ECE 598 Computational Power Systems

Duality, optimality conditions, and electricity pricing

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Last lecture recap

- Look around you and form teams of 2 people (1 min)
- Quickly review your notes or the slide deck (1 min)
- Share your **three** personal highlights with your partner (3 min)
- Get iClicker app ready

Lagrangian function

Primal problem:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & c(\mathbf{x}) \\ \text{subject to} \quad & g_i(\mathbf{x}) \leqslant 0, \quad \forall i = 1, \ldots, m \\ & h_j(\mathbf{x}) = 0, \quad \forall j = 1, \ldots, p \end{aligned}$$

- $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is the vector of decision variable
- $c : \mathbb{R}^n \mapsto \mathbb{R}$ is the objective or cost function
- constraint functions $g_i : \mathbb{R}^n \mapsto \mathbb{R}$ and $h_j : \mathbb{R}^n \mapsto \mathbb{R}$

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Lagrangian function: $\mathcal{L} : \mathbb{R}^{n+m+p} \mapsto \mathbb{R}$

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}) = c(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x}) + \sum_{j=1}^{p} \nu_j h_j(\mathbf{x})$$

- $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m) \succeq \mathbf{0}$ and $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_m)$ are so-called **Lagrange multipliers**
- The objective function augmented with weighted sum of constraint functions

Dual function

Dual function: $\phi : \mathbb{R}^{m+p} \mapsto \mathbb{R}$

$$\phi(\boldsymbol{\lambda}, \boldsymbol{\nu}) = \min_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}) = \min_{\mathbf{x}} c(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x}) + \sum_{j=1}^{p} \nu_j h_j(\mathbf{x})$$

- always concave even for non-convex primal problem
- can be $-\infty$ for some choice of $(\boldsymbol{\lambda}, \boldsymbol{\nu})$

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- always concave even for non-convex primal problem
- can be $-\infty$ for some choice of $(\boldsymbol{\lambda}, \boldsymbol{\nu})$

Example: Linear program (LP)

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & \mathbf{G}\mathbf{x} \leqslant \mathbf{h} \quad : \boldsymbol{\lambda} \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \quad : \boldsymbol{\nu} \end{aligned}$$

Lagrangian function

$$\begin{aligned} \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}) &= \mathbf{c}^\top \mathbf{x} + \boldsymbol{\lambda}^\top (\mathbf{G}\mathbf{x} - \mathbf{h}) + \boldsymbol{\nu}^\top (\mathbf{A}\mathbf{x} - \mathbf{b}) \\ &= (\mathbf{c} + \mathbf{G}^\top \boldsymbol{\lambda} + \mathbf{A}^\top \boldsymbol{\nu})^\top \mathbf{x} - \boldsymbol{\lambda}^\top \mathbf{h} - \boldsymbol{\nu}^\top \mathbf{b} \end{aligned}$$

Dual function

$$\phi(\boldsymbol{\lambda}, \boldsymbol{\nu}) = \begin{cases} -\boldsymbol{\lambda}^\top \mathbf{h} - \boldsymbol{\nu}^\top \mathbf{b}, & \text{if } \mathbf{c} + \mathbf{G}^\top \boldsymbol{\lambda} + \mathbf{A}^\top \boldsymbol{\nu} = \mathbf{0} \\ -\infty (\text{unbounded}), & \text{if otherwise} \end{cases}$$

Weak duality (lower bound property)

If $\boldsymbol{\lambda} \succeq \mathbf{0}$ and $\mathbf{x}$ is primal feasible, then

$$\phi(\boldsymbol{\lambda}, \boldsymbol{\nu}) \leqslant c(\mathbf{x})$$

Proof: if $g_i(\mathbf{x}) \leqslant 0$, $h_i(\mathbf{x}) = 0$, and $\boldsymbol{\lambda} \succeq \mathbf{0}$, then

$$\phi(\boldsymbol{\lambda}, \boldsymbol{\nu}) = \min_{\mathbf{y}} c(\mathbf{y}) + \sum_{i=1}^{m} \lambda_i g_i(\mathbf{y}) + \underbrace{\sum_{j=1}^{p} \nu_j h_j(\mathbf{y})}_{0} \leqslant c(\mathbf{x}) + \underbrace{\sum_{i=1}^{m} \lambda_i g_i(\mathbf{x})}_{\leqslant 0} \leqslant c(\mathbf{x})$$

For dual feasible $(\boldsymbol{\lambda}, \boldsymbol{\nu})$, the dual function provides the lower bound on optimal value!

