ECE 598 Computational Power Systems

Optimal power flow & Locational pricing

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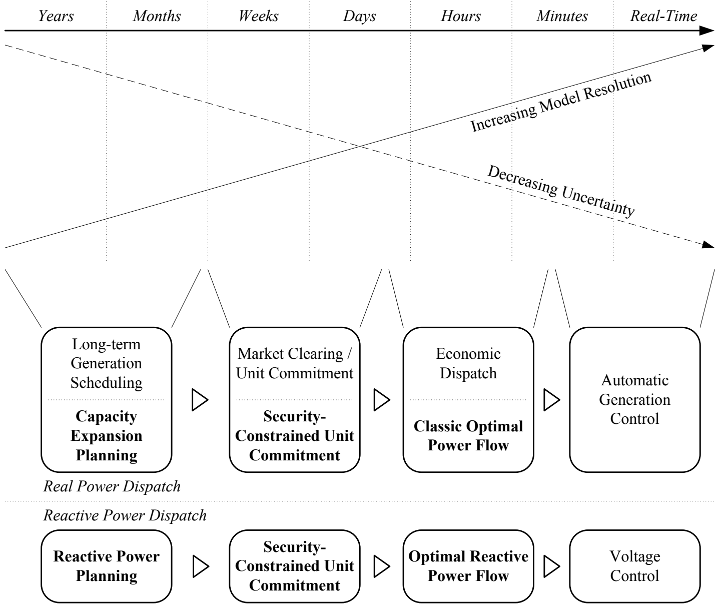
Last lecture recap

- Look around you and form teams of 2 people (1 min)
- Quickly review your notes or the slide deck (1 min)
- Share your **three** personal highlights with your partner (3 min)
- Get iClicker app ready



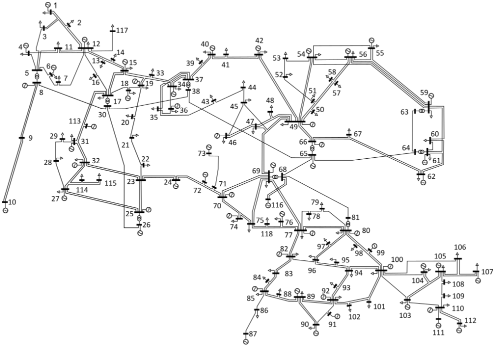
S. Frank and S. Rebennack. An introduction to optimal power flow: Theory, formulation, and examples. IIE Transactions, 2016

1. Power flow models

2. Optimal power flow

3. Locational electricity pricing

Power transmission network as an electric circuit

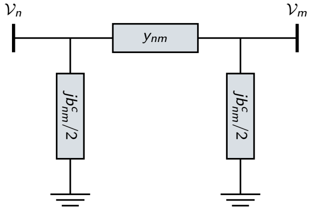


- N nodes (generator/load buses) and E edges (lines, transformers)
- AC voltages and currents as phasors (at nominal frequency)

$$\mathcal{V} = Ve^{j\theta} = \Re[\mathcal{V}] + j\Im[\mathcal{V}]$$

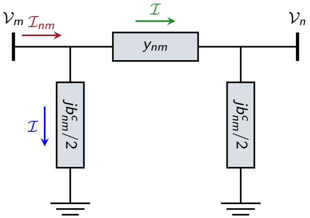
- Ohm's law $\mathcal{V} = \mathcal{Z}\mathcal{I}$

$\pi-$model of transmission lines



- Voltages $\mathcal{V}_n$ and $\mathcal{V}_m$ at line ends
- Line series impedance $z_{nm} = r_{nm} + jx_{nm}$
- Line series admittance $y_{nm} = \frac{1}{z_{nm}} = g_{nm} + jb_{nm}$
- Line series conductance $g_{nm} = \frac{r_{nm}}{r^2_{nm}+x^2_{nm}}$
- Line series susceptance $b_{nm} = \frac{x_{nm}}{r^2_{nm}+x^2_{nm}}$
- Line charging susceptance b^c_{nm}

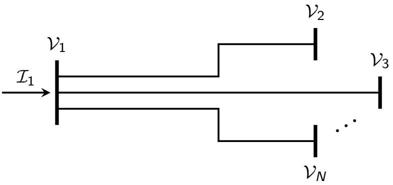
Line currents



$$\begin{aligned}\mathcal{I}_{nm} &= y_{nm}\left(\mathcal{V}_n - \mathcal{V}_m\right) + j\frac{b^c_{nm}}{2}\mathcal{V}_n \\ &= \left(y_{nm} + j\frac{b^c_{nm}}{2}\right)\mathcal{V}_n - y_{nm}\mathcal{V}_m\end{aligned}$$

Kirchoff's current law:

$$\mathcal{I}_1 = \sum_{i=2}^{N}\left(y_{1i} + j\frac{b^c_{1i}}{2}\right)\mathcal{V}_1 - \sum_{i=2}^{N} y_{1i}\mathcal{V}_i$$



Collect currents and voltages $\{\mathcal{I}_i, \mathcal{V}_i\}_{i=1}^N$ into vectors $\mathbf{i}, \mathbf{v} \in \mathbb{C}^N$.
We will ignore transformers and phase shifters (for now).

