Efficient Collision-Resistant Hashing from Worst-Case Assumptions on Cyclic Lattices

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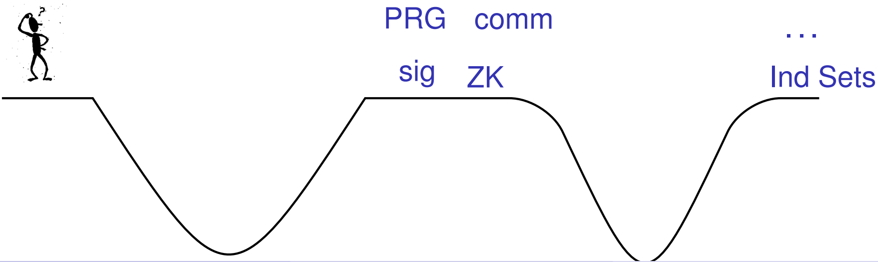
Theory of Cryptography Conference

5 March 2006

One-Way Function (family):

$$a, \quad y = f_a(x) \xrightarrow{\text{hard}} x' \in f_a^{-1}(y)$$

✔ Sufficient for *some* crypto



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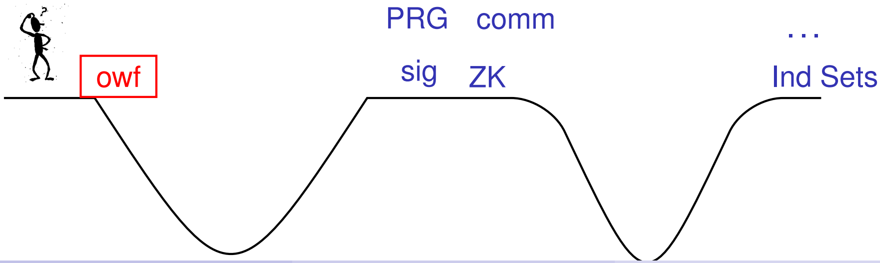

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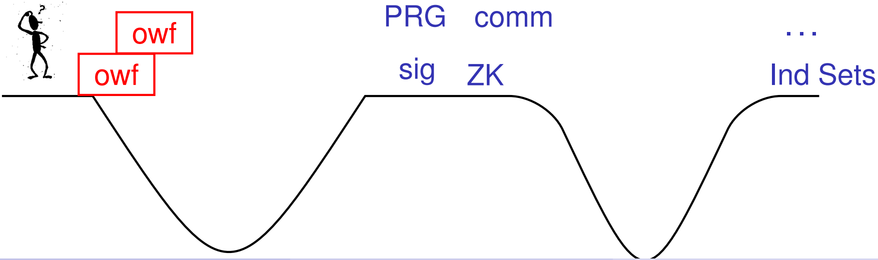

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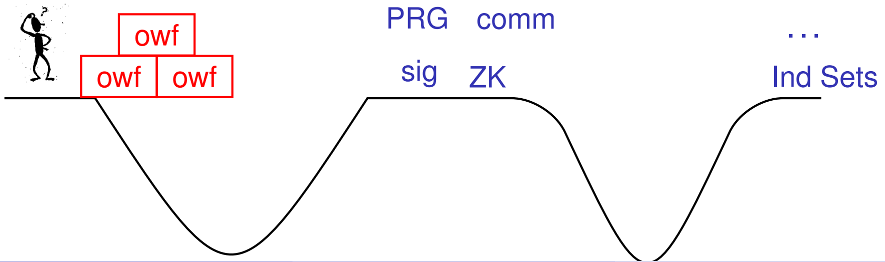

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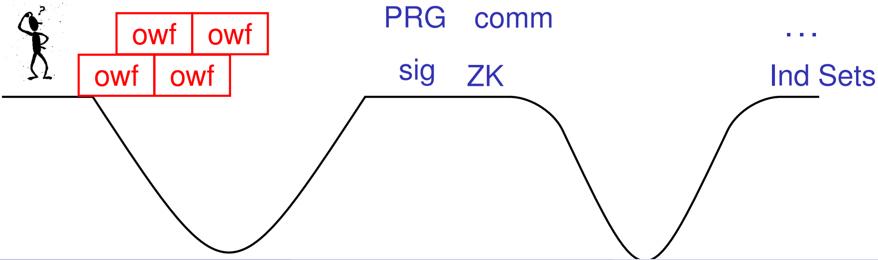

