How to Use a Short Basis: Trapdoors for Hard Lattices and New Cryptographic Constructions

Chris Peikert
SRI

Work with Craig Gentry and Vinod Vaikuntanathan

Digital Signatures

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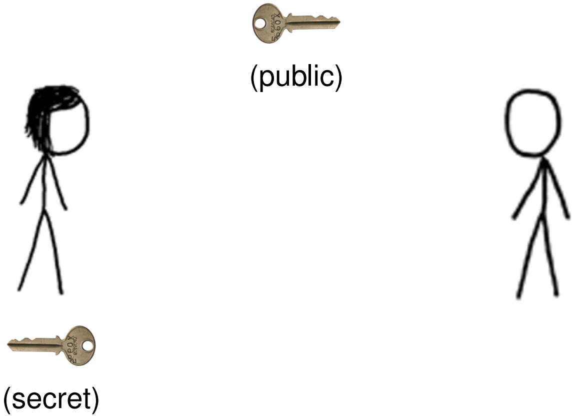

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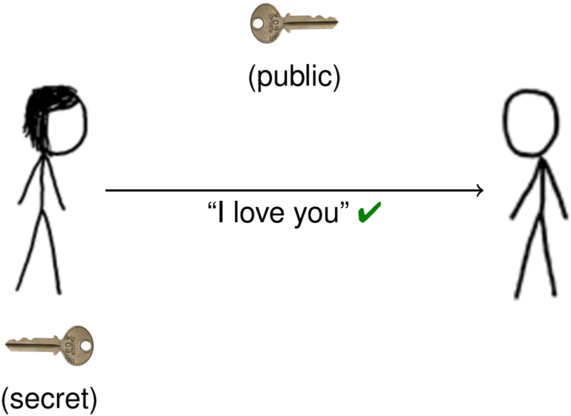

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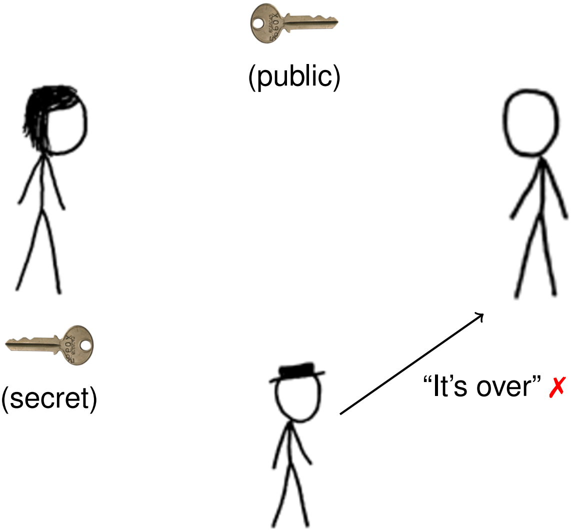

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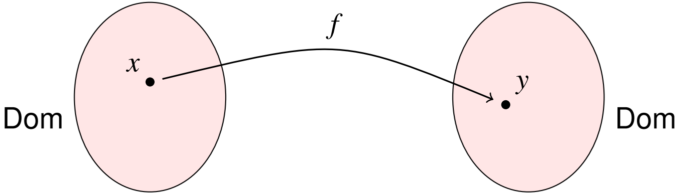

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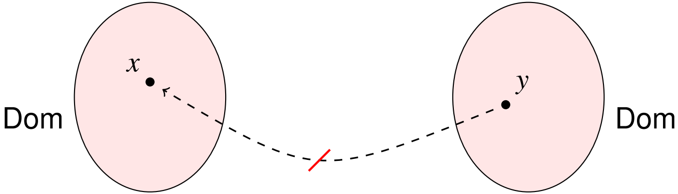

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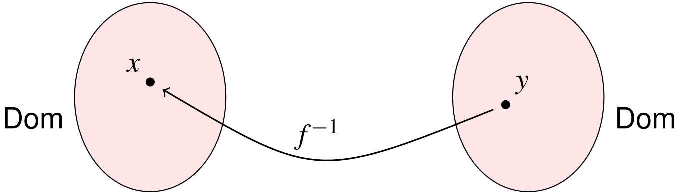