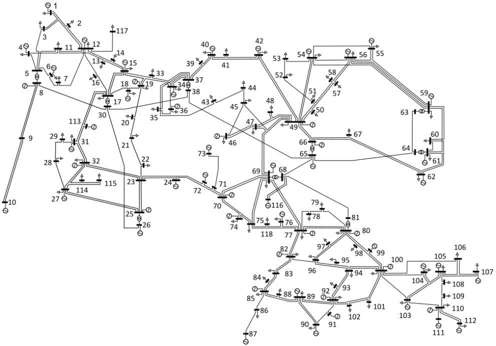
Multivariate Ohm's law

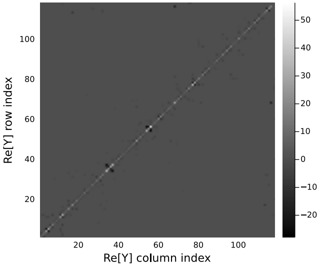
- Currents are linearly related to voltages, i.e., $\mathbf{i} = \mathbf{Y}\mathbf{v}$
- **Bus admittance matrix** is fundamental in power flow analysis

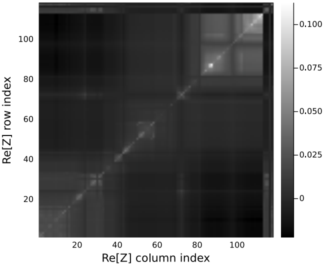
$$Y_{nm} = \begin{cases} \sum_{k \neq n} y_{nk} + j\frac{b^c_{nk}}{2} & , n = m \\ -y_{nm} & , \exists \text{ line } (n, m) \\ 0 & , \text{otherwise} \end{cases}$$

  - symmetric ($Y_{nm} = Y_{mn}$); non-Hermitian ($Y_{nm} \neq Y^*_{mn}$)
  - sparse: efficient computations and storage
  - invertible if $b^c_{nm} \neq 0$ for at least one line; otherwise $\mathbf{Y}\mathbf{1} = \mathbf{0}$
- **Bus impedance matrix** $\mathbf{Z} = \mathbf{Y}^{-1}$ $(\mathbf{v} = \mathbf{Z}\mathbf{i})$
  - non-sparse
  - **not** the matrix of line impedances, i.e., $Z_{nm} \neq z_{nm} = \frac{1}{y_{nm}}$

IEEE 118-Bus system







Complex power

- Power $S_n = S_n^g - S_n^d$ consumed/generated at bus n

  $\{S_i = P_i + jQ_i = \mathcal{V}_i \mathcal{I}_i^*\}_{i=1}^N$ and $\mathbf{i} = \mathbf{Y}\mathbf{v}$

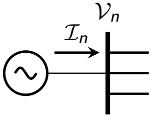
- Eliminate current to get the multivariate power model

  $$\mathbf{s} = \text{diag}(\mathbf{s})\mathbf{Y}^*\mathbf{v}^*$$

  N complex equations in $2N$ complex unknowns

- Similar expressions for power flow on line (n, m)

  $$S_{nm} = \mathcal{V}_m \mathcal{I}_{nm}^*$$



AC power flow model

$$S_n = P_n + jQ_n$$

Voltages in **polar coordinates** ($\theta_{nm} = \theta_n - \theta_m$)

$$P_n = V_n \sum_{m=1}^{N} V_m \left(G_{nm} \cos\theta_{nm} + B_{nm} \sin\theta_{nm} \right)$$

$$Q_n = V_n \sum_{m=1}^{N} V_m \left(G_{nm} \sin\theta_{nm} - B_{nm} \cos\theta_{nm} \right)$$

depends on voltage difference; reference bus $\theta_N = 0$

Voltages in **rectangular coordinates**

$$P_n = \Re[V_n] \sum_{m=1}^{N} \left(\Re[V_m] G_{nm} - \Im[V_m] B_{nm} \right) + \Im[V_n] \sum_{m=1}^{N} \left(\Im[V_m] G_{nm} + \Re[V_m] B_{nm} \right)$$

$$Q_n = \Im[V_n] \sum_{m=1}^{N} \left(\Re[V_m] G_{nm} - \Im[V_m] B_{nm} \right) - \Re[V_n] \sum_{m=1}^{N} \left(\Im[V_m] G_{nm} + \Re[V_m] B_{nm} \right)$$

Solving power flow equations

- There are $2N$ equations and $4N$ variables $\{(P_m, Q_m, V_m, \theta_m)\}_{m=1}^{N}$
- **Problem statement:** Fixing the values of $2N$ variables, find the values of the rest $2N$ unknowns that satisfy the nonlinear power flow equations

Solving power flow equations

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- **Problem statement:** Fixing the values of $2N$ variables, find the values of the rest $2N$ unknowns that satisfy the nonlinear power flow equations
- Given values typically come from
  1. First N_d load buses (PQ buses) (P_n, Q_n)
  2. Next N_g generator buses (PV buses) (P_n, V_n)
  3. Reference bus $(V_N, \theta_N = 0)$
- Number of buses $N = 1 + N_g + N_d$

Solving power flow equations

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- Number of buses $N = 1 + N_g + N_d$