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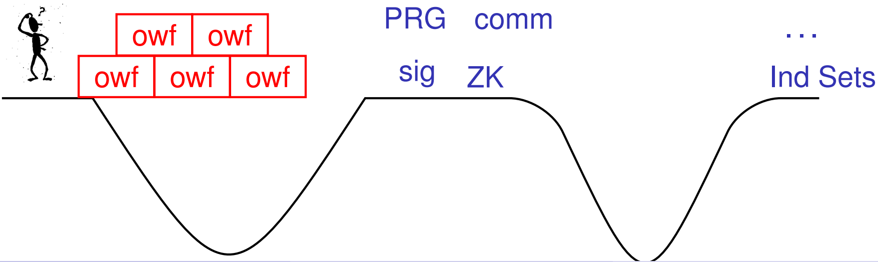

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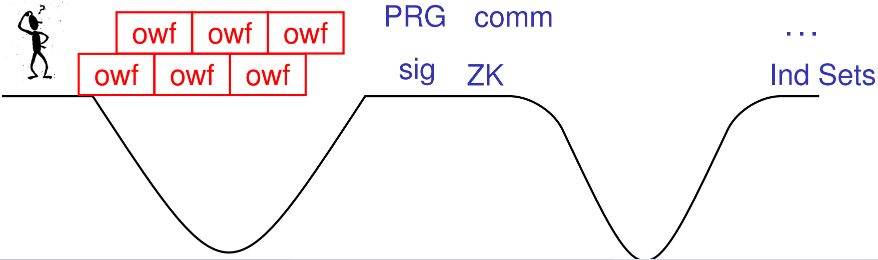

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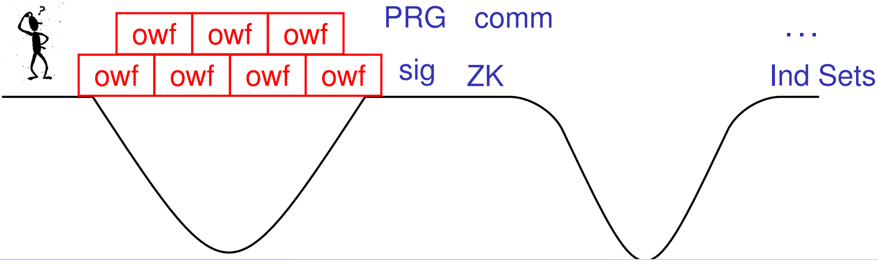

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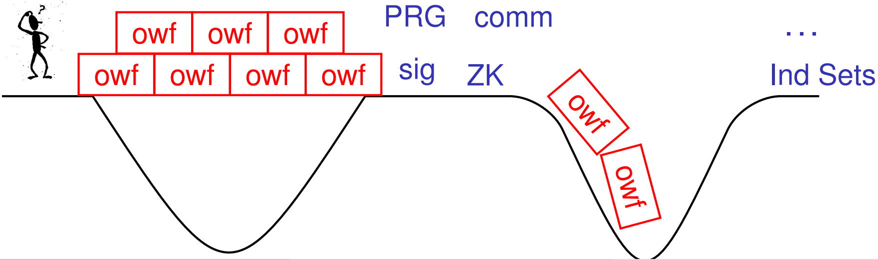
- ✗ Can't realize some notions at all! (black-box)



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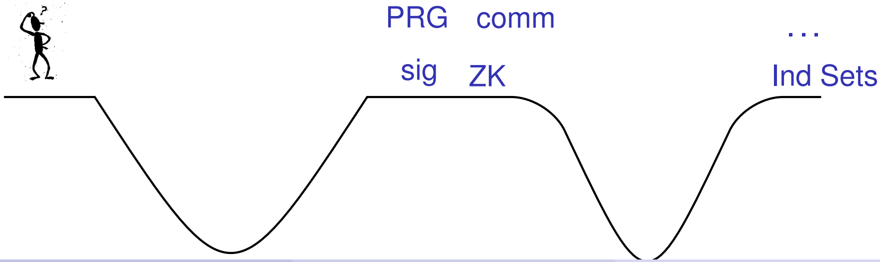