Dual problem

Computes the best lower bound on $c(\mathbf{x}^\star)$:

$$\begin{aligned} \underset{\boldsymbol{\lambda},\boldsymbol{\nu}}{\text{maximize}} \quad & \phi(\boldsymbol{\lambda},\boldsymbol{\nu}) \\ \text{subject to} \quad & \boldsymbol{\lambda} \succeq \mathbf{0} \end{aligned}$$

- Every primal problem has a dual problem
- Concave problem even if the primal problem is non-convex
- **Weak duality**
  - *Non-zero duality gap* $c(\mathbf{x}^\star) - \phi(\boldsymbol{\lambda}^\star,\boldsymbol{\nu}^\star) \geqslant 0$
  - Common for non-convex problems
- **Strong duality**
  - *Zero duality gap* $\phi(\boldsymbol{\lambda}^\star,\boldsymbol{\nu}^\star) - c(\mathbf{x}^\star) = 0$
  - Holds for convex problems (**Slater's condition**)
  - We can use $\boldsymbol{\lambda}^\star,\boldsymbol{\nu}^\star$ to certify optimality of $\mathbf{x}^\star$

Dual problem of linear program (LP)

Primal LP

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & \mathbf{G}\mathbf{x} \leqslant \mathbf{h} \quad : \boldsymbol{\lambda} \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \quad : \boldsymbol{\nu} \end{aligned}$$

Dual problem of linear program (LP)

Primal LP

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & \mathbf{G}\mathbf{x} \leqslant \mathbf{h} \quad : \boldsymbol{\lambda} \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \quad : \boldsymbol{\nu} \end{aligned}$$

Dual problem of LP is also LP

$$\begin{aligned} \underset{\boldsymbol{\lambda},\boldsymbol{\nu}}{\text{maximize}} \quad & -\boldsymbol{\lambda}^\top \mathbf{h} - \boldsymbol{\nu}^\top \mathbf{b} \\ \text{subject to} \quad & \mathbf{c} + \mathbf{G}^\top \boldsymbol{\lambda} + \mathbf{A}^\top \boldsymbol{\nu} = \mathbf{0} \quad : \mathbf{x} \\ & \boldsymbol{\lambda} \succeq \mathbf{0} \end{aligned}$$

- Unless the dual function is unbounded, strong duality holds

$$-\boldsymbol{\lambda}^{\star\top} \mathbf{h} - \boldsymbol{\nu}^{\star\top} \mathbf{b} = \mathbf{c}^\top \mathbf{x}^\star$$

- What is the dual problem of the dual LP?

Dual problem of quadratic program (QP)

Primal QP: assume $\mathbf{Q} \succeq \mathbf{0}$ (positive semidefinite) for simplicity

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{x}^\top \mathbf{Q} \mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leqslant \mathbf{b} \quad : \boldsymbol{\lambda} \end{aligned}$$

Dual problem of quadratic program (QP)

Primal QP: assume $\mathbf{Q} \succeq \mathbf{0}$ (positive semidefinite) for simplicity

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{x}^\top \mathbf{Q}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leqslant \mathbf{b} \quad : \boldsymbol{\lambda} \end{aligned}$$

- Lagrangian function: $\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{x}^\top \mathbf{Q}\mathbf{x} + \boldsymbol{\lambda}^\top (\mathbf{A}\mathbf{x} - \mathbf{b})$
- By definition, the dual function $\phi(\boldsymbol{\lambda}) = \min_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda})$

  Setting $\nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{0}$, yields $\mathbf{x}^\star = -\frac{1}{2}\mathbf{Q}^{-1}\mathbf{A}^\top \boldsymbol{\lambda}$
- The dual function is then $\phi(\boldsymbol{\lambda}) = \mathcal{L}(\mathbf{x}^\star, \boldsymbol{\lambda}) = -\frac{1}{4}\boldsymbol{\lambda}^\top \mathbf{A}\mathbf{Q}^{-1}\mathbf{A}^\top \boldsymbol{\lambda} - \mathbf{b}^\top \boldsymbol{\lambda}$

