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ON THE STRENGTH OF THE INTERPRETATION METHOD

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Abstract. In spite of the fact that true arithmetic reduces to the monadic second-order theory of the real line, Peano arithmetic cannot be interpreted in the monadic second-order theory of the real line.

§0. Introduction. The decision problem for the monadic second-order theory of the real line was posed by Grzegorczyk in 1951 [Gr], and was proved undecidable in 1976 by Shelah [Sh]. Shelah reduced the first-order theory of true arithmetic to the monadic second-order theory of the real line. In other words, he constructed an algorithm that, given a sentence φ in the first-order language of arithmetic, produces a sentence φ' in the monadic second-order language of order such that φ is true in the standard model of arithmetic if and only if φ' is true in the real line. Shelah's proof was analyzed, strengthened and generalized in several papers by the present authors (see [Gu2]). In particular, the following somewhat strange results were proved in [GS]:

Let V be a model of ZFC, B the Boolean algebra of regular open subsets of the real line in V, and V^B the corresponding Boolean-valued model of ZFC. The full second-order V^B-theory of arithmetic reduces to the monadic second-order V-theory of the real line. If V satisfies the continuum hypothesis then the full second-order V^B-theory of the real line reduces to the monadic second-order V-theory of the real line.

In spite of these strong reducibility results, we prove in this paper that Peano arithmetic is not interpretable in the monadic second-order theory of the real line. Actually, we prove a stronger result which is also more convenient to prove. To formulate the stronger result we need a couple of definitions.

DEFINITION 0.1. We define a first-order theory with equality which will be called the *weak set theory*. The signature of the weak set theory consists of one binary predicate symbol P, and the axioms of the weak set theory are as follows:

(a) $\forall x \exists y \forall z[P(z,y) \leftrightarrow z = x]$.

(b) $\forall x \forall y \exists u \forall z[P(z,u) \leftrightarrow (P(z,x) \text{ or } P(z,y))]$.

(c) $\exists x \forall y[\neg P(y,x)]$.

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It is a very weak set theory indeed, and it is easily interpretable in Peano arithmetic; see §3 in this connection.

DEFINITION 0.2. A *chain* is a linearly ordered set. A *colored chain* is a chain with additional unary predicates (the *colors*).

DEFINITION 0.3. A chain C is *short* if every well-ordered subchain of C is countable and every well-ordered subchain of the inverse of C is countable. (Notice that the real line is short.)

MAIN THEOREM. *The weak set theory is not interpretable in the monadic second-order theory of any short colored chain.*

The notion of interpretation is recalled in detail in §1.

The reader may wonder: If Shelah's reduction of true first-order arithmetic to the monadic second-order theory of the real line is not an interpretation, then what is it? A closer look on the reduction reveals that it is a kind of generalized Boolean-valued interpretation. This is made explicit in [GS], where the Boolean-valued character of generalized interpretations is exploited to give the above-mentioned stronger reducibility results. Our Main Theorem shows that the Boolean-valued character of the generalized interpretation is essential.

We are greatly thankful to the referee for the following important remark:

> It should be noted... that this is not the first example in the literature of an undecidable theory T such that true arithmetic or Peano arithmetic or even the weak set theory cannot be interpreted in T. The examples I have in mind are various stable theories which have been shown to be undecidable by Walter Baur, such as the theory of pairs of abelian groups or the theory of modules over certain finite rings. Baur's method is to give a reduction of undecidable word problems into these theories. Clearly any interpretation of the weak set theory into a theory T would give rise to a formula defining a linear ordering on an infinite set in some model of T, forcing such a T to be unstable.

For a specific example, see [Ba].

§1. The notion of interpretation. In this section we recall the well-known notion of interpretation of one first-order theory in another. This notion is a generalization of Tarski's original and slightly more restricted notion [TMR].

It will be assumed in this paper that every first-order language is a first-order language with equality.

DEFINITION 1.1. Let σ be a signature $\langle P_1, P_2, \ldots\rangle$, where each P_i is a predicate symbol of some arity ri. An *interpretation* of σ in a first-order language L' is a sequence

$$I = \langle d, U(v_1), E(v_1, v_2), P'_1(v_1, \ldots, v_{r1}), P'_2(v_1, \ldots, v_{r2}), \ldots\rangle,$$

where:

(a) d is a positive integer (the *dimension*);

(b) $U(v_1)$ and $E(v_1, v_2)$ are L'-formulas (the *universe formula* and the *equality formula* respectively); and

(c) each $P'_i(v_1, \ldots, v_{ri})$ is an L'-formula (the *interpretation* of P_i).