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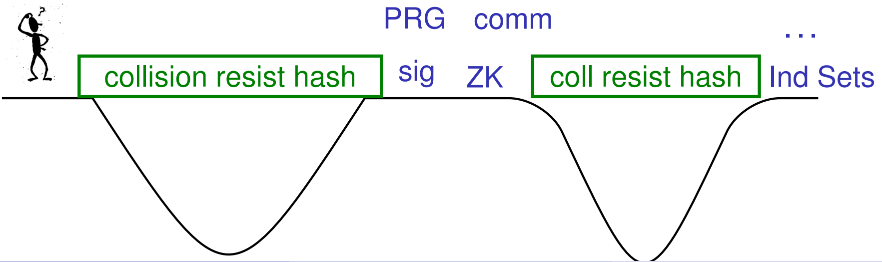

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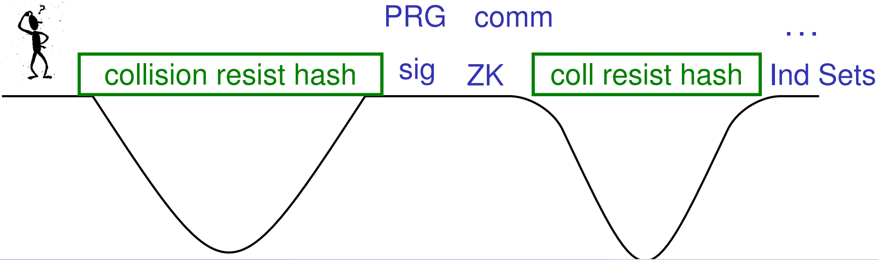

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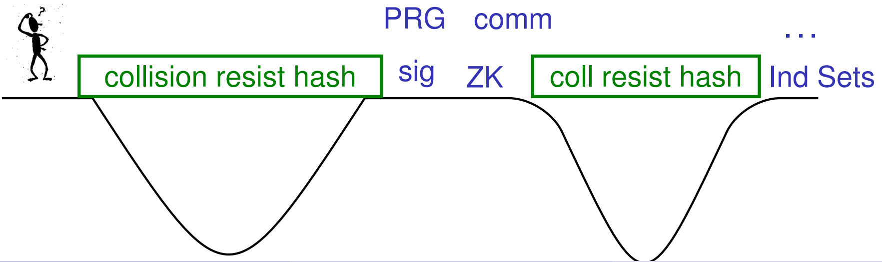
- ☞ MD5, SHA-1 highlight need for sound & efficient hashes



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☞ Our function is a certain kind of knapsack…

Generalized Knapsack Function [Mic02]

Let R be a ring with $+$ and $\times$, and let $S \subseteq R$. For:

- $\mathbf{A} = (\mathbf{a}_1, \ldots, \mathbf{a}_m) \in R^m$ — m "weights": key
- $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_m) \in S^m$ — m "coeffs": input

$$f_{\mathbf{A}}(\mathbf{X}) = \sum_{i=1}^{m} \mathbf{a}_i \times \mathbf{x}_i$$

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Lineage of Cryptographic Knapsacks

Knapsack Function	Security Notion	Efficient?
[Ajt96, GGH97]	collision-resistant	✗
[Mic02]	one-way	✔
Today	collision-resistant	✔✔

Micciancio's Function

- $R = (\mathbb{Z}_p^n, +, \otimes)$, where $\otimes$ is cyclic convolution:

$$\begin{bmatrix} | \\ \mathbf{a} \\ | \end{bmatrix} \otimes \begin{bmatrix} | \\ \mathbf{x} \\ | \end{bmatrix} = \begin{bmatrix} a_0 & a_{n-1} & \cdots & a_1 \\ a_1 & a_0 & \cdots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \cdots & a_0 \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

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Evaluating f

$$\mathbf{A} = \begin{bmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_m \\ | & | & & | \end{bmatrix} \in R^m$$

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"decoding" in cyclic lattices hard to approx in the worst case

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Today: *No!* (But we have a remedy...)

