Practical Bootstrapping in Quasilinear Time

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Fully Homomorphic Encryption [RAD'78,Gen'09]

- FHE lets you do this:

$$\boxed{\mu} \longrightarrow \boxed{\mathsf{Eval}\left(f\ ,\ \boxed{\mu}\right)} \longrightarrow \boxed{f(\mu)}$$

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- Naturally occurring schemes are "somewhat homomorphic" (SHE): they can only evaluate functions of an *a priori* bounded depth.

$$\boxed{\mu} \rightarrow \boxed{\mathsf{Eval}\left(f, \boxed{\mu}\right)} \rightarrow \boxed{f(\mu)} \rightarrow \boxed{\mathsf{Eval}\left(g, \boxed{f(\mu)}\right)} \rightarrow \boxed{g(f(\mu))}$$

Bootstrapping: SHE $\to$ FHE [Gen'09]

- Homomorphically evaluates the SHE decryption function to "refresh" a ciphertext $\boxed{\mu}$, allowing further homomorphic operations.

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  - ⋆ The only known way of obtaining unbounded FHE.
  - ⋆ Goal: Efficiency! Minimize depth d and size s of decryption "circuit."
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- The asymptotically most efficient methods on "packed" ciphertexts [GHS'12a,GHS'12b] are very complex, and appear practically worse than asymptotically slower methods.

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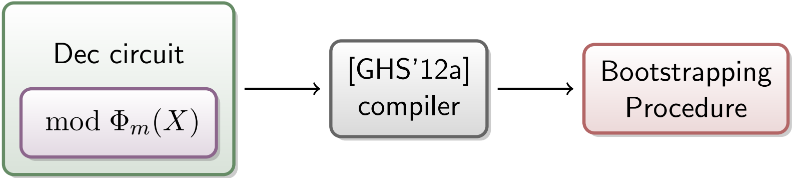

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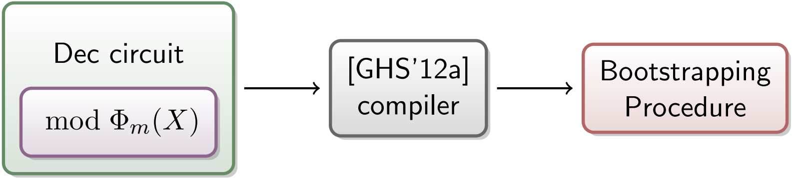

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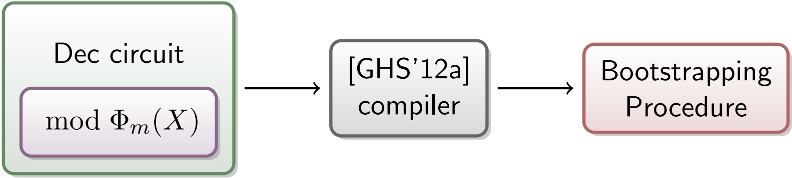

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✗✗ [GHS'12a] compiler is very complex, w/large polylog overhead factor.

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   - ✔ Special purpose, completely algebraic description – no "circuits."
   - ✔ Completely decouples the algebraic structure of SHE plaintext ring from that needed for bootstrapping.

Setting the Stage: Decryption in SHE [LPR'10,BV'11,BGV'12]

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  "Packed" plaintext uses more of R_2, e.g., multiple "slots" [SV'11].

Warm-Up:
Bootstrapping Unpacked Ciphertexts

Bootstrapping Unpacked Ciphertexts: Main Idea

1. Isolate message-carrying coefficient v_0 of $v(X)$ by homomorphically "tracing down" a tower of cyclotomic rings $\mathcal{O}_{2k}/\mathcal{O}_k/\cdots/\mathcal{O}_4/\mathbb{Z}$.
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$$v_0 + v_1 X + v_2 X^2 + \cdots v_{k-1} X^{k-1} \qquad \mathbb{Z}_q[X]/(X^k+1)$$