Here $v_1, v_2, \ldots$ are disjoint d-tuples of distinct variables of L', and the formulas $U(v_1)$, $E(v_1, v_2)$, and $P_i'(v_1, \ldots, v_{ri})$ contain no free variables in addition to those shown. If $U(v_1)$ is logically true we call I *universal.*

Definition 1.1 may be criticized as not sufficiently rigorous. What is exactly the d-tuple v_1 in the universe formula? First of all, the universe formula may have less than d variables. But even if it has exactly d variables, how are they ordered? What is the first variable in v_1, for example? Similarly, what exactly are the d-tuples v_1 and v_2 in the equality formula, and what are the d-tuples $v_1, \ldots, v_{ri}$ in the formula $P_i'(v_1, \ldots, v_{ri})$? One can settle these questions by requiring that v_1 consists of the first d variables of L', v_2 consists of the next d variables, and so on.

Definition 1.1 may also be criticized as not sufficiently general. One way to generalize Definition 1.1 is to allow individual parameters in the interpreting formulas $U(v_1)$, $E(v_1, v_2)$ and $P'(v_1, \ldots, v_r)$. However, the same effect can be achieved by extending L' by means of individual constants. This is why the Main Theorem speaks about *colored* chains. Extending a specific monadic second-order language of colored chains by finitely many individual constants means simply adding finitely many colors to the existing signature. Thus introducing colors allows us to use the simpler definition of interpretation without losing generality.

Another natural way to generalize Definition 1.1 is to allow function symbols and in particular individual constants (which we consider to be 0-ary function symbols) in the signature σ. We will in effect circumvent this generalization by restricting our attention to relational versions of first-order theories.

DEFINITION 1.2. Let T be an arbitrary first-order theory, and let σ be the signature of T. A *relational version* σ^* of σ is the signature that results from replacing every function symbol f in σ by a new predicate symbol, called G_f in this definition, of arity equal to one plus the arity of f. The *functional version* of an arbitrary σ^*-formula φ is the result of replacing every subformula $G_f(x_1, \ldots, x_r, y)$ of φ by the σ-formula $f(x_1, \ldots, x_r) = y$. The *relational version* of T in the signature σ^* is the first-order theory T^* in the signature σ^* such that an arbitrary σ^*-formula φ is a theorem of T^* if and only if the functional version of σ is a theorem of T.

If σ is a signature, let $L(\sigma)$ be the first-order language of σ.

DEFINITION 1.3. Let σ, L' and I be as in Definition 1.1. Fix a function that associates each $L(\sigma)$-variable v with a d-tuple v' of distinct L'-variables in such a way that if u and v are different $L(\sigma)$-variables then the tuples u' and v' share no common variables. By induction, we define the *I-translation* φ' of an arbitrary $L(\sigma)$-formula φ:

(a) $(x = y)' = E(x', y')$.

(b) If P is a predicate symbol of arity r in L, then $(P(x_1, \ldots, x_r))' = P'(x_1', \ldots, x_r')$.

(c) $(\neg\varphi)' = \neg(\varphi')$, $(\varphi \mathbin{\&} \psi)' = (\varphi' \mathbin{\&} \psi')$, and $(\varphi \vee \psi)' = (\varphi' \vee \psi')$.

(d) $(\forall x \varphi(x))' = \forall x'[U(x') \to \varphi'(x')]$, and $(\exists x \varphi(x))' = \exists x'[U(x') \mathbin{\&} \varphi'(x')]$.

The function that associates variables of one first-order language $L1$ with d-tuples of another first-order language $L2$ can be fixed once for all. For example, it may associate the first $L1$-variable with the tuple of the first d $L2$-variables, the second $L2$-variable with the tuple of the next d $L2$-variables, and so on.

If T is a first-order theory let $L(T)$ be the language of T. As usual, a first-order formula φ is called *closed* if it has no free variables.

DEFINITION 1.4. Let T and T' be first-order theories such that the signature of T consists of predicate symbols, and T' is consistent and complete. Let I be an interpretation of the signature of T in the language of T', and let $U(x)$ be the universe formula of I. I is an *interpretation* of T in T' if

(a) the formula $\exists x U(x)$ is a theorem of T', and

(b) the I-translation of every closed theorem of T is a theorem of T'.