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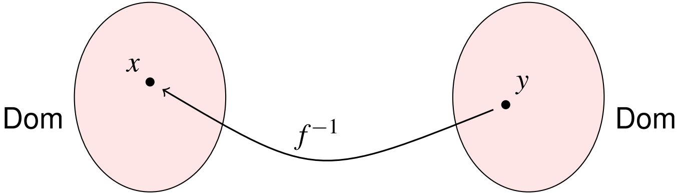

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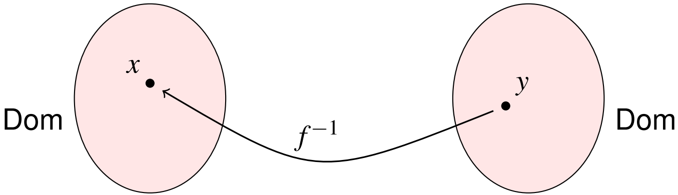

- Candidates: [RSA78,Rabin79,Paillier99]
  - ✔ "General assumption"
  - ✔ Applications: digital signatures, OT, NIZK, ...
- All rely on hardness of factoring
  - ✗ Complex: 2048-bit exponentiation
  - ✗ Lack of diversity
  - ✗ Broken by quantum algorithms [Shor]

Lattice-Based Cryptography

What's To Like

- Simple & efficient: linear ops, small integers
- Resist subexp & quantum attacks (so far)
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1. One-way & collision-resistant functions [Ajtai,. . . ,MicciancioRegev]
2. Public-key encryption [AjtaiDwork,Regev]
3. Recent developments [LyubMicc,PeikWat,. . .]

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What's Missing

- Everything else!

  Practical signatures, protocols, "advanced" crypto, ...

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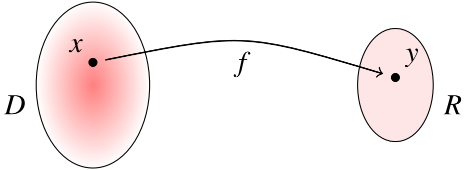

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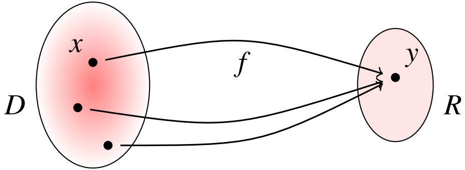

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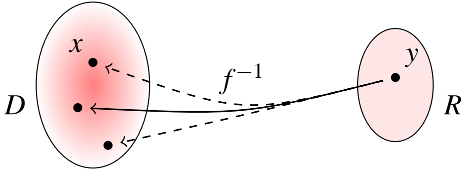

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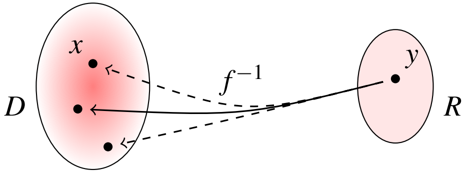

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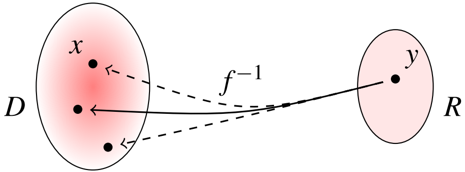

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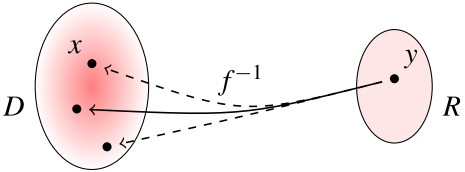

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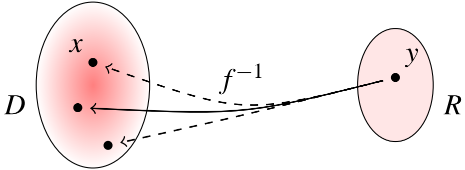

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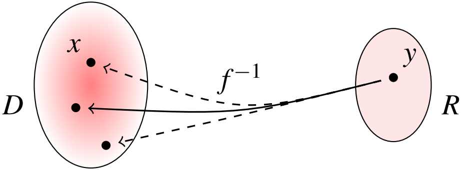