- Resultant $2N$ power flow equations

$$P_n = V_n \sum_{m=1}^{N} V_m \left(G_{nm} \cos\theta_{nm} + B_{nm} \sin\theta_{nm}\right), \quad \forall n = 1, \ldots, N_d + N_g = N - 1$$
$$Q_n = V_n \sum_{m=1}^{N} V_m \left(G_{nm} \sin\theta_{nm} - B_{nm} \cos\theta_{nm}\right), \quad \forall n = 1, \ldots, N_d$$

- Equations in $\{(V_n, \theta_n)\}_{n=1}^{N}$ solved recursively (Gauss-Seidel, Newton, FDPF)

- Once voltages $\{(V_n^\star, \theta_n^\star)\}_{n=1}^{N}$ are found, any other quantity (injections, flows, currents, losses) can be calculated

DC power flow model

Only active power flow $$P_n = V_n \sum_{m=1}^{N} V_m \left(G_{nm} \cos \theta_{nm} + B_{nm} \sin \theta_{nm} \right)$$

Assumptions

A1 Low r/x ratios in transmission lines (1/5-1/10 for 220-400kV)

$$r_{nm} \ll x_{nm} \quad \rightarrow \quad g_{nm} \ll b_{nm} \quad \rightarrow \quad \mathbf{G} \approx \mathbf{0} \quad \text{and} \quad b_{nm} = \frac{x_{nm}}{r_{nm}^2 + x_{nm}^2}$$

A2 Small angle difference $\sin(\theta_n - \theta_m) \approx \theta_n - \theta_m$

A3 Voltage magnitudes $V_n \approx 1$

DC power flow model

$$P_n \approx \sum_{m:n\sim m} P_{nm} = \sum_{m:n\sim m} b_{nm}(\theta_n - \theta_m)$$

B matrix

Power injections (and flows) relate linearly to phase differences

$$P_n = \sum_{m:n\sim m} b_{nm}(\theta_n - \theta_m)$$

Multivariate power flow model: $\mathbf{p} = \mathbf{B}\boldsymbol{\theta}$

DC bus admittance matrix: (different from matrix $\mathbf{B}$ in $\mathbf{Y} = \mathbf{G} + j\mathbf{B}$)

$$B_{nm} = \begin{cases} \sum_{k\neq n} b_{nk} & , n = m \\ -b_{nm} & , \exists \text{ line } (n, m) \\ 0 & , \text{otherwise} \end{cases}$$

- Real, symmetric, sparse, and positive semidefinite
- Lossless lines: $\mathbf{B1} = \mathbf{0} \Rightarrow \mathbf{1}^\top \mathbf{p} = 0 \quad (\mathbf{1}^\top(\mathbf{p}^g - \mathbf{p}^d) = 0)$
- Oftentimes further simplify $b_{nm} = \frac{x_{nm}}{r_{nm}^2 + x_{nm}^2} \approx \frac{1}{x_{nm}}$

Formulation in **rectangular coordinates**

- Collect nodal voltages in rectangular coordinates in $\mathbf{v} \in \mathbb{C}^N$

$$\mathbf{v} = \begin{bmatrix} \Re[\mathcal{V}_1] + j\Im[\mathcal{V}_1] & \dots & \Re[\mathcal{V}_N] + j\Im[\mathcal{V}_N] \end{bmatrix}^\top$$

- Power injections and squared voltage magnitudes are quadratic functions of $\mathbf{v}$:

$$\begin{aligned} P_n(\mathbf{v}) &= \mathbf{v}^H \mathbf{M}_{P_n} \mathbf{v} \\ Q_n(\mathbf{v}) &= \mathbf{v}^H \mathbf{M}_{Q_n} \mathbf{v} \\ V_n^2(\mathbf{v}) &= \mathbf{v}^H \mathbf{M}_{V_n} \mathbf{v} \end{aligned}$$

  where matrices $\mathbf{M}$ are Hermitian symmetric ($\mathbf{M} = \mathbf{M}^H$)

- Every bus contributes two quadratic constraints (active and reactive power) on $\mathbf{v}$

Finding $\mathbf{M}$ matrices

- Voltage magnitude ($\mathbf{e}_n$ is the n-th canonical vector)

$$V_n^2(\mathbf{v}) = \mathcal{V}_n^* \mathcal{V}_n = \mathbf{v}^H \mathbf{e}_n \mathbf{e}_n^\top \mathbf{v} \quad \Rightarrow \quad \mathbf{M}_{V_n} = \mathbf{e}_n \mathbf{e}_n^\top$$

- Complex power injection

$$S_n = P_n + jQ_n = \mathcal{V}_n \mathcal{I}_n^* = (\mathbf{v}^\top \mathbf{e}_n)(\mathbf{e}_n^\top \mathbf{i}^*) = \mathbf{v}^\top \mathbf{e}_n \mathbf{e}_n^\top \mathbf{Y}^* \mathbf{v}^* = \mathbf{v}^H \mathbf{Y}^* \mathbf{e}_n \mathbf{e}_n^\top \mathbf{v}$$