  Concave quadratic; all $\boldsymbol{\lambda} \succeq \mathbf{0}$ are dual feasible

Dual of primal QP is also QP:

$$\begin{aligned} \underset{\boldsymbol{\lambda}}{\text{maximize}} \quad & -\frac{1}{4}\boldsymbol{\lambda}^\top \mathbf{A}\mathbf{Q}^{-1}\mathbf{A}^\top \boldsymbol{\lambda} - \mathbf{b}^\top \boldsymbol{\lambda} \\ \text{subject to} \quad & \boldsymbol{\lambda} \succeq \mathbf{0} \end{aligned}$$

How do I become proficient in duality?

- Write down a simple problem, i.e.,

$$\begin{aligned} \underset{x_1, x_2}{\text{minimize}} \quad & \frac{1}{2}(x_1 - 1)^2 + (x_2 + 2)^2 \\ \text{subject to} \quad & 0.5x_1 + x_2 = 10 \\ & 0 \leqslant x_1 \leqslant 10 \\ & 0 \leqslant x_2 \leqslant 10 \end{aligned}$$

- Vectorize the problem
- Write down the Lagrangian function
- Write down the dual function
- Formulate the dual problem
- Code it in Julia, verify strong duality
- Come to my **office hours** Wednesday 2:00 to 3:30 PM

Applications of duality in power systems

- Wholesale electricity market clearing (duals are electricity prices)
- Congestion management (duals indicate the lines operating at the limit)
- Long-term grid planning (duals indicate grid bottlenecks and guide investments)
- Cross-boarder coordination of power grids (duals are coordination variables)
- Volt/VAr control (duals are used to update reactive power set-points)
- Emission accounting (duals provide marginal emission factors of el. loads)
- Carbon policy (dual of carbon tax constraint is the optimal carbon tax)
- Energy storage optimization (duals represent the marginal value of energy stored)
- Convex relaxations of the non-convex power flow formulations
- Certifying optimality, bounding decision errors, etc.

And many-many more applications of duality ...

Complementarity slackness

Suppose that $\mathbf{x}^\star, \boldsymbol{\lambda}^\star$, and $\boldsymbol{\nu}^\star$ are primal-dual optimal with zero duality gap

$$c(\mathbf{x}^\star) = \phi(\boldsymbol{\lambda}^\star, \boldsymbol{\nu}^\star) = \min_{\mathbf{y}} c(\mathbf{y}) + \sum_{i=1}^{m} \lambda_i^\star g_i(\mathbf{y}) + \sum_{j=1}^{p} \nu_j^\star h_j(\mathbf{y})$$

$$= c(\mathbf{x}^\star) + \sum_{i=1}^{m} \lambda_i^\star g_i(\mathbf{x}^\star)$$

When does this hold?

Complementarity slackness condition:

$$\sum_{i=1}^{m} \lambda_i^\star g_i(\mathbf{x}^\star) = 0 \quad \Rightarrow \quad \lambda_i^\star g_i(\mathbf{x}^\star) = 0,\ \forall i = 1, \ldots, m$$

- if i-th constraint inactive at optimum, then $\lambda_i^\star = 0$
- if $\lambda_i^\star > 0$, i-th constraint active at optimum

Lagrangian optimality

Suppose that

- constraint functions $g_i : \mathbb{R}^n \mapsto \mathbb{R}$ and $h_j : \mathbb{R}^n \mapsto \mathbb{R}$ are differentiable
- $\mathbf{x}^\star, \boldsymbol{\lambda}^\star$, and $\boldsymbol{\nu}^\star$ are primal-dual optimal with zero duality gap

Due to complementarity slackness

$$\mathcal{L}(\mathbf{x}^\star, \boldsymbol{\lambda}^\star, \boldsymbol{\nu}^\star) = \min_{\mathbf{y}} \mathcal{L}(\mathbf{y}, \boldsymbol{\lambda}^\star, \boldsymbol{\nu}^\star)$$