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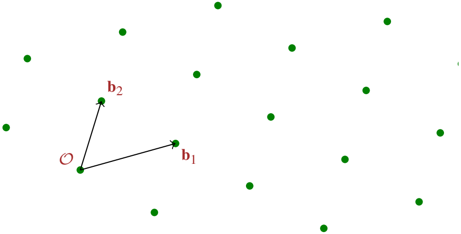
New Algorithmic Tool

- "Oblivious decoder" on lattices

Lattices

A lattice $\mathcal{L} \subset \mathbb{R}^n$ having basis $\mathbf{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is:

$$\mathcal{L} \quad = \quad \sum_{i=1}^{n} (\mathbb{Z} \cdot \mathbf{b}_i)$$



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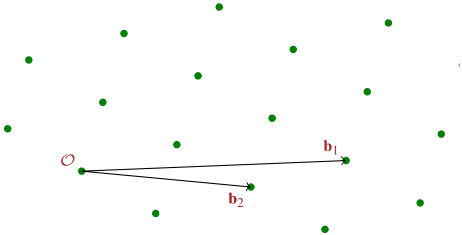

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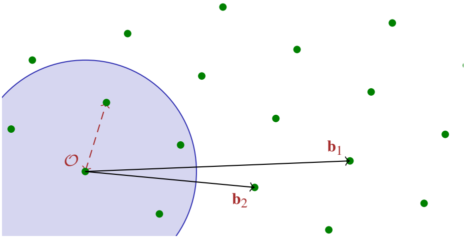

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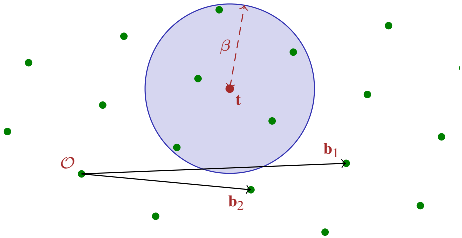

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Absolute Distance Decoding (ADD$_\beta$)

- Given $\mathbf{B}$ and target $\mathbf{t} \in \mathbb{R}^n$, find some $\mathbf{v} \in \mathcal{L}$ within distance β.

Complexity of Lattice Problems

SVP$_\gamma$ in the Worst Case

$\gamma =$	$O(1)$	$\text{poly}(n)$	2^n
	NP-hard	2^n time	$\text{poly}(n)$ time
	[Ajt,Mic,Kho]	[AKS]	[LLL,Sch]

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Bottom Line

- On random lattices, SVP_γ and ADD_β seem exponentially hard

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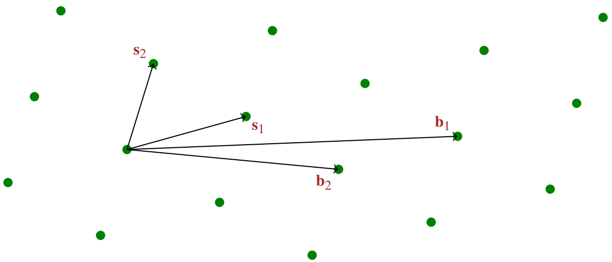

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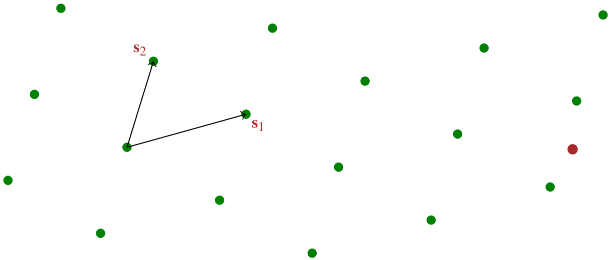

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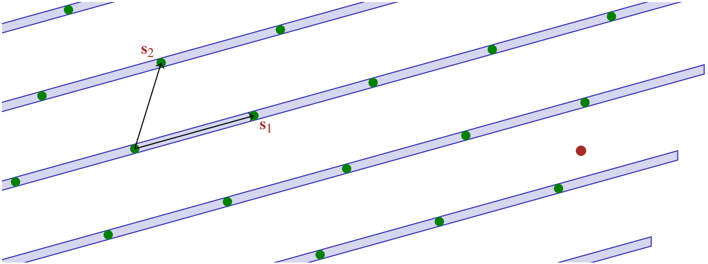