T is *interpretable* in T' if there is an interpretation of T in T'.

DEFINITION 1.5. Let T, T', I, and $U(x)$ be as in Definition 1.3 except T' may be incomplete. Let T'' be the extension of T' by an additional axiom $\exists x U(x)$. I is an *interpretation* of T in T' if

(a) T'' is consistent, and

(b) the I-translation of every closed theorem of T is a theorem of T''.

T is *interpretable* in T' if there is an interpretation of T in T'.

It is easy to generalize Definition 1.1 to the case of arbitrary σ: the interpretation of a function symbol f of arity r is a formula $\varphi(v_1,\ldots,v_r,v_{r+1})$ intended to translate the equality $f(v_1,\ldots,v_r) = v_{r+1}$. The generalization of Definition 1.3 is a little messy but nevertheless pretty obvious. Definition 1.4 does not need to be generalized; just omit the restriction that σ consists of predicate symbols only. It seems more convenient, however, to deal with relational versions of theories where functions are presented by their graphs.

LEMMA 1.1. *Let T be an arbitrary consistent first-order theory, and let T^* be a relational version of T. Then T and T^* are interpretable in each other.*

The proof is easy. $\square$

In the rest of this paper it is supposed that the signature of every first-order theory consists of predicate symbols.

DEFINITION 1.6. Let $\sim$ be an equivalence relation on a nonempty set A, and let R be a relation of some arity r on A. The equivalence relation $\sim$ *respects* R if for all elements $a_1,\ldots,a_r$, $b_1,\ldots,b_r$ of A, $[R(a_1,\ldots,a_r) \,\&\, (a_1 \sim b_1) \,\&\, \cdots \,\&\, (a_r \sim b_r)]$ implies $R(b_1,\ldots,b_r)$.

DEFINITION 1.7. Let σ, I and L' be as in Definition 1.1. Let M be a model for L' and

(a) $U^* = \{x\colon x$ is a d-tuple of elements of M and $U(x)$ holds in $M\}$;

(b) $E^* = \{(x,y)\colon x \in U^*, y \in U^*$ and $E(x,y)$ holds in $M\}$; and

(c) if P is a predicate symbol of arity r in σ, then $P^* = \{(x_1,\ldots,x_r)\colon$ each x_i belongs to U^* and $P'(x_1,\ldots,x_r)$ holds in $M\}$.

The interpretation I *respects* the structure M if U^* is not empty, E^* is an equivalence relation, and E^* respects every P^*.

LEMMA 1.2. *Any interpretation of a first-order theory T in a consistent complete first-order theory T' respects every model of T'.*

The proof is easy. $\square$

DEFINITION 1.8. Let σ, I and L' be as in Definition 1.1, and let M, U^*, E^*, and P^* be as in Definition 1.7. We suppose that I respects M and define a model for L which will be called the *I-image* of M and will be denoted $I(M)$. Elements of $I(M)$ are equivalence classes $x/E^* = \{y \in U^*\colon xE^*y\}$ of E^* (where x ranges over U^*). If P is a predicate symbol of arity r in σ then P is interpreted in $I(M)$ as the relation $\{(x_1/E^*,\ldots,x_r/E^*)\colon (x_1,\ldots,x_r) \in P^*\}$.

LEMMA 1.3. *Let $I = (d, U(v_1), E(v_1,v_2),\ldots)$ be an interpretation of a signature σ in*

the first-order language of a structure M. Suppose that I respects M. Let
$\varphi(v_1,\ldots,v_l)$ *be an arbitrary* $L(\sigma)$*-formula,*
$\varphi'(v'_1,\ldots,v'_l)$ *be the I-translation of* $\varphi(v_1,\ldots,v_l)$,
$U^* = \{x: x$ *is a d-tuple of elements of M and* $U(x)$ *holds in* $M\}$,
$E^* = \{(x,y): x \in U^*, y \in U^*$ *and* $E(x,y)$ *holds in* $M\}$, *and*
$x_1,\ldots,x_l$ *belong to* U^*.
Then $\varphi'(x_1,\ldots,x_l)$ *holds in M if and only if* $\varphi(x_1/E^*,\ldots,x_l/E^*)$ *holds in* $I(M)$.