That is, $\mathbf{x}^\star$ minimizes $\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}^\star, \boldsymbol{\nu}^\star)$. Therefore, $\nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x}^\star, \boldsymbol{\lambda}^\star, \boldsymbol{\nu}^\star) = \mathbf{0}$

$$\nabla_{\mathbf{x}} c(\mathbf{x}^\star) + \sum_{i=1}^{m} \lambda_i^\star \nabla_{\mathbf{x}} g_i(\mathbf{x}^\star) + \sum_{j=1}^{p} \nu_j^\star \nabla_{\mathbf{x}} h_j(\mathbf{x}^\star) = \mathbf{0}$$

(**Lagrangian optimality conditions**)

Karush–Kuhn–Tucker (KKT) conditions

Suppose that

- constraint functions $g_i : \mathbb{R}^n \mapsto \mathbb{R}$ and $h_j : \mathbb{R}^n \mapsto \mathbb{R}$ are differentiable
- $\mathbf{x}^\star, \boldsymbol{\lambda}^\star$, and $\boldsymbol{\nu}^\star$ are primal-dual optimal with zero duality gap

The **KKT conditions** are:

1. Primal feasibility: $g_i(\mathbf{x}^\star) \leqslant 0,\ \forall i = 1, \ldots, m$ and $h_j(\mathbf{x}^\star) = 0,\ \forall j = 1, \ldots, p$
2. Dual feasibility: $\boldsymbol{\lambda}^\star \succeq \mathbf{0}$
3. Lagrangian optimality:

$$\nabla_{\mathbf{x}} c(\mathbf{x}^\star) + \sum_{i=1}^{m} \lambda_i^\star \nabla_{\mathbf{x}} g_i(\mathbf{x}^\star) + \sum_{j=1}^{p} \nu_j^\star \nabla_{\mathbf{x}} h_j(\mathbf{x}^\star) = \mathbf{0}$$

4. Complementary slackness: $\lambda_i^\star g_i(\mathbf{x}^\star) = 0,\ \forall i = 1, \ldots, m$

Conversely: for convex problems, if $\mathbf{x}^\star, \boldsymbol{\lambda}^\star$, and $\boldsymbol{\nu}^\star$ satisfy the KKT conditions, then they are primal-dual optimal and strong duality holds

Duality of the economic dispatch problem

$$
\begin{array}{lll}
\underset{\mathbf{p}=(p_1,\ldots,p_n)}{\text{minimize}} & c(\mathbf{p}) & \textit{dispatch cost} \\
\text{subject to} & \underline{\mathbf{p}} \leqslant \mathbf{p} \leqslant \overline{\mathbf{p}} \;:\; \underline{\boldsymbol{\lambda}}, \overline{\boldsymbol{\lambda}} & \textit{dispatch limits} \\
& \mathbf{1}^\top \mathbf{p} = d \quad : \nu & \textit{power balance}
\end{array}
$$

Duality of the economic dispatch problem

$$\begin{aligned}
&\underset{\mathbf{p}=(p_1,\dots,p_n)}{\text{minimize}} && c(\mathbf{p}) && \textit{dispatch cost} \\
&\text{subject to} && \underline{\mathbf{p}} \leqslant \mathbf{p} \leqslant \overline{\mathbf{p}} \ : \underline{\boldsymbol{\lambda}}, \overline{\boldsymbol{\lambda}} && \textit{dispatch limits} \\
& && \mathbf{1}^\top \mathbf{p} = d \quad : \nu && \textit{power balance}
\end{aligned}$$

- **Lagrangian function**
- **Dual function**
- **Dual problem**
- **Complementarity slackness**
- **KKT optimality conditions**

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$$
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\text{subject to} \quad & \underline{\mathbf{p}} \leqslant \mathbf{p} \leqslant \overline{\mathbf{p}} \ : \underline{\boldsymbol{\lambda}}, \overline{\boldsymbol{\lambda}} && \textit{dispatch limits} \\
& \mathbf{1}^\top \mathbf{p} = d \quad : \nu && \textit{power balance}
\end{aligned}
$$

- **Lagrangian function**

$$
\mathcal{L}(\mathbf{p}, \underline{\boldsymbol{\lambda}}, \overline{\boldsymbol{\lambda}}, \nu) = c(\mathbf{p}) + \underline{\boldsymbol{\lambda}}^\top (\underline{\mathbf{p}} - \mathbf{p}) + \overline{\boldsymbol{\lambda}}^\top (\mathbf{p} - \overline{\mathbf{p}}) + \nu(\mathbf{1}^\top \mathbf{p} - d)
$$