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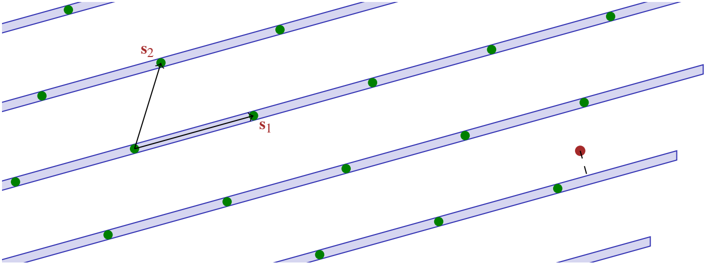

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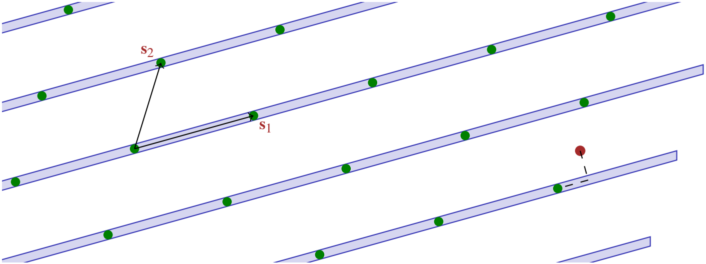

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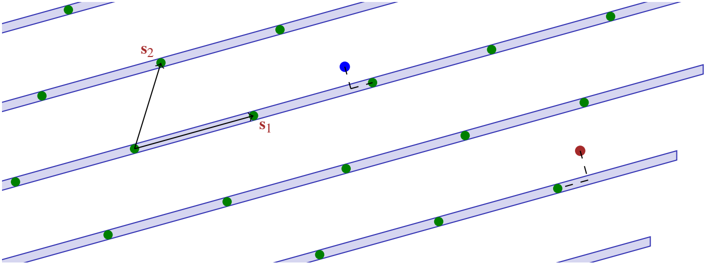

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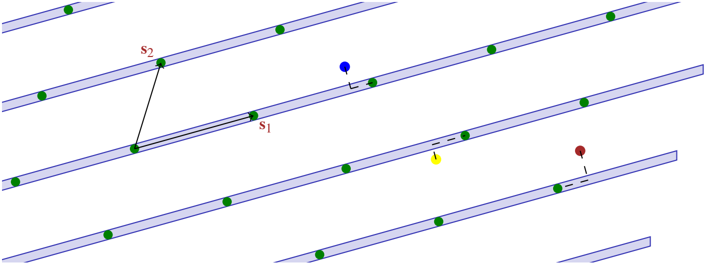

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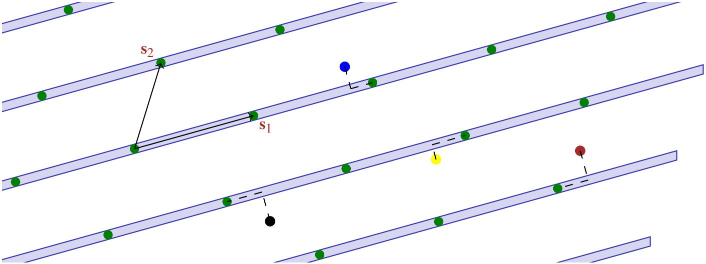

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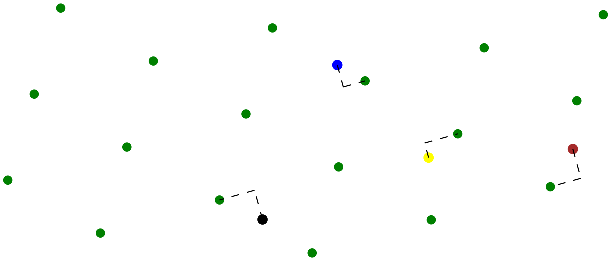