PROOF. An obvious induction on φ. □

LEMMA 1.4. *If I is an interpretation of a first-order theory T in the first-order theory of a structure M, then the I-image of M is a model for T.*

PROOF. Let φ be any closed theorem of T. Since I interprets T in the theory of M, the I-translation φ' of φ holds in M. By Lemma 1.3, φ holds in $I(M)$. □

THEOREM 1.1. *Suppose that I is an interpretation of a first-order theory T in the first-order theory of some structure M, and J is an interpretation of the signature of T in the first-order language of some structure N such that J respects N. If the I-image of M is isomorphic to the J-image of N, then J interprets T in the theory of N.*

PROOF. Since J respects N, the universe formula of J is satisfiable in N. It remains to show that the J-translation of an arbitrary closed theorem φ of T holds in N. By Lemma 1.4, φ holds in $I(M)$. Since $J(N)$ is isomorphic to $I(N)$, φ holds in $J(N)$. By Lemma 1.3, the J-translation of φ holds in N. □

§2. Colored short chains. It will be convenient for us to deal only with first-order theories. The monadic (as well as full) second-order theory of any first-order structure can be viewed in a natural way as the first-order theory of some associated structure. In this section, we define the monadic second-order theory of a colored chain C as the first-order theory of an appropriate associated structure $\hat{C}$ and then prove that if an arbitrary first-order theory is interpretable in the monadic second-order theory of a colored short chain, then it is one-dimensionally and universally interpretable in the monadic second-order theory of some noncolored short chain.

DEFINITION 2.1. The *monadic second-order theory* of a chain C with colors $A_0,\ldots,A_{m-1}$ is the first order theory of the associated structure

$$\hat{C} = \langle \mathrm{PS}(C), \subset, <, \mathrm{Empty}, P_0,\ldots,P_{m-1}\rangle$$

where $\mathrm{PS}(C)$ is the power set of C, $\subset$ is the proper inclusion relation, $<$ is the binary relation $\{(\{x\},\{y\}): x,y$ are elements of C and $x<y$ in $C\}$, Empty is the unary relation $\{\varnothing\}$, and $P_0,\ldots,P_{m-1}$ are the singleton unary relations $\{A_0\},\ldots,\{A_{m-1}\}$.

REMARK. The relation Empty seems superfluous because it is easily definable from the other relations, but it is needed for the composition theorem of §4. The colors are presented in the associated structure by unary relations rather then individual constants because of our commitment to deal with relational versions of first-order theories.

We will be interested only in the monadic second-order theories of chains, and not in their first-order theories. This fact allows us the following abbreviation.

ABBREVIATION. Let C be a colored or uncolored chain. An *interpretation* of T in C is an interpretation of T in the monadic second-order theory of C. A first-order theory T is *interpretable* in C if T is interpretable in the monadic second-order theory of C.

Recall that an interpretation is called *universal* if its universe formula is logically true.

THEOREM 2.1. *If a first-order theory T is interpretable in any colored short chain, then there is a one-dimensional universal interpretation of T in some noncolored short chain.*

PROOF. Let σ be the signature of T. To simplify the exposition, we suppose that σ consists of one binary predicate symbol P. In our application of Theorem 2.1, T will be the weak set theory whose signature consists of one binary predicate symbol.

LEMMA 2.1. *If T is interpretable in a colored chain C, then there is a universal interpretation of T in the extension of C by one additional color.*

PROOF. Let $I = (d, U, E, P')$ be an interpretation of T in C. We may suppose that both $U(X)$ and $U(Y)$ hold in C if either $E(X, Y)$ or $P'(X, Y)$ holds in C. Let A be any subset of C satisfying the universe formula U, and let $C\$$ be the extension of C by the additional color A. The desired interpretation $J = (d, \mathrm{TRUE}, E\$, P\$)$ where $E\$(X, Y)$ is the disjunction of the formulas

$$\neg U(X) \mathbin{\&} [U(Y) \rightarrow E(A, Y)],$$
$$[U(X) \rightarrow E(A, X)] \mathbin{\&} \neg U(Y),$$
$$U(X) \mathbin{\&} U(Y) \mathbin{\&} E(X, Y),$$

and $P\$(X, Y, Z)$ is the formula

$$\exists X'Y'Z'[U(X') \mathbin{\&} U(Y') \mathbin{\&} U(Z') \\ \mathbin{\&} E\$(X, X') \mathbin{\&} E\$(Y, Y') \mathbin{\&} E\$(Z, Z') \mathbin{\&} P'(X', Y', Z')].$$

It is easy to see that J respects the structure $C\$$ and the J-image of $C\$$ is isomorphic to the I-image of C. Now use Theorem 1.1. Lemma 2.1 is proved. $\square$

Recall that a chain C is *short* if every well-ordered subchain of C is countable and every well-ordered subchain of the reverse of C is also countable. Short chains were introduced in [Gu1]. The class of short chains happens to be closed under many operations.

