \overline{\boldsymbol{\theta}} \right\|_2^2$$

update the dual variable

for $i = 1, \ldots, n$ **do**

$$\boldsymbol{\mu}_i^k = \boldsymbol{\mu}_i^{k-1} + \rho(\boldsymbol{\theta}_i^k - \overline{\boldsymbol{\theta}}^k)$$

end for

end for

- Iterate until the primal residuals $\|\boldsymbol{\theta}_i^k - \overline{\boldsymbol{\theta}}^k\| \leqslant \varepsilon_{\text{tol}}\ \forall i$
- Suffices to communicate only with neighbors
- Which sub-problems admit closed-form solutions?

Decentralizing DC-OPF

$$
\begin{aligned}
\underset{\underline{\mathbf{p}}\leqslant\mathbf{p}\leqslant\overline{\mathbf{p}}}{\text{minimize}} \quad & c(\mathbf{p}) \\
\text{subject to} \quad & \mathbf{1}^\top(\mathbf{p}-\mathbf{d}) = 0 && : \lambda \\
& \mathbf{F}(\mathbf{p}-\mathbf{d}) \leqslant \bar{\mathbf{f}} && : \overline{\boldsymbol{\mu}} \\
& -\mathbf{F}(\mathbf{p}-\mathbf{d}) \leqslant \bar{\mathbf{f}} && : \underline{\boldsymbol{\mu}}
\end{aligned}
$$

- Dualize the coupling constraints, regroup the terms to form LMPs
- Add regularization terms for coupling constraints with some $\rho > 0$
- Why power flow-related terms use the max operator? E.g., $\max\{-1, 0\} = 0$.

Decentralizing DC-OPF

$$\underset{\underline{\mathbf{p}}\leqslant\mathbf{p}\leqslant\overline{\mathbf{p}}}{\text{minimize}} \quad c(\mathbf{p}) - \underbrace{(\mathbf{1}\lambda - \mathbf{F}^\top\overline{\boldsymbol{\mu}} + \mathbf{F}^\top\underline{\boldsymbol{\mu}})^\top}_{\text{locational marginal price } \boldsymbol{\pi}}(\mathbf{p}-\mathbf{d})$$

$$\begin{aligned} \text{subject to} \quad \mathbf{1}^\top(\mathbf{p}-\mathbf{d}) &= 0 && : \lambda \\ \mathbf{F}(\mathbf{p}-\mathbf{d}) &\leqslant \overline{\mathbf{f}} && : \overline{\boldsymbol{\mu}} \\ -\mathbf{F}(\mathbf{p}-\mathbf{d}) &\leqslant \overline{\mathbf{f}} && : \underline{\boldsymbol{\mu}} \end{aligned}$$

- Dualize the coupling constraints, regroup the terms to form LMPs
- Add regularization terms for coupling constraints with some $\rho > 0$
- Why power flow-related terms use the max operator? E.g., $\max\{-1, 0\} = 0$.

Decentralizing DC-OPF

$$
\begin{aligned}
\underset{\underline{\mathbf{p}} \leqslant \mathbf{p} \leqslant \overline{\mathbf{p}}}{\text{minimize}} \quad & c(\mathbf{p}) - \underbrace{(\mathbf{1}\lambda - \mathbf{F}^\top \overline{\boldsymbol{\mu}} + \mathbf{F}^\top \underline{\boldsymbol{\mu}})^\top}_{\text{locational marginal price } \boldsymbol{\pi}} (\mathbf{p} - \mathbf{d}) + \frac{\rho}{2} \left\| \mathbf{1}^\top (\mathbf{p} - \mathbf{d}) \right\|_2^2 \\
& + \frac{\rho}{2} \left\| \max \left\{ \mathbf{F}(\mathbf{p} - \mathbf{d}) - \bar{\mathbf{f}}, \mathbf{0} \right\} \right\|_2^2 + \frac{\rho}{2} \left\| \max \left\{ \overline{-\mathbf{f}} - \mathbf{F}(\mathbf{p} - \mathbf{d}), \mathbf{0} \right\} \right\|_2^2 \\
\text{subject to} \quad & \mathbf{1}^\top (\mathbf{p} - \mathbf{d}) = 0 \quad : \lambda \\
& \mathbf{F}(\mathbf{p} - \mathbf{d}) \leqslant \bar{\mathbf{f}} \quad : \overline{\boldsymbol{\mu}} \\
& -\mathbf{F}(\mathbf{p} - \mathbf{d}) \leqslant \bar{\mathbf{f}} \quad : \underline{\boldsymbol{\mu}}
\end{aligned}
$$