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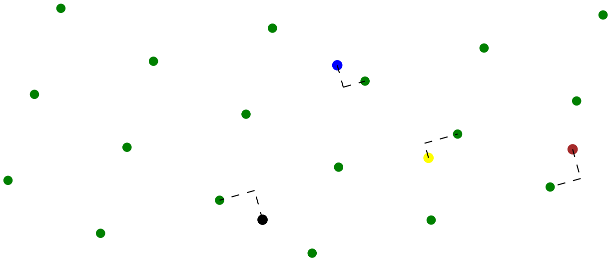

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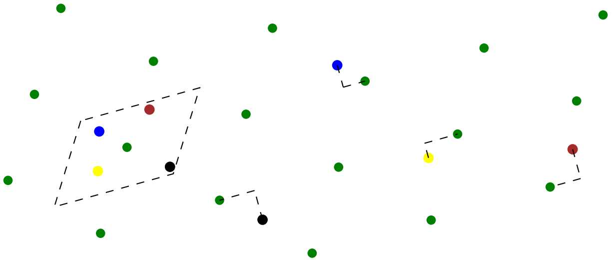

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2. Secret key leakage
   - Total break after several signatures [NguyenRegev]

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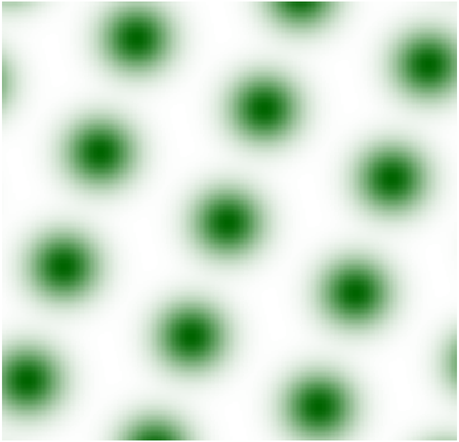

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"Uniform" in $\mathbb{R}^n$ when std dev $\geq$ shortest basis

[Regev,MicciancioRegev]

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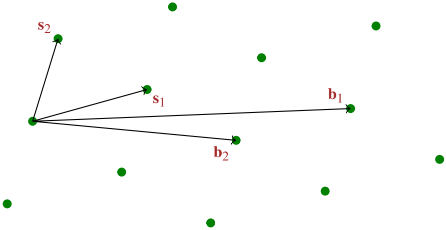

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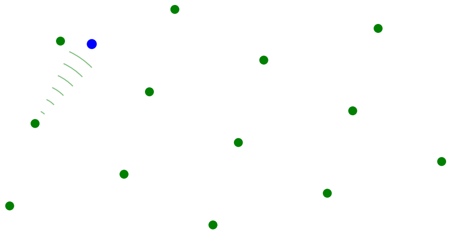

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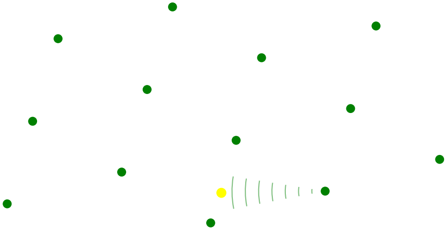

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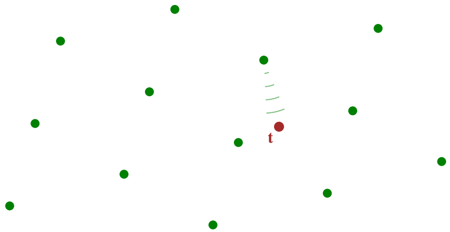

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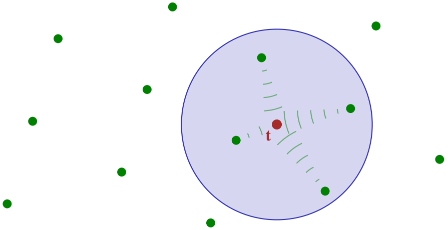