- Active and reactive power then take the form

$$P_n = \frac{1}{2}(S_n + S_n^*) = \mathbf{v}^H \mathbf{M}_{P_n} \mathbf{v} \quad \text{where} \quad \mathbf{M}_{P_n} = \frac{1}{2}(\mathbf{Y}^* \mathbf{e}_n \mathbf{e}_n^\top + \mathbf{e}_n \mathbf{e}_n^\top \mathbf{Y}^*)$$

$$Q_n = \frac{1}{2}(S_n - S_n^*) = \mathbf{v}^H \mathbf{M}_{Q_n} \mathbf{v} \quad \text{where} \quad \mathbf{M}_{Q_n} = \frac{1}{2}(\mathbf{Y}^* \mathbf{e}_n \mathbf{e}_n^\top - \mathbf{e}_n \mathbf{e}_n^\top \mathbf{Y}^*)$$

Power flow as a feasibility problem

- System state as solution of feasibility problem

$$\begin{aligned} \text{find} \quad & \mathbf{v} \\ \text{s.t.} \quad & \mathbf{v}^H \mathbf{M}_k \mathbf{v} = s_k, \quad \forall k = 1, \ldots, 2N \end{aligned} \qquad \left[\text{note: } \mathbf{v}^H \mathbf{M}_k \mathbf{v} = \mathrm{Tr}(\mathbf{M}_k \mathbf{v}\mathbf{v}^H)\right]$$

- Introduce matrix variable $\mathbf{V} = \mathbf{v}\mathbf{v}^H$

$$\begin{aligned} \text{find} \quad & (\mathbf{v}, \mathbf{V}) \\ \text{s.t.} \quad & \mathrm{Tr}(\mathbf{M}_k \mathbf{V}) = s_k, \quad \forall k = 1, \ldots, 2N \\ & \mathbf{V} = \mathbf{v}\mathbf{v}^H \end{aligned}$$

- Eliminate variable $\mathbf{v}$; *non-convex* problem due to rank constraint

$$\begin{aligned} \text{find} \quad & (\mathbf{V}) \\ \text{s.t.} \quad & \mathrm{Tr}(\mathbf{M}_k \mathbf{V}) = s_k, \quad \forall k = 1, \ldots, 2N \\ & \mathbf{V} \succeq \mathbf{0}, \ \mathrm{rank}(\mathbf{V}) = 1 \end{aligned}$$

Semidefinite program relaxation

- Drop rank constraint to get semidefinite program (SDP)

$$\begin{aligned} \text{find} \quad & (\mathbf{V}) \\ \text{s.t.} \quad & \mathrm{Tr}(\mathbf{M}_k \mathbf{V}) = s_k, \quad \forall k = 1, \ldots, 2N \\ & \mathbf{V} \succeq \mathbf{0} \end{aligned}$$

  which is a convex problem

- If the solution $\mathbf{V}^\star$ is rank-1, the relaxation is said to be exact

- If exact, find $\mathbf{v}^\star$ from $\mathbf{V}^\star = \mathbf{v}^\star \mathbf{v}^{\star H}$

- Relaxation is oftentimes exact under practical system conditions!

Optimal power flow (OPF) using semidefinite relaxation

- OPF problem:

$$\begin{aligned} \underset{\mathbf{V}\succeq\mathbf{0}}{\text{minimize}} \quad & \mathrm{Tr}(\mathbf{MV}) \\ \text{subject to} \quad & \mathrm{Tr}(\mathbf{M}_k\mathbf{V}) = s_k, \quad \forall k = 1, \ldots, 2N \end{aligned}$$

- Design matrix $\mathbf{M}$ to strengthen the relaxation (favor rank-1 solutions)[1]:
  - selecting $\mathbf{M} = \mathbf{Y}^H\mathbf{Y}$ minimizes $\|\mathbf{i}\|_2^2$
  - selecting $\mathbf{M} = \mathbf{B}$ minimizes losses
  - both yield the "high-voltage solution" of the power flow equations

- Use $\sum_{n=1}^{N} c_n \mathrm{Tr}(\mathbf{M}_{P_n}\mathbf{V})$ as an objective to minimize the dispatch cost[2]

- Incorporate squared voltage bounds as $\underline{\mathbf{v}}^2 \leqslant \mathbf{M}_{V_n}\mathbf{V} \leqslant \bar{\mathbf{v}}^2$

[1] R. Madani, J. Lavaei, and R. Baldick. Convexification of power flow problem over arbitrary networks. IEEE CDC 2015

[2] J. Lavaei and S. Low. Zero duality gap in optimal power flow problem. IEEE Trans. on Power Systems. 2012

Formulation in **polar coordinates**

- AC power flow model

$$P_n(\mathbf{v},\boldsymbol{\theta}) = V_n \sum_{m=1}^{N} V_m \left(G_{nm}\cos\theta_{nm} + B_{nm}\sin\theta_{nm}\right), \quad \forall n = 1,\ldots,N$$
$$Q_n(\mathbf{v},\boldsymbol{\theta}) = V_n \sum_{m=1}^{N} V_m \left(G_{nm}\sin\theta_{nm} - B_{nm}\cos\theta_{nm}\right), \quad \forall n = 1,\ldots,N$$