- Dualize the coupling constraints, regroup the terms to form LMPs
- Add regularization terms for coupling constraints with some $\rho > 0$
- Why power flow-related terms use the max operator? E.g., $\max\{-1, 0\} = 0$.

Exchange ADMM: preparation

- At every iteration k, the central agent computes LMPs

$$\pi^k = \mathbf{1}\lambda^k - \mathbf{F}^\top \overline{\boldsymbol{\mu}}^k + \mathbf{F}^\top \underline{\boldsymbol{\mu}}^k \in \mathbb{R}^n$$

- and *private* constraint mismatch for each agent $i = 1, \ldots, n$

$$\begin{aligned}
\Delta p_i^k &= \mathbf{1}^\top(\mathbf{p}^k - \mathbf{d}) - p_i^k \in \mathbb{R} \\
\Delta f_{+i}^k &= \mathbf{F}(\mathbf{p}^k - \mathbf{d}) - \bar{\mathbf{f}} - \mathbf{F}_i^\top p_i^k \in \mathbb{R}^e \\
\Delta f_{-i}^k &= -\bar{\mathbf{f}} - \mathbf{F}(\mathbf{p}^k - \mathbf{d}) - \mathbf{F}_i^\top p_i^k \in \mathbb{R}^e
\end{aligned}$$

Exchange ADMM: algorithm

for $k = 1, \ldots, K$ **do**

update dispatch decision in response to LMP and constraint mismatch

for $i = 1, \ldots, n$ **do**

$$p_i^k = \underset{\underline{p}_i \leqslant\ p_i \leqslant \overline{p}_i}{\operatorname{argmin}} \quad c_i(p_i) - \pi_i^{k-1} p_i + \frac{\rho}{2} \left\| \Delta p_i^{k-1} - p_i \right\|_2^2$$

$$+ \frac{\rho}{2} \left\| \max\left\{ \Delta f_{+i}^{k-1} - \mathbf{F}_i^\top p_i, 0 \right\} \right\|_2^2 + \frac{\rho}{2} \left\| \max\left\{ \Delta f_{-i}^{k-1} - \mathbf{F}_i^\top p_i, 0 \right\} \right\|_2^2$$

end for

update dual variables

for $i = 1, \ldots, e$ **do**

$$\overline{\boldsymbol{\mu}}_i^k = \max\left\{ \overline{\boldsymbol{\mu}}_i^{k-1} + \rho(\mathbf{F}(\mathbf{p}^k - \mathbf{d}) - \bar{\mathbf{f}}), \mathbf{0} \right\}$$

$$\underline{\boldsymbol{\mu}}_i^k = \max\left\{ \underline{\boldsymbol{\mu}}_i^{k-1} + \rho(-\bar{\mathbf{f}} - \mathbf{F}(\mathbf{p}^k - \mathbf{d})), \mathbf{0} \right\}$$

end for

$$\lambda^k = \lambda^{k-1} + \varrho(\mathbf{1}^\top (\mathbf{p}^k - \mathbf{d}))$$

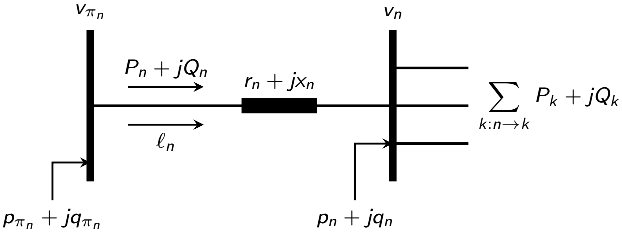
update auxiliary variables

$$\pi^k, \Delta p_i^k, \Delta f_{+i}^k, \Delta f_{-i}^k$$

end for

- Iterate until all primal residuals reach tolerance ε_{tol}
- Requires centralized update of LMPs and constraint mismatches
- Does the sub-problem admit a closed-form solution?