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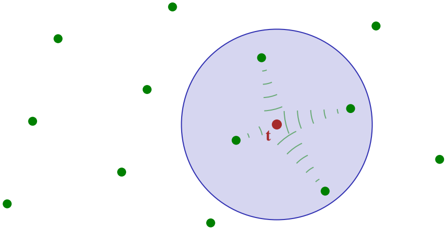
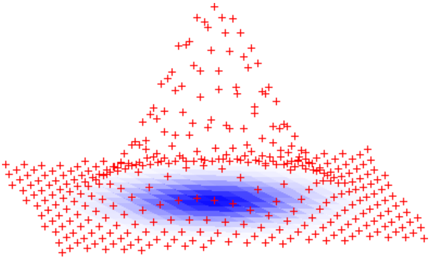


- Conditional distribution is "discrete Gaussian" $D_{\mathcal{L},\mathbf{t}}$



Analysis tool in [Ban,AR,Reg,MR,Pei,. . .]

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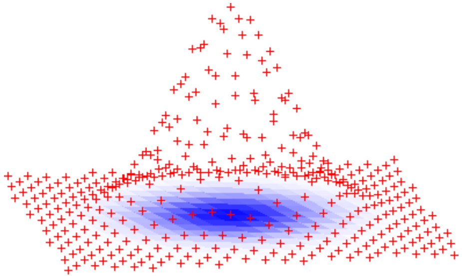

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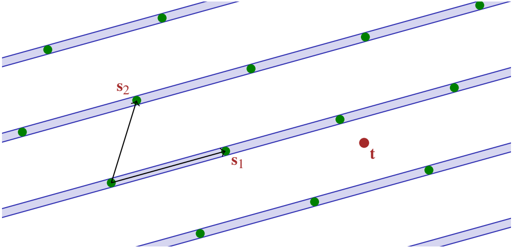

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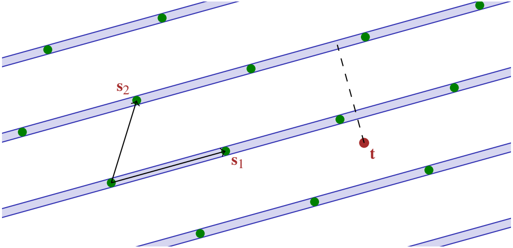

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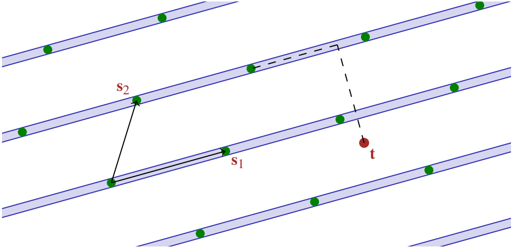

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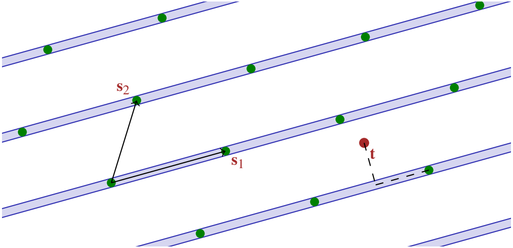

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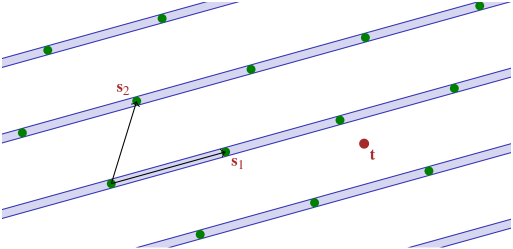

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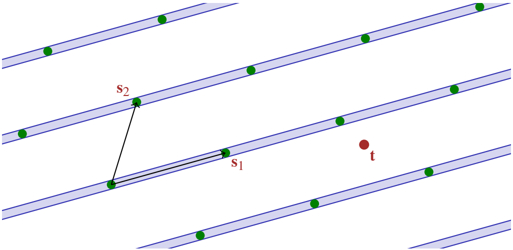

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[This work]: std dev $\geq \max\|\tilde{\mathbf{s}}_i\| \Rightarrow$ samples $D_{\mathcal{L},\mathbf{t}}$ exactly*