Duality of the economic dispatch problem

$$\begin{array}{lll} \underset{\mathbf{p}=(p_1,\ldots,p_n)}{\text{minimize}} & c(\mathbf{p}) & \textit{dispatch cost} \\ \text{subject to} & \underline{\mathbf{p}} \leqslant \mathbf{p} \leqslant \overline{\mathbf{p}} \;\; : \underline{\boldsymbol{\lambda}}, \overline{\boldsymbol{\lambda}} & \textit{dispatch limits} \\ & \mathbf{1}^\top \mathbf{p} = d \quad : \nu & \textit{power balance} \end{array}$$

- **Dual function**

$$\phi(\underline{\boldsymbol{\lambda}}, \overline{\boldsymbol{\lambda}}, \nu) = \min_{\mathbf{p}} \mathcal{L}(\mathbf{p}, \underline{\boldsymbol{\lambda}}, \overline{\boldsymbol{\lambda}}, \nu) = \mathcal{L}(\mathbf{p}^\star, \underline{\boldsymbol{\lambda}}, \overline{\boldsymbol{\lambda}}, \nu)$$

For linear cost $c(\mathbf{p}) = \mathbf{c}^\top \mathbf{p}$

$$\phi(\underline{\boldsymbol{\lambda}}, \overline{\boldsymbol{\lambda}}, \nu) = \begin{cases} \overline{\boldsymbol{\lambda}}^\top \overline{\mathbf{p}} - \underline{\boldsymbol{\lambda}}^\top \underline{\mathbf{p}} - \nu d, & \text{if } \mathbf{c} - \underline{\boldsymbol{\lambda}} + \overline{\boldsymbol{\lambda}} + \mathbf{1}\nu = \mathbf{0} \\ -\infty(\text{unbounded}), & \text{if otherwise} \end{cases}$$

For quadratic cost $c(\mathbf{p}) = \mathbf{p}^\top \mathbf{C} \mathbf{p} + \mathbf{c}^\top \mathbf{p} \quad (\mathbf{C} \succeq \mathbf{0})$

$$\mathbf{p}^\star = -\frac{1}{2} \mathbf{C}^{-1} (\mathbf{c} - \underline{\boldsymbol{\lambda}} + \overline{\boldsymbol{\lambda}} + \mathbf{1}\nu)$$

$$\phi(\underline{\boldsymbol{\lambda}}, \overline{\boldsymbol{\lambda}}, \nu) = -\frac{1}{2} \left\| \mathbf{c} - \underline{\boldsymbol{\lambda}} + \overline{\boldsymbol{\lambda}} + \mathbf{1}\nu \right\|_{\mathbf{C}^{-1}}^2 + \overline{\boldsymbol{\lambda}}^\top \overline{\mathbf{p}} - \underline{\boldsymbol{\lambda}}^\top \underline{\mathbf{p}} - \nu d$$

Duality of the economic dispatch problem

$$\begin{array}{lll}\underset{\mathbf{p}=(p_1,\dots,p_n)}{\text{minimize}} & c(\mathbf{p}) & \textit{dispatch cost}\\ \text{subject to} & \underline{\mathbf{p}} \leqslant \mathbf{p} \leqslant \overline{\mathbf{p}} \ : \underline{\boldsymbol{\lambda}}, \overline{\boldsymbol{\lambda}} & \textit{dispatch limits}\\ & \mathbf{1}^\top \mathbf{p} = d \quad : \nu & \textit{power balance}\end{array}$$

- **Dual problem**

For linear economic dispatch

$$\begin{array}{ll}\underset{\underline{\boldsymbol{\lambda}},\overline{\boldsymbol{\lambda}},\boldsymbol{\nu}}{\text{maximize}} & \overline{\boldsymbol{\lambda}}^\top \overline{\mathbf{p}} - \underline{\boldsymbol{\lambda}}^\top \underline{\mathbf{p}} - \nu d\\ \text{subject to} & \mathbf{c} - \underline{\boldsymbol{\lambda}} + \overline{\boldsymbol{\lambda}} + \mathbf{1}\nu = \mathbf{0}\\ & \underline{\boldsymbol{\lambda}}, \overline{\boldsymbol{\lambda}} \succeq \mathbf{0}\end{array}$$