Relaxed branch flow model



$$\sum_{k:n\to k} P_k = p_n + P_n - r_n\ell_n$$

$$\sum_{k:n\to k} Q_k = q_n + Q_n - x_n\ell_n$$

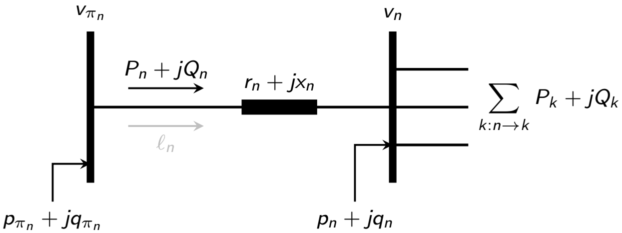
$$v_n = v_{\pi_n} - 2r_nP_n - 2x_nQ_n + (r_n^2 + x_n^2)\ell_n$$

$$\ell_n = \frac{P_n^2 + Q_n^2}{v_{\pi_n}}$$

- Radial networks (tree-like topology)
- Squared $v_n = |V_n|^2$ and $\ell_n = |I_n|^2$
- Voltage and current phases are dropped

Linearized distribution flow (LinDistFlow)

- Approximate model to overcome the complexity of quadratic equations
- Derived from forward DistFlow model upon dropping terms related to losses
- Voltage drop and flows are approximately linearly related to power injections



$$\sum_{k:n\to k} P_k \approx p_n + P_n - r_n\ell_n$$

$$\sum_{k:n\to k} Q_k \approx q_n + Q_n - x_n\ell_n$$

$$v_n \approx v_{\pi_n} - 2r_nP_n - 2x_nQ_n + (r_n^2 + x_n^2)\ell_n$$

- Over-estimates squared voltage magnitudes, under-estiamtes line flows

LinDistFlow in compact form

- Matrix-vector notation using branch-bus incidence matrix
- Does not explicitly model active and reactive power flows
- Relates voltages to injections using symmetric positive definite matrices $\mathbf{R}$ and $\mathbf{X}$

$$\mathbf{v} = v_0\mathbf{1} + \mathbf{R}\mathbf{p} + \mathbf{X}\mathbf{q}$$

- Voltage v_0 at the substation (root node). Typically, $v_0 = 1$ p.u.
- Voltage $\mathbf{v}$ and power injection $(\mathbf{p}, \mathbf{q})$ at non-root node

See Lecture 11 of Prof. Kekatos's course for details

`https://engineering.purdue.edu/~kekatos/pdsa.html`

Distribution AC-OPF based on LinDistFlow model

- Optimize distributed energy resources (DER) to keep voltages in the limits
- Active power injections are given, but reactive power injections can be controlled
- Optimize DER set-points on LinDistFlow equations to keep voltage within limits

$$\begin{aligned} \underset{\mathbf{v},\mathbf{q}}{\text{minimize}} \quad & \frac{1}{2}\mathbf{q}^\top \mathbf{C}\mathbf{q} \\ \text{subject to} \quad & \mathbf{v} = v_0\mathbf{1} + \mathbf{R}\mathbf{p} + \mathbf{X}(\widehat{\mathbf{q}} + \mathbf{q}) \\ & \underline{\mathbf{v}} \leqslant \mathbf{v} \leqslant \overline{\mathbf{v}} \\ & \underline{\mathbf{q}} \leqslant \mathbf{q} \leqslant \overline{\mathbf{q}} \end{aligned}$$

- Simplified AC power flow modeling, but often used in practice
- Matrix **C** prioritizes DERs for voltage control (e.g., based on capacity)

Dualizing distribution AC-OPF

$$
\begin{aligned}
\underset{\mathbf{v},\mathbf{q}}{\text{minimize}} \quad & \frac{1}{2}\mathbf{q}^\top \mathbf{C}\mathbf{q} \\
\text{subject to} \quad & \mathbf{v} = v_0 \mathbf{1} + \mathbf{R}\mathbf{p} + \mathbf{X}(\widehat{\mathbf{q}} + \mathbf{q}) \quad : \boldsymbol{\lambda} \\
& \underline{\mathbf{v}} \leqslant \mathbf{v} \leqslant \overline{\mathbf{v}} \\
& \underline{\mathbf{q}} \leqslant \mathbf{q} \leqslant \overline{\mathbf{q}}
\end{aligned}
$$

- Dualize the coupling voltage constraint
- Add regularization terms for coupling constraints with some $\rho > 0$
- Is the resulting problem separable?