Collisions via an Algebraic View

Ring $R = \mathbb{Z}_p^n$ under $\otimes$ has algebraic structure:

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Yields a collision:

$$\mathbf{a}_i(\alpha) \cdot \underbrace{(\alpha^{n-1} + \cdots + 1)}_{\mathbf{x}_i} \quad = \quad \mathbf{a}_i(\alpha) \cdot \underbrace{0}_{\mathbf{x}_i'} \quad \bmod (\alpha^n - 1)$$

Works because $\mathbb{Z}_p[\alpha]/(\alpha^n - 1)$ is *not* an integral domain.

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Choose n prime.

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(Cyclic) Lattices

Let $\mathbf{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \subset \mathbb{Z}^n$ be linearly independent.
The lattice $\mathcal{L}(\mathbf{B}) \subset \mathbb{Z}^n$ having basis $\mathbf{B}$ is:

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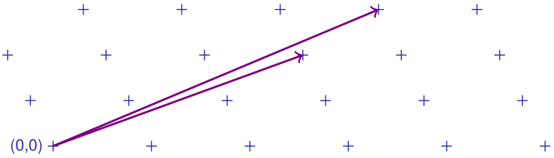

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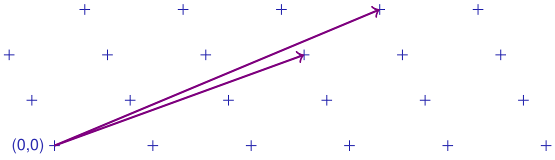
Lattice Λ is cyclic if $\mathbf{x} \in \Lambda \Rightarrow \mathrm{rot}(\mathbf{x}) \in \Lambda$.
For $\mathbf{x} = (x_0, \ldots, x_{n-1})$: $\quad \mathrm{rot}(\mathbf{x}) = (x_{n-1}, x_0, \ldots, x_{n-2})$.



(Cyclic) Lattices

Let $\mathbf{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \subset \mathbb{Z}^n$ be linearly independent.
The lattice $\mathcal{L}(\mathbf{B}) \subset \mathbb{Z}^n$ having basis $\mathbf{B}$ is:

$$\mathcal{L}(\mathbf{B}) = \left\{ \sum_{i=1}^{d} c_i \mathbf{b}_i \mid \forall\, i, c_i \in \mathbb{Z} \right\}.$$

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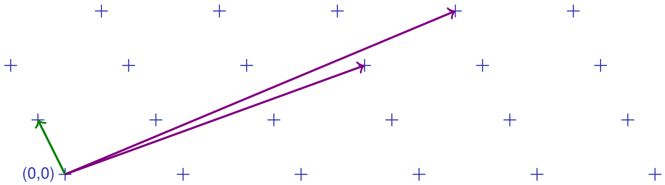
Cyclic lattices are closed under convolution with any $\mathbf{v} \in \mathbb{Z}^n$:

$$\mathbf{x} \otimes \mathbf{v} = \begin{pmatrix} x_0 & x_{n-1} & \cdots & x_1 \\ x_1 & x_0 & \cdots & x_2 \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-1} & x_{n-2} & \cdots & x_0 \end{pmatrix} \cdot \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix} \in \Lambda.$$

Complexity of Shortest Vector

Shortest Vector Problem (SVP)

Given $\mathbf{B}$, find $\mathbf{v} \in \mathcal{L}(\mathbf{B}), \mathbf{v} \neq 0$ s.t. $\|\mathbf{v}\|$ (approx) minimal.



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Our Assumption

For prime dimensions n, SVP hard to approx
to within $\tilde{\Theta}(n)$ in *cyclic* lattices, *in the worst case*.

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For any polynomial $\Phi(\alpha) \mid (\alpha^n - 1)$, define the linear subspace:

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Corollary: $H_{\alpha-1}$ is "hard-core" for SVP.

Worst-Case to Average-Case Reduction

Solve SVP in $H_{\alpha-1}$

For *any* $\mathbf{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \subset \mathbb{Z}^n$ generating lattice Λ,
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- ✔ "Bad" oracle answers are very rare (with elementary proof). (Integral domain.)
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