For quadratic economic dispatch

$$\begin{array}{ll}\underset{\underline{\boldsymbol{\lambda}},\overline{\boldsymbol{\lambda}},\boldsymbol{\nu}}{\text{maximize}} & -\frac{1}{2}\left\|\mathbf{c} - \underline{\boldsymbol{\lambda}} + \overline{\boldsymbol{\lambda}} + \mathbf{1}\nu\right\|^2_{\mathbf{C}^{-1}} + \overline{\boldsymbol{\lambda}}^\top \overline{\mathbf{p}} - \underline{\boldsymbol{\lambda}}^\top \underline{\mathbf{p}} - \nu d\\ \text{subject to} & \underline{\boldsymbol{\lambda}}, \overline{\boldsymbol{\lambda}} \succeq \mathbf{0}\end{array}$$

Duality of the economic dispatch problem

$$
\begin{array}{lll}
\underset{\mathbf{p}=(p_1,\ldots,p_n)}{\text{minimize}} & c(\mathbf{p}) & \textit{dispatch cost} \\
\text{subject to} & \underline{\mathbf{p}} \leqslant \mathbf{p} \leqslant \overline{\mathbf{p}} \;\; : \underline{\boldsymbol{\lambda}}, \overline{\boldsymbol{\lambda}} & \textit{dispatch limits} \\
& \mathbf{1}^\top \mathbf{p} = d \quad : \nu & \textit{power balance}
\end{array}
$$

- **Complementarity slackness**

$$
\underline{\lambda}_i(\underline{p}_i - p_i) = 0, \quad \forall i = 1, \ldots, n
$$

$$
\overline{\lambda}_i(p_i - \overline{p}_i) = 0, \quad \forall i = 1, \ldots, n
$$

Duality of the economic dispatch problem

$$
\begin{array}{llr}
\underset{\mathbf{p}=(p_1,\ldots,p_n)}{\text{minimize}} & c(\mathbf{p}) & \textit{dispatch cost} \\
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 & \mathbf{1}^\top \mathbf{p} = d \quad : \nu & \textit{power balance}
\end{array}
$$

- **KKT optimality conditions**

  Primal feasibility: $\underline{\mathbf{p}} \leqslant \mathbf{p} \leqslant \overline{\mathbf{p}},\ \mathbf{1}^\top \mathbf{p} = d$

  Dual feasibility: $\underline{\boldsymbol{\lambda}}, \overline{\boldsymbol{\lambda}} \succeq \mathbf{0}$

  Lagrangian optimality: $\nabla_{\mathbf{x}} c(\mathbf{x}) - \underline{\boldsymbol{\lambda}} + \overline{\boldsymbol{\lambda}} + \mathbf{1}\nu = \mathbf{0}$

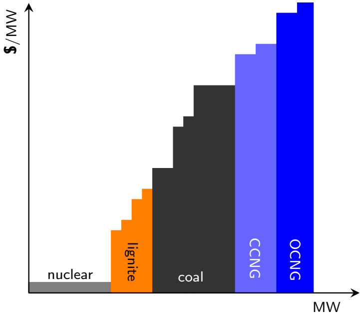
  Complementary slackness: $\underline{\boldsymbol{\lambda}} \circ (\underline{\mathbf{p}} - \mathbf{p}) = 0,\ \overline{\boldsymbol{\lambda}} \circ (\mathbf{p} - \overline{\mathbf{p}}) = 0$

“We still have an electricity market that is designed in a way like it was necessary twenty years ago when we started to bring in the renewables [. . .] Today, the market is completely different and **this market system does not work any more**”. Ursula von der Leyen, 8 June 2022

“People are being charged for their electricity prices on the basis of the top marginal gas price, and that is frankly ludicrous. We need to get rid of that system.”
Boris Johnson, 25 June 2022

“You have skyrocketing electricity prices that no longer have anything to do with electricity production costs, it follows gas, it’s absurd” Emmanuel Macron, 28 June 2022