Dualizing distribution AC-OPF

$$
\begin{aligned}
\underset{\mathbf{v},\mathbf{q}}{\text{minimize}} \quad & \frac{1}{2}\mathbf{q}^\top \mathbf{C}\mathbf{q} + \boldsymbol{\lambda}^\top (v_0\mathbf{1} + \mathbf{R}\mathbf{p} + \mathbf{X}(\widehat{\mathbf{q}} + \mathbf{q}) - \mathbf{v}) \\
\text{subject to} \quad & \mathbf{v} = v_0\mathbf{1} + \mathbf{R}\mathbf{p} + \mathbf{X}(\widehat{\mathbf{q}} + \mathbf{q}) \quad : \boldsymbol{\lambda} \\
& \underline{\mathbf{v}} \leqslant \mathbf{v} \leqslant \overline{\mathbf{v}} \\
& \underline{\mathbf{q}} \leqslant \mathbf{q} \leqslant \overline{\mathbf{q}}
\end{aligned}
$$

- Dualize the coupling voltage constraint
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Dualizing distribution AC-OPF

$$\begin{aligned}
\underset{\mathbf{v},\mathbf{q}}{\text{minimize}} \quad & \frac{1}{2}\mathbf{q}^\top \mathbf{C}\mathbf{q} + \boldsymbol{\lambda}^\top (v_0\mathbf{1} + \mathbf{R}\mathbf{p} + \mathbf{X}(\widehat{\mathbf{q}} + \mathbf{q}) - \mathbf{v}) + \frac{\rho}{2}\left\| v_0\mathbf{1} + \mathbf{R}\mathbf{p} + \mathbf{X}(\widehat{\mathbf{q}} + \mathbf{q}) - \mathbf{v}\right\|_2^2 \\
\text{subject to} \quad & \mathbf{v} = v_0\mathbf{1} + \mathbf{R}\mathbf{p} + \mathbf{X}(\widehat{\mathbf{q}} + \mathbf{q}) \quad : \boldsymbol{\lambda} \\
& \underline{\mathbf{v}} \leqslant \mathbf{v} \leqslant \overline{\mathbf{v}} \\
& \underline{\mathbf{q}} \leqslant \mathbf{q} \leqslant \overline{\mathbf{q}}
\end{aligned}$$

- Dualize the coupling voltage constraint
- Add regularization terms for coupling constraints with some $\rho > 0$
- Is the resulting problem separable?

Voltage sub-problem

$$\underset{\mathbf{v},\mathbf{q}}{\text{minimize}} \quad \frac{1}{2}\mathbf{q}^\top \mathbf{C}\mathbf{q} + \boldsymbol{\lambda}^\top (v_0\mathbf{1} + \mathbf{R}\mathbf{p} + \mathbf{X}(\widehat{\mathbf{q}} + \mathbf{q}) - \mathbf{v}) + \frac{\rho}{2}\left\|v_0\mathbf{1} + \mathbf{R}\mathbf{p} + \mathbf{X}(\widehat{\mathbf{q}} + \mathbf{q}) - \mathbf{v}\right\|_2^2$$

Sub-problem for voltage variable:

$$\begin{aligned} \mathbf{v}^k = \underset{\mathbf{v}}{\text{argmin}} \quad & -\boldsymbol{\lambda}^{k-1\top}\mathbf{v} + \frac{\rho}{2}\left\|v_0\mathbf{1} + \mathbf{R}\mathbf{p} + \mathbf{X}(\widehat{\mathbf{q}} + \mathbf{q}^{k-1}) - \mathbf{v}\right\|_2^2 \\ \text{subject to} \quad & \underline{\mathbf{v}} \leqslant \mathbf{v} \leqslant \overline{\mathbf{v}} \end{aligned}$$

Does it have a closed-form solution?

