COMPLEX NUMBERS AND PHASORS

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I. Abstract

The purpose of this document is review for EECS 216 students the concept of *phasors*–complex numbers used to represent sinusoids, to simplify the math of manipulating sinusoids. Phasors should have been learned in EECS 215. It also includes a brief primer on complex numbers and their manipulations.

A. Table of contents by sections:

1. Abstract (you're reading this now)
2. Complex numbers: Magnitude, phase, real and imaginary parts
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7. Phasors: Real-world applications (borrowing from EECS 215)

B. Motivation for using Phasors (other than against the EECS 216 instructor)

What do complex numbers have to do with electrical engineering? What good are they? Before you say, "Beam me up, Scotty," consider the following circuits problem:

- <u>Circuits problem:</u> A voltage source $5\cos(6t)$ is connected in series with a 4Ω resistor and a $\frac{1}{2}H$ inductor. The goal is to compute the current $i(t)$ which flows through all three devices.

- <u>Hard Way:</u> In EECS 215 or 314 you learned that Kirchoff's voltage law leads to $5\cos(6t) - 4i - \frac{1}{2}\frac{di}{dt} = 0$.
  - In Math 216 you will learn to solve this using the *trial solution* $i(t) = A\cos(6t) + B\sin(6t)$.
  - Substituting this gives $5\cos(6t) - 4A\cos(6t) - 4B\sin(6t) + \frac{1}{2}6A\sin(6t) - \frac{1}{2}6B\cos(6t) = 0$.
  - Collecting terms gives $[5 - 4A - 3B]\cos(6t) + [-4B + 3A]\sin(6t) = 0$. Set $t = 0$ and $t = \frac{\pi}{12} \rightarrow$
  - $5 = 4A + 3B$ and $0 = 3A - 4B$. Solving$\rightarrow A = 0.8$ and $B = 0.6$, so that $i(t) = 0.8\cos(6t) + 0.6\sin(6t)$.
  - Using the cosine addition formula three times, this simplifies to the answer $i(t) = \cos(6t - 37^o)$.
- Are we having fun yet? But if we use complex numbers and phasors, we can solve the problem this way:
- <u>Easy Way:</u> Phasor $I = 5/[4 + j(6)(1/2)] = 5/(4 + j3) = 1e^{-j37^o} \rightarrow i(t) = \cos(6t - 37^o)$.

Hard way or easy way? You make the call! Want to learn more about the easy way? Read on...

II. Complex numbers: Magnitude, phase, real and imaginary parts

A. You're in EECS Now!

You've seen complex numbers before. For example, solving the quadratic equation $z^2 - 6z + 13 = 0$ using the quadratic formula results in the complex number $3 + 2j$ and its complex conjugate $3 - 2j$ where...

Wait a minute! j? It should be i! Wrong. You're in EECS now, and in EECS i denotes current and $j = \sqrt{-1}$ (note the problem in the section above). Get used to it! Fortunately, Matlab accepts either i or j as $\sqrt{-1}$. We now return you to your regularly scheduled program of complex numbers review.

B. Euler's formula

(Leonhard) Euler's formula relates complex exponentials and trig functions. It states that

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{1}$$

The easiest way to derive it is to set $x = j\theta$ in the power series for e^x:

$$e^{j\theta} = 1 + (j\theta) + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \ldots = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots + j\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots\right) = \cos\theta + j\sin\theta \tag{2}$$

This isn't a rigorous derivation, but it will do for EECS 216.

Note the following immediate implications (these are worth writing on your exam cheat sheet):

- $e^{j\pi/2} = j$ and $e^{j3\pi/2} = e^{-j\pi/2} = -j$ (to visualize this, see below on Argand diagrams);
- $e^{j\pi} = -1$ and $e^{j\pi} + 1 = 0$ cogently relates the five most important numbers in mathematics;
- A few people wax rhapsodic about this formula (I think they need to get lives).