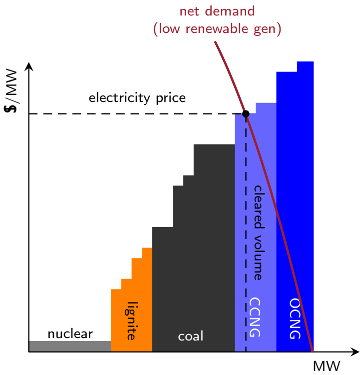
Electricity pricing via economic dispatch



- Merit-order (supply) curve: cost-based ranking of power producers
- Net demand = total electricity demand − renewable generation
- The price is at the intersection of net demand and supply

`https://neon.energy/marginal-pricing`

Electricity pricing via economic dispatch



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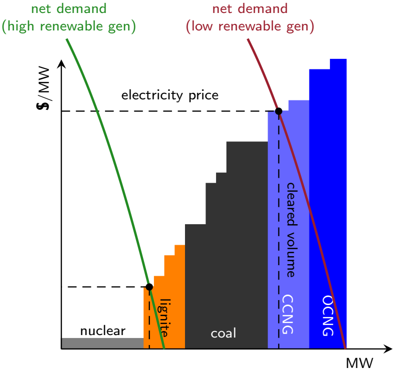

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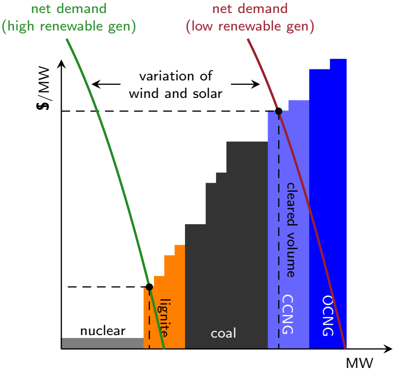
Electricity pricing via economic dispatch



- Merit-order (supply) curve: cost-based ranking of power producers
- Net demand = total electricity demand − renewable generation
- The price is at the intersection of net demand and supply

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Electricity pricing via economic dispatch (cont'd)

- In the optimum, the Lagrangian function equals to the optimal dispatch cost

$$\mathcal{L}(\mathbf{p}^\star, \underline{\boldsymbol{\lambda}}^\star, \overline{\boldsymbol{\lambda}}^\star, \nu^\star) = c(\mathbf{p}^\star)$$

- Electricity price = change of dispatch costs due to marginal change of demand ζ

$$\begin{aligned}
&\frac{\partial}{\partial \zeta}\mathcal{L}(\mathbf{p}^\star, \underline{\boldsymbol{\lambda}}^\star, \overline{\boldsymbol{\lambda}}^\star, \nu^\star, \zeta) \\
&\quad = \frac{\partial}{\partial \zeta}\Big(c(\mathbf{p}^\star) + \underline{\boldsymbol{\lambda}}^{\star\top}(\underline{\mathbf{p}} - \mathbf{p}^\star) + \overline{\boldsymbol{\lambda}}^{\star\top}(\mathbf{p}^\star - \overline{\mathbf{p}}) + \nu^\star(\mathbf{1}^\top\mathbf{p}^\star - d + \zeta)\Big) = \nu^\star
\end{aligned}$$

The dual variable of the power balance constraint is the marginal system cost

- Let generator j be a marginal generator. From Lagrangian optimality

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial p_j} &= c_j - \underline{\lambda}_j^\star + \overline{\lambda}_j^\star - \nu^\star \qquad \text{(linear cost)} \\
&= c_j - \nu^\star = 0 \quad \Rightarrow \nu^\star = c_j
\end{aligned}$$

Electricity price is the cost of the marginal generator

Equilibrium interpretation of the economic dispatch problem

Equilibrium formulation of electricity market clearing:

1. Optimal generator response $p_i^\star : \mathbb{R} \mapsto \mathbb{R}$ to electricity prices

$$p_i^\star(\nu) = \underset{\underline{p}_i \leqslant p_i \leqslant \overline{p}_i}{\operatorname{argmax}} \; \nu p_i - c_i(p_i), \quad \forall i = 1, \ldots, n$$

2. Power balance (equilibrium) condition

$$\sum_{i=1}^{n} p_i^\star(\nu^\star) = d$$

Equilibrium price $\nu^\star$ financially incentivizes generators to meet the demand.