- **Classic AC-OPF** problem formulation

$$\begin{aligned}
\underset{\mathbf{p}^g,\mathbf{q}^g,\mathbf{v},\boldsymbol{\theta}}{\text{minimize}} \quad & c(\mathbf{p}^g) && \textit{generation cost} \\
\text{subject to} \quad & \mathbf{p}(\mathbf{v},\boldsymbol{\theta}) = \mathbf{p}^g - \mathbf{p}^d && \textit{active power flow} \\
& \mathbf{q}(\mathbf{v},\boldsymbol{\theta}) = \mathbf{q}^g - \mathbf{q}^d && \textit{reactive power flow} \\
& \underline{\mathbf{p}}^g \leqslant \mathbf{p}^g \leqslant \overline{\mathbf{p}}^g && \textit{min/max gen p-limits} \\
& \underline{\mathbf{q}}^g \leqslant \mathbf{q}^g \leqslant \overline{\mathbf{q}}^g && \textit{min/max gen q-limits} \\
& \underline{\mathbf{v}} \leqslant \mathbf{v} \leqslant \overline{\mathbf{v}} && \textit{min/max voltage mag limits} \\
& \underline{\boldsymbol{\theta}} \leqslant \boldsymbol{\theta} \leqslant \overline{\boldsymbol{\theta}} && \textit{min/max voltage angle limits}
\end{aligned}$$

- Minimize generation cost subject to power flow equations and variable limits

Security-constrained AC-OPF (SC-AC-OPF)

$$
\begin{aligned}
\underset{\mathbf{p}^g,\mathbf{q}^g,\mathbf{v}_0,\boldsymbol{\theta}_0,\mathbf{v}_c,\boldsymbol{\theta}_c}{\text{minimize}} \quad & c(\mathbf{p}^g) && \textit{generation cost} \\
\text{subject to} \quad & \mathbf{p}_0(\mathbf{v}_0,\boldsymbol{\theta}_0) = \mathbf{p}^g - \mathbf{p}^d && \textit{nominal act. power flow} \\
& \mathbf{q}_0(\mathbf{v}_0,\boldsymbol{\theta}_0) = \mathbf{q}^g - \mathbf{q}^d && \textit{nominal rea. power flow} \\
& \mathbf{p}_c(\mathbf{v}_c,\boldsymbol{\theta}_c) = \mathbf{p}^g - \mathbf{p}^d,\ \forall c = 1,\ldots,N_c && \textit{post-contingency act. power flow} \\
& \mathbf{q}_c(\mathbf{v}_c,\boldsymbol{\theta}_c) = \mathbf{q}^g - \mathbf{q}^d,\ \forall c = 1,\ldots,N_c && \textit{post-contingency rea. power flow} \\
& + \text{limits on optimization variables}
\end{aligned}
$$

- Constraints for the nominal and all contingency scenarios
- Line outage: $\mathbf{p}_c()$ and $\mathbf{q}_c()$ include new admittances $\mathbf{Y}_c$
- One dispatch $(\mathbf{p}^g, \mathbf{q}^g)$ is computed for the nominal and all contingency scenarios
- SC-AC-OPF costs $\geqslant$ classic AC-OPF (why? what is N_c for line outages?)
- Similarly, generator outage security constraints are added to SC-AC-OPF

DC-OPF ($b\theta-$formulation)

$$
\begin{aligned}
\underset{\mathbf{p},\boldsymbol{\theta}}{\text{minimize}} \quad & c(\mathbf{p}) && \textit{generation cost} \\
\text{subject to} \quad & \mathbf{B}\boldsymbol{\theta} = \mathbf{p} - \mathbf{d} && \textit{active power balance} \\
& \underline{\mathbf{p}} \leqslant \mathbf{p} \leqslant \overline{\mathbf{p}} && \textit{min/max gen p-limits} \\
& -\overline{f}_{nm} \leqslant b_{nm}(\theta_n - \theta_m) \leqslant \overline{f}_{nm} && \textit{power flow limits}
\end{aligned}
$$

- New notation: $\mathbf{p}^g \rightarrow \mathbf{p}$ and $\mathbf{p}^d \rightarrow \mathbf{d}$
- Acts on the DC power flow approximation
- Active power only; reactive power disregarded
- Double-sided power flow constraints (why?)

DC-OPF (PTDF formulation)

- Formulate the DC-OPF problem in one variable $\mathbf{p}^g$ only
- Use matrix $\mathbf{F} \in \mathbb{R}^{E \times N}$ of power transfer distribution factors (PTDF)
  - how the power flow in line e changes w.r.t. to the change of power injection at node n?
  - obtained by manipulating the DC bus admittance matrix $\mathbf{B}$ (see today's tutorial)
- Power flows $f = \mathbf{F}(\mathbf{p} - \mathbf{d})$ (distribution of net injections across power lines)
- The new DC-OPF formulation

$$
\begin{aligned}
\underset{\mathbf{p}}{\text{minimize}} \quad & c(\mathbf{p}) && \textit{generation cost} \\
\text{subject to} \quad & \mathbf{1}^\top(\mathbf{p} - \mathbf{d}) = 0 && \textit{active power balance} \\
& |\mathbf{F}(\mathbf{p} - \mathbf{d})| \leqslant \overline{f} && \textit{power flow limits} \\
& \underline{\mathbf{p}} \leqslant \mathbf{p} \leqslant \overline{\mathbf{p}} && \textit{min/max gen p-limits}
\end{aligned}
$$

- Less variables than in $b\theta-$formulation, but requires more memory to store and operate with matrix $\mathbf{F}$
- Often used for *locational marginal pricing* in high-voltage grids