C. Definitions of Sine and Cosine in Terms of $e^{j\theta}$

By adding and subtracting Euler's formula and its complex conjugate, we get

$$\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta}); \quad \sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta}) \tag{3}$$

These are sometimes called the "inverse Euler formulae." lecture notes. A mathematician would call them *definitions* of $\cos\theta$ and $\sin\theta$ (never mind all that opposite-over-hypotenuse stuff...)

These are important formulae of which you will make extensive use in EECS 216. To remember them:

- $\cos\theta$ is an even (symmetric) function of θ, so changing the sign of θ should not change the definition;
- $\sin\theta$ is an odd (anti-symmetric) function, so changing the sign of θ should change the sign in the definition;
- The j in the denominator of $\sin\theta$ is tricky; you simply have to remember it.

D. Polar and Rectangular Forms of Complex Numbers

Multiplying Euler's formula by a constant $M > 0$ yields the two forms of complex numbers:

$$z = \overbrace{\underbrace{M}_{\textbf{MAG}}\ \underbrace{e^{j\theta}}_{\textbf{ARG}}}^{\textbf{POLAR}} = \overbrace{\underbrace{M\cos\theta}_{\textbf{REAL}} + j\underbrace{M\sin\theta}_{\textbf{IMAG}}}^{\textbf{RECTANGULAR}} \tag{4}$$

This allows us to define two ways of representing a complex number:

- Its **polar** form $z = Me^{j\theta}$, in which M=its **magnitude** and θ=its **phase**; and

- Its **rectangular** form $z = M\cos\theta + jM\sin\theta = x + jy$, in which:

- $x = M\cos\theta$=its **real part** and $y = M\sin\theta$=its **imaginary part**.

The following notation is used to represent the above four parts of a complex number z:

- **Magnitude:** $M = |z|$.   **Phase:** $\theta = \arg[z]$ (*argument of z* is another term for *phase of z*)

- **Real part:** $x = Re[z]$.   **Imaginary part:** $y = Im[z]$. **Warning:** $Im[3 + j4] = 4$ **NOT** $j4$!

- Polar form is sometimes written using: $z = Me^{j\theta} = M\angle\theta$ (this is easier to read since θ is larger).

III. Complex numbers: Polar-to-Rectangular Conversion and vice-versa

A. Procedures: Polar-to-Rectangular Conversion and vice-versa

The above equation allows easy conversion between the polar and rectangular forms of a complex number:

- Given the **polar** form $z = Me^{j\theta}$, its **rectangular** form $z = x + jy$ is computed using:

 – $x = M\cos\theta = Re[z]$=**real** part of z;

 – $y = M\sin\theta = Im[z]$=**imaginary** part of z.

- Given the **rectangular** form $z = x + jy$, its **polar** form $z = Me^{j\theta}$ is computed using:

 – $M = \sqrt{x^2 + y^2} = |z|$=**magnitude** of z;

 – $\theta = \arctan(\frac{y}{x}) = \tan^{-1}(\frac{y}{x})$=**phase** of z.   **IF** $x < 0$ then use $\theta = \arctan(\frac{y}{x}) + \pi = \tan^{-1}(\frac{y}{x}) + \pi$.

What's this about $x < 0$? Try converting $z = -1 - j$ to polar form:

- $M = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$ and $\theta = \arctan(\frac{-1}{-1}) = \frac{\pi}{4} \rightarrow -1 - j = \sqrt{2}e^{j\pi/4}$. NOT! (try $P \rightarrow R$).

- The problem is that there are **two** solutions to the always-correct $\tan\theta = \frac{y}{x}$, 180° apart.

- You have to choose the correct one. The sign of the real part x tells you which one is correct.

Your calculator has, somewhere on it, both an $R \rightarrow P$ key or function and a $P \rightarrow R$ key or function.

Please don't ask me to find it for you; some of your calculators look like the dashboard on the space shuttle!