Voltage sub-problem

$$\underset{\mathbf{v},\mathbf{q}}{\text{minimize}} \quad \frac{1}{2}\mathbf{q}^\top \mathbf{C}\mathbf{q} + \boldsymbol{\lambda}^\top (v_0\mathbf{1} + \mathbf{R}\mathbf{p} + \mathbf{X}(\widehat{\mathbf{q}} + \mathbf{q}) - \mathbf{v}) + \frac{\rho}{2}\left\| v_0\mathbf{1} + \mathbf{R}\mathbf{p} + \mathbf{X}(\widehat{\mathbf{q}} + \mathbf{q}) - \mathbf{v} \right\|_2^2$$

Sub-problem for voltage variable:

$$\begin{aligned} \mathbf{v}^k = \underset{\mathbf{v}}{\text{argmin}} & \quad -\boldsymbol{\lambda}^{k-1\top}\mathbf{v} + \frac{\rho}{2}\left\| v_0\mathbf{1} + \mathbf{R}\mathbf{p} + \mathbf{X}(\widehat{\mathbf{q}} + \mathbf{q}^{k-1}) - \mathbf{v} \right\|_2^2 \\ \text{subject to} & \quad \underline{\mathbf{v}} \leqslant \mathbf{v} \leqslant \overline{\mathbf{v}} \end{aligned}$$

Does it have a closed-form solution? **Yes!**

Step 1: $\tilde{\mathbf{v}}^k = \frac{1}{\rho}\boldsymbol{\lambda}^{k-1} + v_0\mathbf{1} + \mathbf{R}\mathbf{p} + \mathbf{X}(\widehat{\mathbf{q}} + \mathbf{q}^{k-1})$ *from the derivative*

Step 2: $\mathbf{v}^k = \max\{\underline{\mathbf{v}}, \min\{\tilde{\mathbf{v}}^k, \overline{\mathbf{v}}\}\}$ *projection of $\tilde{\mathbf{v}}^k$ on the feasible voltage range*

No optimization is required!

Reactive power sub-problem

$$\underset{\mathbf{q},\mathbf{v}}{\text{minimize}} \quad \frac{1}{2}\mathbf{q}^\top \mathbf{C}\mathbf{q} + \boldsymbol{\lambda}^\top (v_0\mathbf{1} + \mathbf{R}\mathbf{p} + \mathbf{X}(\widehat{\mathbf{q}} + \mathbf{q}) - \mathbf{v}) + \frac{\rho}{2}\|v_0\mathbf{1} + \mathbf{R}\mathbf{p} + \mathbf{X}(\widehat{\mathbf{q}} + \mathbf{q}) - \mathbf{v}\|_2^2$$

Sub-problem for reactive power variable:

$$\begin{aligned} \mathbf{q}^k = \underset{\mathbf{q}}{\operatorname{argmin}} \quad & \frac{1}{2}\mathbf{q}^\top \mathbf{C}\mathbf{q} + \boldsymbol{\lambda}^{k-1\top}\mathbf{X}\mathbf{q} + \frac{\rho}{2}\left\|v_0\mathbf{1} + \mathbf{R}\mathbf{p} + \mathbf{X}(\widehat{\mathbf{q}} + \mathbf{q}) - \mathbf{v}^{k-1}\right\|_2^2 \\ \text{subject to} \quad & \underline{\mathbf{q}} \leqslant \mathbf{q} \leqslant \overline{\mathbf{q}} \end{aligned}$$

Does it have a closed-form solution?