Duality of DC-OPF

- Focus on the **coupling** constraints (i.e., linking generators and loads):

$$\begin{aligned}\underset{\underline{\mathbf{p}}\leqslant\mathbf{p}\leqslant\overline{\mathbf{p}}}{\text{minimize}} \quad & c(\mathbf{p}) \\ \text{subject to} \quad & \mathbf{1}^\top(\mathbf{p}-\mathbf{d}) = 0 \quad &: \lambda \\ & \mathbf{F}(\mathbf{p}-\mathbf{d}) \leqslant \overline{\mathbf{f}} \quad &: \overline{\boldsymbol{\mu}} \\ & -\mathbf{F}(\mathbf{p}-\mathbf{d}) \leqslant \overline{\mathbf{f}} \quad &: \underline{\boldsymbol{\mu}}\end{aligned}$$

- **Partial Lagrangian function** (dualize the coupling constraints only):

$$\begin{aligned}\max_{\lambda,\overline{\boldsymbol{\mu}},\underline{\boldsymbol{\mu}}} \min_{\underline{\mathbf{p}}\leqslant\mathbf{p}\leqslant\overline{\mathbf{p}}} \quad \mathcal{L}(\mathbf{p},\lambda,\overline{\boldsymbol{\mu}},\underline{\boldsymbol{\mu}}) =& c(\mathbf{p}) - \lambda\mathbf{1}^\top(\mathbf{p}-\mathbf{d}) \\ & + \overline{\boldsymbol{\mu}}^\top(\mathbf{F}(\mathbf{p}-\mathbf{d}) - \overline{\mathbf{f}}) + \underline{\boldsymbol{\mu}}^\top(-\mathbf{F}(\mathbf{p}-\mathbf{d}) - \overline{\mathbf{f}})\end{aligned}$$

- Group terms corresponding to dispatch **p**, demand **d** and line limits $\overline{\mathbf{f}}$:

$$\mathcal{L} = \mathcal{L}^{\mathbf{p}} + \mathcal{L}^{\mathbf{d}} + \mathcal{L}^{\mathbf{f}}, \quad \text{where}$$

$$\begin{aligned}\mathcal{L}^{\mathbf{p}}(\mathbf{p},\lambda,\overline{\boldsymbol{\mu}},\underline{\boldsymbol{\mu}}) &= c(\mathbf{p}) - (\mathbf{1}\lambda - \mathbf{F}^\top\overline{\boldsymbol{\mu}} + \mathbf{F}^\top\underline{\boldsymbol{\mu}})^\top\mathbf{p} \\ \mathcal{L}^{\mathbf{d}}(\lambda,\overline{\boldsymbol{\mu}},\underline{\boldsymbol{\mu}}) &= (\mathbf{1}\lambda - \mathbf{F}^\top\overline{\boldsymbol{\mu}} + \mathbf{F}^\top\underline{\boldsymbol{\mu}})^\top\mathbf{d} \\ \mathcal{L}^{\mathbf{f}}(\overline{\boldsymbol{\mu}},\underline{\boldsymbol{\mu}}) &= -(\underline{\boldsymbol{\mu}} + \overline{\boldsymbol{\mu}})^\top\overline{\mathbf{f}}\end{aligned}$$

Power dispatch **p** and demand **d** share the same multiplier but with oposite signs

Locational marginal prices (LMPs)

$$\boldsymbol{\pi}^\star(\lambda^\star, \overline{\boldsymbol{\mu}}^\star, \underline{\boldsymbol{\mu}}^\star) = \underbrace{\mathbf{1}\lambda^\star}_{\text{uniform}} - \underbrace{\mathbf{F}^\top(\overline{\boldsymbol{\mu}}^\star - \underline{\boldsymbol{\mu}}^\star)}_{\text{congestion}} \in \mathbb{R}^N$$

- $\pi_n^\star$ is the cost of supplying the next unit of demand at node n
- in case of congestion ($\overline{\boldsymbol{\mu}}^\star > 0$ or $\overline{\boldsymbol{\mu}}^\star > 0$), electricity price varies across the grid
- price at the reference bus is $\lambda^\star$ (the ref. column of $\mathbf{F}$ is zero)

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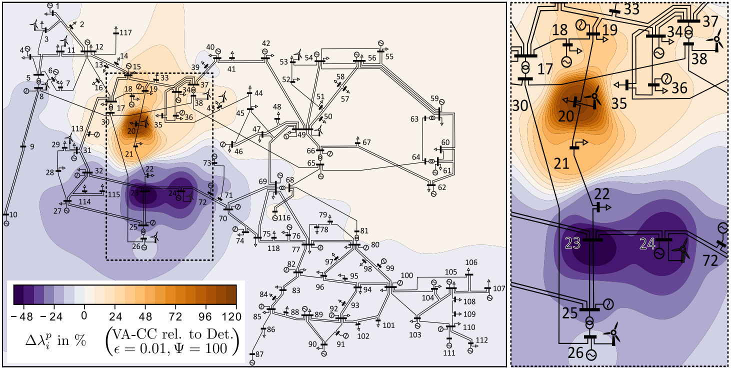