Find them, and learn to use them–you will use them a **lot** in EECS 216.

B. Examples: Polar-to-Rectangular Conversion and vice-versa

$\underline{3 + j4 = 5e^{j0.927}}$:	$3 = 5\cos(0.927)$	$4 = 5\sin(0.927)$	$5 = \sqrt{3^2 + 4^2}$	$0.927 = \arctan(\frac{4}{3})$
$\underline{1 + j = \sqrt{2}e^{j0.785}}$:	$1 = \sqrt{2}\cos(0.785)$	$1 = \sqrt{2}\sin(0.785)$	$\sqrt{2} = \sqrt{1^2 + 1^2}$	$0.785 = \arctan(\frac{1}{1})$
$\underline{-5 + j12 = 13e^{j1.966}}$:	$-5 = 13\cos(1.966)$	$12 = 13\sin(1.966)$	$13 = \sqrt{5^2 + 12^2}$	$1.966 = \arctan(\frac{12}{-5}) + \pi$

You can *visualize* these using an *Argand diagram*, which is just a plot of imaginary part vs. real part of a complex number. For example, $z = 3 + j4 = 5e^{j0.927}$ is plotted at rectangular coordinates $(3, 4)$ and polar coordinates $(5, 0.927)$, where 0.927 is the angle in radians measured counterclockwise from the positive real axis (try it). This is an *excellent* way of picking up errors, especially in the phase being off by π radians=180.º

IV. COMPLEX NUMBERS: ADDITION, SUBTRACTION, MULTIPLICATION, DIVISION

So why bother with rectangular-to-polar or polar-to-rectangular conversions? Here's why:

- To multiply or divide complex numbers, convert them to polar form and use $Me^{j\theta}Ne^{j\phi} = (MN)e^{j(\theta+\phi)}$;
- In particular, note that the magnitude of a product is the product of the magnitudes;
- To add or subtract, convert to rectangular form and use $(a + jb) + (c + jd) = (a + c) + j(b + d)$.

  The following two examples (side-by-side) show how to do this:

$5e^{j0.927}$	$+$	$\sqrt{2}e^{j0.785}$	$=$	$6.4e^{j0.896}$	$3 + j4$	$\times$	$1 + j$	$=$	$-1 + j7$
$\Downarrow P \to R$		$\Downarrow P \to R$		$\Uparrow R \to P$	$\Downarrow R \to P$		$\Downarrow R \to P$		$\Uparrow P \to R$
$3 + j4$	$+$	$1 + j$	$=$	$4 + j5$	$5e^{j0.927}$	$\times$	$\sqrt{2}e^{j0.785}$	$=$	$5\sqrt{2}e^{j1.71}$
add	**in**	**rectangular**		**form**	**multiply**	**in**	**polar**		**form**

Note that we have already seen all of the $P \to R$ and $R \to P$ conversions used here.

Yes, for this simple example you could compute

$$(3 + j4)(1 + j) = ((3 \cdot 1) - (4 \cdot 1)) + j((4 \cdot 1) + (3 \cdot 1)) = -1 + j7 \tag{5}$$

but for more complicated examples you *won't* want to do this.

Another example:

$$\text{Compute} \quad 2\sqrt{3}e^{j\pi/6} + 2e^{-j\pi/3} \quad \text{by} \quad \text{converting polar to rectangular} :$$
$$= 2\sqrt{3}\cos(\pi/6) + j2\sqrt{3}\sin(\pi/6) \quad + \quad 2\cos(\pi/3) + j2\sin(-\pi/3)$$
$$= 2\sqrt{3}(\sqrt{3}/2) + j2\sqrt{3}(1/2) \quad + \quad 2(1/2) - j2(\sqrt{3}/2) = 4(!) \tag{6}$$