Reactive power sub-problem

$$\underset{\mathbf{q},\mathbf{v}}{\text{minimize}} \quad \frac{1}{2}\mathbf{q}^\top \mathbf{C}\mathbf{q} + \boldsymbol{\lambda}^\top (v_0\mathbf{1} + \mathbf{R}\mathbf{p} + \mathbf{X}(\widehat{\mathbf{q}} + \mathbf{q}) - \mathbf{v}) + \frac{\rho}{2}\left\| v_0\mathbf{1} + \mathbf{R}\mathbf{p} + \mathbf{X}(\widehat{\mathbf{q}} + \mathbf{q}) - \mathbf{v}\right\|_2^2$$

Sub-problem for reactive power variable:

$$\begin{aligned} \mathbf{q}^k = \underset{\mathbf{q}}{\text{argmin}} \quad & \frac{1}{2}\mathbf{q}^\top \mathbf{C}\mathbf{q} + \boldsymbol{\lambda}^{k-1\top}\mathbf{X}\mathbf{q} + \frac{\rho}{2}\left\| v_0\mathbf{1} + \mathbf{R}\mathbf{p} + \mathbf{X}(\widehat{\mathbf{q}} + \mathbf{q}) - \mathbf{v}^{k-1}\right\|_2^2 \\ \text{subject to} \quad & \underline{\mathbf{q}} \leqslant \mathbf{q} \leqslant \overline{\mathbf{q}} \end{aligned}$$

Does it have a closed-form solution? **Yes!**

Step 1: $\tilde{\mathbf{q}}^k = -\left(\mathbf{I} + \rho\mathbf{X}^\top\mathbf{X}\right)^{-1}\mathbf{X}^\top\left(\boldsymbol{\lambda}^{k-1} + \rho\left(v_0\mathbf{1} + \mathbf{R}\mathbf{p} + \mathbf{X}\widehat{\mathbf{q}} - \mathbf{v}^{k-1}\right)\right)$

Step 2: $\mathbf{q}^k = \max\{\underline{\mathbf{q}}, \min\{\tilde{\mathbf{q}}^k, \overline{\mathbf{q}}\}\}$

No optimization is required!

ADMM for distribution AC-OPF

for $k = 1, \dots, K$ **do**

Voltage update

$$\tilde{\mathbf{v}}^k = \frac{1}{\rho}\boldsymbol{\lambda}^{k-1} + v_0\mathbf{1} + \mathbf{R}\mathbf{p} + \mathbf{X}(\widehat{\mathbf{q}} + \mathbf{q}^{k-1})$$

$$\mathbf{v}^k = \max\{\underline{\mathbf{v}}, \min\{\tilde{\mathbf{v}}^k, \overline{\mathbf{v}}\}\}$$

Reactive power update

$$\tilde{\mathbf{q}}^k = -\left(\mathbf{I} + \rho\mathbf{X}^\top\mathbf{X}\right)^{-1}\mathbf{X}^\top\left(\boldsymbol{\lambda}^{k-1} + \rho\left(v_0\mathbf{1} + \mathbf{R}\mathbf{p} + \mathbf{X}\widehat{\mathbf{q}} - \mathbf{v}^{k-1}\right)\right)$$

$$\mathbf{q}^k = \max\{\underline{\mathbf{q}}, \min\{\tilde{\mathbf{q}}^k, \overline{\mathbf{q}}\}\}$$

Dual variable update

$$\boldsymbol{\lambda}^k = \boldsymbol{\lambda}^{k-1} + \rho(v_0\mathbf{1} + \mathbf{R}\mathbf{p} + \mathbf{X}(\widehat{\mathbf{q}} + \mathbf{q}^k) - \mathbf{v}^k)$$

end for

Iterate until the primal residual reaches tolerance ε_{tol}

Resources

- Boyd, S., *et al*. (2011). Distributed optimization and statistical learning via the alternating direction method of multipliers. Foundations and Trends in Machine learning, 3(1), 1-122. [§2,§3,§7]

- Wright, J., & Ma, Y. (2022). High-dimensional data analysis with low-dimensional models: Principles, computation, and applications. Cambridge University Press. [§8.4,§8.5]

- Molzahn, Daniel K., et al. "A survey of distributed optimization and control algorithms for electric power systems." IEEE Transactions on Smart Grid 8.6 (2017): 2941-2962.

Tutorial: ADMM for distribution OPF

- 15-bus radial distribution network
- 14 loads, 5 redDER
- Use `tutorial_4_ADMM_AC_OPF`
- Re-write the voltage and reactive power updates using their respective closed forms