Mieth *et al.* Risk- and Variance-Aware Electricity Pricing. PSCC 2020

Equilibrium interpretation

- Equilibrium problem:

  Power suppliers minimize negative profits:

  $$\mathbf{p}(\boldsymbol{\pi}^\star) = \underset{\underline{\mathbf{p}} \leqslant \mathbf{p} \leqslant \overline{\mathbf{p}}}{\text{argmin}} \quad \mathcal{L}^{\mathbf{p}}(\mathbf{p}, \boldsymbol{\pi}^\star) = c(\mathbf{p}) - \boldsymbol{\pi}^{\star\top}\mathbf{p}$$

  Market operator finds such prices $\boldsymbol{\pi}^\star(\lambda^\star, \overline{\boldsymbol{\mu}}^\star, \underline{\boldsymbol{\mu}}^\star)$ that satisfy:

  $$\mathbf{1}^\top(\mathbf{p}(\boldsymbol{\pi}^\star) - \mathbf{d}) = 0 \quad \text{and} \quad |\mathbf{F}(\mathbf{p}(\boldsymbol{\pi}^\star) - \mathbf{d})| \leqslant \overline{f}$$

- Once equilibrium is found, power suppliers are paid as above and:

  Inelastic demands are charged with $\boldsymbol{\pi}^\star \circ \mathbf{d}$

  Transmission operator collects congestion rent $(\underline{\boldsymbol{\mu}} + \overline{\boldsymbol{\mu}})^\top \overline{\mathbf{f}}$

- No congestion ($\overline{\boldsymbol{\mu}}^\star = \underline{\boldsymbol{\mu}}^\star = \mathbf{0}$) $\rightarrow$ problem reduces to economic dispatch

- Elastic demand $\rightarrow$ utility-maximization problem for each demand

Some desirable market properties - Part I

- **Market efficiency:** Equilibrium LMPs yield the least-cost dispatch

  Proof: Extrapolate the solution to Assignment 1 (Problem 2) to the network-constrained case.

- **Cost recovery:** for fully dispatchable generators (i.e., $\underline{\mathbf{p}} = \mathbf{0}$)

  Proof:

  primal problem

  $$\begin{aligned} \underset{\mathbf{p}}{\text{maximize}} \quad & \boldsymbol{\pi}^{\star\top}\mathbf{p} - \mathbf{c}^\top\mathbf{p} \\ \text{subject to} \quad & \underline{\mathbf{p}} \leqslant \mathbf{p} \leqslant \overline{\mathbf{p}} \end{aligned}$$

  dual problem

  $$\begin{aligned} \underset{\overline{\boldsymbol{\vartheta}}, \underline{\boldsymbol{\vartheta}} \geqslant \mathbf{0}}{\text{minimize}} \quad & \overline{\boldsymbol{\vartheta}}^\top \overline{\mathbf{p}} - \underline{\boldsymbol{\vartheta}}^\top \underline{\mathbf{p}} \\ \text{subject to} \quad & \mathbf{c} - \boldsymbol{\pi}^\star + \overline{\boldsymbol{\vartheta}} - \underline{\boldsymbol{\vartheta}} = \mathbf{0} \end{aligned}$$

  Strong duality: $\boldsymbol{\pi}^{\star\top}\mathbf{p}^\star - \mathbf{c}^\top\mathbf{p}^\star = \overline{\boldsymbol{\vartheta}}^{\star\top}\overline{\mathbf{p}} - \underline{\boldsymbol{\vartheta}}^{\star\top}\underline{\mathbf{p}}$

  $\overline{\boldsymbol{\vartheta}}^\star, \underline{\boldsymbol{\vartheta}}^\star \geqslant \mathbf{0}$ and $\underline{\mathbf{p}} = \mathbf{0} \Rightarrow$ the profit is non-negative $\Rightarrow$ cost recovery

- What holds for aggregate also holds for individual generators

- On your own: repeat the same steps for quadratic costs ...

Some desirable market properties - Part II

- **Revenue adequacy** in markets with fully dispatchable units (i.e., $\underline{\mathbf{p}} = \mathbf{0}$):
  Market operator does not run into deficit (demand charges $\geqslant$ generator revenues)

  Proof: The Lagrangian function in optimality decomposes into

  $$\underbrace{\mathcal{L}(\mathbf{p}^\star, \lambda^\star, \overline{\boldsymbol{\mu}}^\star, \underline{\boldsymbol{\mu}}^\star)}_{=c(\mathbf{p}^\star)} = \underbrace{c(\mathbf{p}^\star) - \boldsymbol{\pi}^{\star\top}\mathbf{p}^\star}_{\text{minus profit}} + \underbrace{\boldsymbol{\pi}^{\star\top}\mathbf{d}}_{\text{charges}} - \underbrace{\bar{\mathbf{f}}^\top(\overline{\boldsymbol{\mu}}^\star + \underline{\boldsymbol{\mu}}^\star)}_{\text{congestion rent}}$$

  $$\Longleftrightarrow$$

  $$\boldsymbol{\pi}^{\star\top}\mathbf{d} - \boldsymbol{\pi}^{\star\top}\mathbf{p}^\star = \bar{\mathbf{f}}^\top(\overline{\boldsymbol{\mu}}^\star + \underline{\boldsymbol{\mu}}^\star)$$