Another example, entirely in the polar domain:

$$(2\sqrt{3}e^{j\pi/6})/(2e^{-j\pi/3}) = \frac{2\sqrt{3}}{2}e^{j(\pi/6-(-\pi/3))} = \sqrt{3}e^{j\pi/2} = j\sqrt{3} \tag{7}$$

V. COMPLEX NUMBERS: COMPLEX MANIPULATIONS

A. Complex Conjugates

The **complex conjugate** z^* of z is $z^* = x - jy = Me^{-j\theta} = M\angle - \theta$. This turns out to be useful:

- $Re[z] = \frac{1}{2}(z + z^*)$: We can get the real part by adding the complex conjugate and halving;

- $|z|^2 = zz^*$: We can get magnitude by multiplying by the complex conjugate and taking a square root;

- $e^{j2\text{ARG}[z]} = \frac{z}{z^*}$: We can get the phase by dividing by the complex conjugate and taking half the log;

- Also, $|z^*| = |z|; \quad arg[z^*] = -arg[z]; \quad z = Me^{j\theta} \to z^* = Me^{-j\theta}; \quad \frac{1}{z} = \frac{1}{M}e^{-j\theta}; \quad -z = Me^{j(\theta\pm\pi)}$.

B. Complex Expressions

Try this: compute $|\frac{5(8+j)(8+j6)(5+j12)(5+j10)}{26(7+j4)(7+j24)(2+j11)}|$ (!) Are you kidding me? And yet it's quite easy:

$$\left|\frac{5(8+j)(8+j6)(5+j12)(5+j10)}{26(7+j4)(7+j24)(2+j11)}\right| = \frac{5|8+j|\cdot|8+j6|\cdot|5+j12|\cdot|5+j10|}{26|7+j4|\cdot|7+j24|\cdot|2+j11|} = \sqrt{\frac{(25)(65)(100)(169)(125)}{(26)^2(65)(625)(125)}} = 1$$

(8)

Two very important rules:

- Magnitude of the product of complex numbers is product of magnitudes.

- Argument of the product of complex numbers is the sum of arguments.

The following identity is true to an overall sign. Why? (Try doing this solely in rectangular form!)

$$\frac{(a+jb)(c+jd)}{(e+jf)(g+jh)} = \sqrt{\frac{(a^2+b^2)(c^2+d^2)}{(e^2+f^2)(g^2+h^2)}}\exp j[\tan^{-1}(b/a) + \tan^{-1}(d/c) - \tan^{-1}(f/e) - \tan^{-1}(h/g)]$$ (9)

VI. PHASORS: PHASOR-TO-SINUSOID CONVERSIONS AND VICE-VERSA

A. Procedures: Phasor-to-sinusoid conversions and vice-versa

The bottom line: A **phasor** is a complex number used to represent a sinusoid. In particular:

sinusoid $\quad x(t) = M\cos(\omega t + \theta), -\infty < t < \infty \quad$ **function of time**

phasor $\quad X = Me^{j\theta} = M\cos\theta + jM\sin\theta \quad$ **complex number**

What do these two things have to do with each other? Consider the following:

- $x(t) = Re[Xe^{j\omega t}] = Re[Me^{j\theta}e^{j\omega t}] = Re[Me^{j(\omega t+\theta)}] = M\cos(\omega t + \theta)$.

- This relates the phasor X with the sinusoidal signal $x(t)$.

- Now apply this trick to the sum of two sinusoids at the same frequency:

- $A\cos(\omega t + \theta) + B\cos(\omega t + \phi) = Re[Ae^{j(\omega t+\theta)} + Be^{j(\omega t+\phi)}] = Re[e^{j\omega t}(Ae^{j\theta} + Be^{j\phi})]$

- Note the crucial step: **The complex exponential $e^{j\omega t}$ factors out of the sum!**

- Now let $Ae^{j\theta} + Be^{j\phi} = Ce^{j\psi}$ for some C and ψ (as we were doing before).