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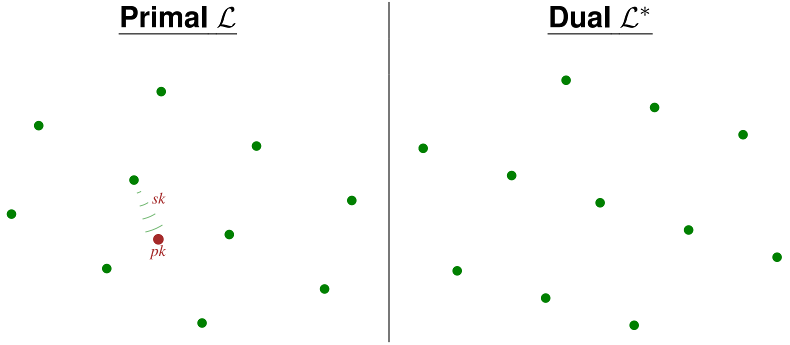
Cryptosystem with Master Trapdoor



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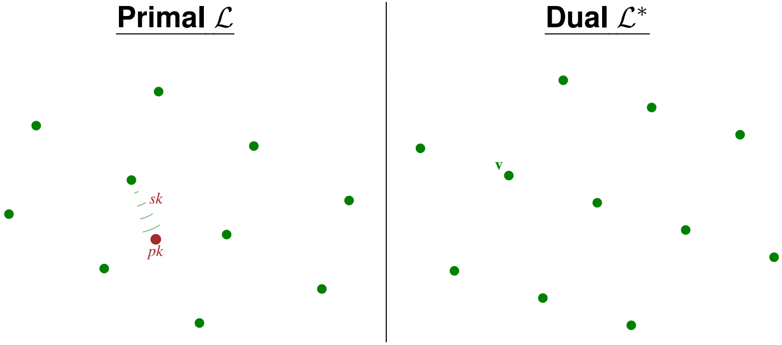

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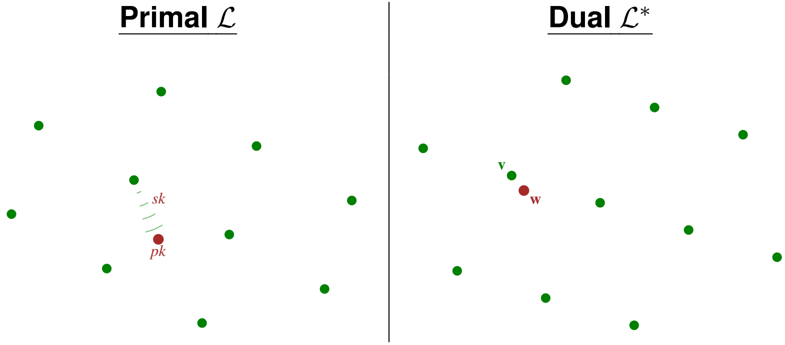

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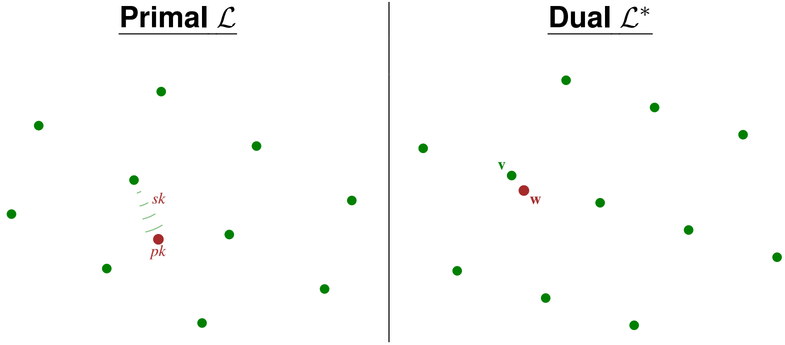

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- For $\mathbf{w} \approx \mathbf{v}$: $\langle \mathbf{v}, pk \rangle \approx \langle \mathbf{w}, sk \rangle \bmod 1$ "quasi"-agreement
- Security: decoding $\mathbf{w}$, a.k.a. "learning with errors"
  - Quantum worst-case connection [Regev]
  - Now: classical worst-case hardness [P]

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Thanks!

(Artwork courtesy of xkcd.org)