$$v_0 + 0X + v_2 X^2 + \cdots 0 X^{k-1} \qquad \mathbb{Z}_q[X^2]/(X^k+1)$$

$$\vdots$$

$$v_0 + v_{k/4} X^{k/4} + \cdots + v_{3k/4} X^{3k/4} \qquad \mathbb{Z}_q[X^{k/4}]/(X^k+1)$$

$$v_0 + v_{k/2} X^{k/2} \qquad \mathbb{Z}_q[X^{k/2}]/(X^k+1)$$

$$v_0 \qquad \mathbb{Z}_q$$

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$$\begin{array}{cc}
v_0 + v_1 X + v_2 X^2 + \cdots v_{k-1} X^{k-1} & \mathbb{Z}_q[X]/(X^k+1) \\
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v_0 & \mathbb{Z}_q
\end{array}$$

2. Homomorphically "round" $v_0 \in \mathbb{Z}_q$ to the message bit $\lfloor \frac{2}{q} \cdot v_0 \rceil \in \mathbb{Z}_2$.

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- "Product" $\mathbb{Z}$-basis of $\mathcal{O}_k$:

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- Let $v = v_0 \cdot 1 + v_1 \cdot \zeta_i \in \mathcal{O}_i$ for $v_0, v_1 \in \mathcal{O}_{i/2}$.

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- More generally, $\mathrm{Tr}_{\mathcal{O}_i/\mathcal{O}_{i'}}$ sums the automorphisms of $\mathcal{O}_i$ that fix $\mathcal{O}_{i'}$. Key facts:
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Recall: $R = \mathcal{O}_k$, and $v = c_0 + c_1 \cdot s \approx \frac{q}{2}\mu \in R_q$ for message $\mu \in \mathbb{Z}_2 \subseteq R_2$.

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⋆⋆ Now have an encryption of $\lfloor v_0 \rceil = \mu$. Done!

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Detail #2: each $\mathrm{Tr}(\mathcal{O}_i) = 2\mathcal{O}_{i/2}$, so lift to plaintext modulus $2q$, then halve result.

Main Result:
Bootstrapping Packed Ciphertexts

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4. Homomorphically reverse-map $\mathbb{Z}_2$-slots back to B-coeffs:

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(Akin to homomorphic DFT^{-1}.)

Algebra: Slots and CRT Sets

- Let $1 = \ell_0 | \ell_1 | \ell_2 | \cdots$ (all odd), and $S^{(i)} = \mathcal{O}_{\ell_i} = \mathbb{Z}[\zeta_{\ell_i}]$.
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- Similarly for $S_q \cong \bigoplus_j (S/\mathfrak{p}_j^{\lg q})$.

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- Choose S so that S_q has $\geq \deg(R/\mathbb{Z})$ $\mathbb{Z}_q$-slots, via:

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$$(v_j) \in \mathbb{Z}_q^{k/2} \longmapsto \sum v_j \cdot c_j \bmod q$$

  for an appropriate CRT set $C = \{c_j\} \subset S$ of size $k/2$.

- Our goal: homomorphically map $\sum v_j \cdot b_j \in R_q \longmapsto \sum v_j \cdot c_j \in S_q$.

  Equivalently, evaluate the $\mathbb{Z}$-linear* map $L \colon R \to S$ defined by

$$L(b_j) = c_j.$$

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Goal for Remainder of Talk

- Extend ring-switching to (efficiently) handle $\mathbb{Z}$-linear maps $L\colon R \to S$.

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- Proof: define $\bar{L}$ by $\bar{L}(r \otimes s) = L(r) \cdot s \in S$.

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Enhanced Ring-Switching, Efficiently

Key Ideas

- The $\mathbb{Z}$-linear $L \colon R \to S$ given by $L(B) = C$ is "highly structured," because B, C are product sets.

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- Ensure small compositums $T^{(i)} = H^{(i-1)} + H^{(i)}$ via large gcd's: replace prime factors of k with those of ℓ, one at a time.

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Toy Example

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- In general, switch through $\leq \log(\deg(R/\mathbb{Z})) = \log(\lambda)$ hybrid rings, one for each prime factor of k.

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Thanks!