- Then: $Re[e^{j\omega t}(Ae^{j\theta} + Be^{j\phi})] = Re[e^{j\omega t}(Ce^{j\psi})] = C\cos(\omega t + \psi)$.

- **The trigonometry equation $A\cos(\omega t + \theta) + B\cos(\omega t + \phi) = C\cos(\omega t + \psi)$ (ugh)**

- **Is equivalent to the complex number equation $Ae^{j\theta} + Be^{j\phi} = Ce^{j\psi}$ (OK!)**

How does this work in practice? Try some examples:

B. Examples: Phasor-to-sinusoid conversions and vice-versa

- Example #1: Simplify $5\cos(\omega t + 53^o) + \sqrt{2}\cos(\omega t + 45^o)$

- The hard way: Use the cosine addition formula three times (the third time to simplify the result)(ugh).

- The easy way: **Phasors:** $5e^{j53^o} + \sqrt{2}e^{j45^o} = (3 + j4) + (1 + j) = (4 + j5) = 6.4e^{j51^o}$.

- So $5\cos(\omega t + 53^o) + \sqrt{2}\cos(\omega t + 45^o) = 6.4\cos(\omega t + 51^o)$.

- Example #2: Simplify $\cos(\omega t + 30^o) + \cos(\omega t + 150^o)) + \cos(\omega t - 90^o)$.

- The hard way: Use the cosine addition formula three times (the third time to simplify the result)(ugh).

- The easy way: **Phasors:** $e^{j30^o} + e^{j150^o} + e^{-j90^o} = 0$ (think resultant; draw an Argand diagram).

- So $\cos(\omega t + 30^o) + \cos(\omega t + 150^o)) + \cos(\omega t - 90^o) = 0$. You didn't even need your calculator!

- This fact is used in three-phase current (still used in some rural areas) so current flow is conserved.

- I could have made it a little harder by making the last term $\sin(\omega t) = \cos(\omega t - 90^o)$.

- Use the trig identity $\sin(\omega t + \theta) = \cos(\omega t + \theta - \pi/2)$ as a definition, as necessary.

VII. PHASORS: REAL-WORLD APPLICATIONS (BORROWING FROM EECS 215)

- Problem #1: A voltage source $26\cos(50t)$ is connected to a small motor. The motor is modelled by a 12Ω resistor (coil resistance) in series with a $0.1H$ inductor (coil inductance effects).

- Compute the current $i(t)$ passing through the motor and the average power dissipated in the motor.

- The current phasor I and average power P can be written as:

- $I = \frac{26+j0}{12+j50(0.1)} = \frac{26\angle 0}{13\angle 22.6^o} = 2\angle -22.6^o \rightarrow i(t) = 2\cos(50t - 22.6^o)$. Peak current=2 amps (a lot!)

- $P = \frac{1}{2}Re[VI^*] = \frac{1}{2}Re[26(2\angle -22.6^o)] = 26\cos(22.6^o) = 24$ watts.

This next problem is about *power factor correction*, which uses *lots* of complex numbers and phasors:

- Problem #2: A blender motor is modelled by a 30Ω resistor (modelling the coil resistance) in series with a $\frac{40}{377}H$ inductor (modelling the inductive effects of the coil). What power is dissipated by the motor?

- The blender is plugged into a wall socket which delivers a sinusoidal voltage at 120 volts **rms** at 60 Hertz. This implies an amplitude of $A = 120\sqrt{2} \approx 170$ volts (did you know that peak voltage is 170 volts, not 120?) and a frequency of $\omega = 2\pi 60 \approx 377\frac{\text{RAD}}{\text{SEC}}$. Tell your aunt/uncle engineer you know about the number 377.

- rms current phasor $= I = \frac{120+j0}{30+j(2\pi 60)(40/377)} = \frac{120+j0}{30+j40} = 2.4\angle -53^o \rightarrow i(t) = 2.4\sqrt{2}\cos(377t - 53^o)$.