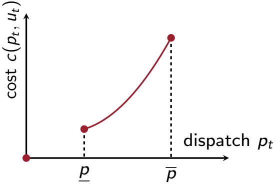
  We thus only need to show that the congestion rent is non-negative

  From dual feasibility conditions $\overline{\boldsymbol{\mu}}^\star, \underline{\boldsymbol{\mu}}^\star \geqslant \mathbf{0} \Rightarrow \boldsymbol{\pi}^{\star\top}\mathbf{d} - \boldsymbol{\pi}^{\star\top}\mathbf{p}^\star \geqslant 0$

  Since the congestion rent is non-negative, the revenue adequacy holds

Unit commitment (UC) problem

$$
\begin{aligned}
\underset{\mathbf{p}_t,\mathbf{u}_t}{\text{minimize}} \quad & \sum_{t=1}^{T} c(\mathbf{p}_t, \mathbf{u}_t) \\
\text{subject to} \quad & \mathbf{1}^\top(\mathbf{p}_t - \mathbf{d}_t) = 0, \\
& \mathbf{F}(\mathbf{p}_t - \mathbf{d}_t) \leqslant \overline{f}, \\
& -\mathbf{F}(\mathbf{p}_t - \mathbf{d}_t) \leqslant \overline{f}, \\
& \mathbf{u}_t \circ \underline{\mathbf{p}} \leqslant \mathbf{p}_t \leqslant \mathbf{u}_t \circ \overline{\mathbf{p}}, \\
& + \textit{other constraints} \;\; \forall t = 1, \ldots, T
\end{aligned}
$$



- Binary unit commitment decisions $\mathbf{u}_t \in \{0,1\}^N$
- Discontinuous generator cost functions $\Rightarrow$ non-convex problem
- Because of discontinuity, the dual variables do not exist!
- Mixed-integer (MI) linear (or quadratic) program
- How to price electricity using unit commitment?

Unit commitment (UC) problem

$$
\begin{aligned}
\underset{\mathbf{p}_t,\mathbf{u}_t}{\text{minimize}} \quad & \sum_{t=1}^{T} c(\mathbf{p}_t, \mathbf{u}_t) \\
\text{subject to} \quad & \mathbf{1}^\top(\mathbf{p}_t - \mathbf{d}_t) = 0, \qquad : \cancel{\lambda_t} \\
& \mathbf{F}(\mathbf{p}_t - \mathbf{d}_t) \leqslant \overline{f}, \qquad : \cancel{\overline{\mu}_t} \\
& -\mathbf{F}(\mathbf{p}_t - \mathbf{d}_t) \leqslant \overline{f}, \qquad : \cancel{\underline{\mu}_t} \\
& \mathbf{u}_t \circ \underline{\mathbf{p}} \leqslant \mathbf{p}_t \leqslant \mathbf{u}_t \circ \overline{\mathbf{p}}, \\
& + \textit{other constraints} \;\; \forall t = 1, \ldots, T
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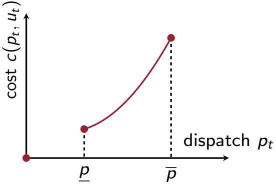

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Electricity pricing with discontinuous costs

- Let $\mathbf{u}_1^\star, \ldots, \mathbf{u}_T^\star$ be the optimal UC decisions (e.g., after solving UC with MI solver)

- Formulate the relaxed problem

$$
\begin{aligned}
\underset{\mathbf{p}_t,\mathbf{u}_t}{\text{minimize}} \quad & \sum_{t=1}^{T} c(\mathbf{p}_t, \mathbf{u}_t) \\
\text{subject to} \quad & \mathbf{1}^\top(\mathbf{p}_t - \mathbf{d}_t) = 0 && : \lambda_t \\
& \mathbf{F}(\mathbf{p}_t - \mathbf{d}_t) \leqslant \overline{f} && : \overline{\boldsymbol{\mu}}_t \\
& -\mathbf{F}(\mathbf{p}_t - \mathbf{d}_t) \leqslant \overline{f} && : \underline{\boldsymbol{\mu}}_t \\
& \mathbf{u}_t \circ \underline{\mathbf{p}} \leqslant \mathbf{p}_t \leqslant \mathbf{u}_t \circ \overline{\mathbf{p}} \\
& \mathbf{u}_t = \mathbf{u}_t^\star && : \boldsymbol{\vartheta}_t
\end{aligned}
$$

- This is a convex optimization problem with the dual solution[3]:
  - Prices $\boldsymbol{\pi}_t^\star(\lambda_t^\star, \overline{\boldsymbol{\mu}}_t^\star, \underline{\boldsymbol{\mu}}_t^\star)$ and ${\boldsymbol{\vartheta}_t}^\star$ solve a competitive equilibrium with disc. cost
  - Uplift payment ${\boldsymbol{\vartheta}_t}^\star \circ \mathbf{u}_t$ to remunerate for the costs that can not be recovered by LMPs
  - This is the case of not fully dispatchable ($\underline{\mathbf{p}} \neq \mathbf{0}$) units (see desirable market property 2)

- UC is solved days ahead to compute uplifts; OPF is solved later to price electricity

[3] O'Neill *et al.* Efficient market-clearing prices in markets with nonconvexities. EJOR, 2005.