- Average power dissipated $= P = Re[V_{rms}I_{rms}^*] = Re[(120)(2.4e^{-j53^o})] = 172.8$ watts.

- **Problem:** The voltage and current are 53° out of phase, so the motor draws more current than it should.

- **So what?** Detroit Edison will charge you big-time for this! And they will be right! Why?

  – Detroit Edison's generators need to generate more current than they should have to;

  – Need physically bigger (thicker and heavier) transmission lines to carry the current;

    – More current→greater losses and heating (power dissipation) due to line resistance;

    – Hotter transmission lines expand and sag, touching something; this causes blackouts.

- **So what do we do?** Hook a capacitor C in parallel with the motor. RMS current phasor:

- $I_{rms} = 120[1/(30 + j40) + j2\pi(60)C)]$.

- What value of C makes the phase of I_{rms} zero? You should obtain $C = 42.4 \times 10^{-6} = 42.4\mu F$.

- Then $I_{rms} = 1.44\angle 0$ and average power dissipated$=Re[V_{rms}I_{rms}^*] = (120)(1.44) = 172.8$ watts.

- **But note:** The current amplitude has dropped from $2.4\sqrt{2}$ amps to $1.44\sqrt{2}$ amps!

- We have almost *halved* the peak current, while maintaining average power=172.8 watts.

A. Miscellaneous Problems

    The above problem shows why you care about problems like these:

- <u>Problem #1:</u> $2\cos(\omega t + \frac{\pi}{3}) + A\cos(\omega t + \frac{5\pi}{4}) = B\cos(\omega t)$ for what value of A?

- That is, what should the amplitude A be to make the left side be in phase with a pure cosine?

- **Phasors:** $2e^{j\pi/3} + Ae^{j5\pi/4} = B \rightarrow 0 = Im[2e^{j\pi/3} + Ae^{j5\pi/4}] = 2\sin(\frac{\pi}{3}) + A\sin(\frac{5\pi}{4}) \rightarrow A = \sqrt{6}$.

- Note this is the same problem as this: For what A is $2e^{j\pi/3} + Ae^{j5\pi/4}$ purely real?

- <u>Problem #2:</u> The amplitude of $5\cos(7t + 0.927) + A\sin(7t)$ is 3 for what value of A?

- **Phasors:** $5e^{j0.927} + Ae^{-j\pi/2} = 3e^{j\theta} \rightarrow (3 + j4) - jA = 3 + j(4 - A)$ has magnitude=3 only if $A = 4$.

- Note that in general this type of problem will have two solutions or none.

- <u>Problem #3:</u> Derive the trig identity $\cos(3\theta) = \cos^3\theta - 3\cos\theta\sin^2\theta$ without using any trig.

- Cube Euler's formula: $e^{j3\theta} = (\cos\theta + j\sin\theta)^3 = \cos^3\theta + 3j\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - j\sin^3\theta$.

- Equating real parts gives the desired trig identity. Also: try equating imaginary parts.

- <u>Problem #4:</u> Compute and put in rectangular form $\left(\frac{\sqrt{2}}{2}(1 + j)\right)^{1003}$.   This is a stunt,

- $\left(\frac{\sqrt{2}}{2}(1 + j)\right)^{1003} = (1e^{j\pi/4})^{1003} = e^{j1000\pi/4}e^{j3\pi/4} = e^{j3\pi/4} = \frac{\sqrt{2}}{2}(-1 + j)$.   but illustrative.

- <u>Problem #5:</u> Compute the five 5th roots of 32, i.e., the five solutions of $z^5 - 32 = 0$.

- $\{2e^{j2\pi1/5}, 2e^{j2\pi2/5}, 2e^{j2\pi3/5}, 2e^{j2\pi4/5}, 2e^{j2\pi5/5} = 2\}$. HINT: Draw an Argand diagram.