LEMMA 2.2. *The monadic theory of every colored short chain C is universally interpretable in some noncolored short chain D.*

PROOF. Suppose that C is a short chain with colors $A_0, \ldots, A_{m-1}$ so that the associated structure $\hat{C} = \langle \mathrm{PS}(C), \subset, <, \mathrm{Empty}, P_1, \ldots, P_{m-1} \rangle$, where each $P_i = \{A_i\}$. Let INT be the ordinal type of integers. For each point $c \in C$, let $M(c)$ be a chain of the type $(\mathrm{INT} + 1) + n(c) + (1 + \mathrm{INT})$, where $n(c)$ is the number $\sum\{2^i \alpha_i(c) : i < m\}$ with $\alpha_i(c) = 1$ if $c \in A_i$ and $\alpha_i(c) = 0$ otherwise. $M(c)$ has a unique point without an immediate predecessor (we will identify this point with c itself) and a unique point without an immediate successor; the interval strictly between the two points contains exactly $n(c)$ points. Without loss of generality, the chains $M(c)$ are disjoint. The desired noncolored short chain D is the chain $\sum\{M(c) : c \in C\}$ obtained from C by replacing every point c with the chain $M(c)$. Obviously, C is a definable subchain of D. For each $X \subseteq D$, let $X' = X \cap C$.

We now describe the desired universal interpretation I. The dimension of I is 1. The equality formula is $X' = Y'$; the interpretations of the formulas $X \subset Y$, $X < Y$ and $\mathrm{Empty}(X)$ are the formulas $X' \subset Y'$, $X' < Y'$ and $\mathrm{Empty}(X')$. The interpretation of $P_i(X)$ expresses in D that an arbitrary $c \in C$ belongs to X' if and only if

$\alpha_i(c) = 1$. It is easy to see that I respects $\hat{D}$ and the I-image of $\hat{D}$ is isomorphic to $\hat{C}$. Now use Theorem 1.1. Lemma 2.2 is proved. $\square$

It remains to prove that if T is universally interpretable in a short chain C then there is a one-dimensional universal interpretation of T in some short chain. Let $I = \langle d, \text{TRUE}, E, P' \rangle$ be a universal interpretation of T in some short chain C. If $d = 1$ then there is nothing to prove, so suppose that $d > 1$.

Let S be the power set of C and S^d the cartesian product of d copies of S. Every element X of S^d is a d-tuple $(X_0, \ldots, X_{d-1})$ of subsets of C called the components of X; X_i will be called the ith component of X, the 0th component will be called *main*, and the rest of the components will be called *auxiliary*. Let C_d be the structure $\langle S^d, \subset, S_0, F_0, \ldots, F_{d-1}, <, \text{Empty} \rangle$, where $\subset$ is the relation $\{(X, Y)\colon$ every component of X is included in the respective component of $Y\}$, $S_0 = \{X\colon$ all auxiliary components of X are empty$\}$, $F_i(X, Y)$ means that $X \in S_0$ and the main component of X is the ith component of Y, $<$ is the relation $\{(X, Y)\colon X \in S_0, Y \in S_0$ and there are elements $a < b$ of C such that $\{a\}$ is the main component of X and $\{b\}$ is the main component of $Y\}$, and Empty(X) means that all components of X are empty.

In a natural way, the interpretation I gives rise to a universal one-dimensional interpretation $I_d = \langle 1, \text{TRUE}, E_d, P_d \rangle$ of T in C_d, where $E_d(X, Y)$ says that there are $X_0, \ldots, X_{d-1}, Y_0, \ldots, Y_{d-1}$ in S_0 such that

$$F_0(X_0, X), \ldots, F_{d-1}(X_{d-1}, X) \text{ hold,}$$
$$F_0(Y_0, Y), \ldots, F_{d-1}(Y_{d-1}, Y) \text{ hold,}$$

and the tuple of the main components of $X_0, \ldots, X_{d-1}, Y_0, \ldots, Y_{d-1}$ satisfies E; and $P_d(X, Y, Z)$ says that there are $X_0, \ldots, X_{d-1}, Y_0, \ldots, Y_{d-1}, Z_0, \ldots, Z_{d-1}$ in S_0 such that

$$F_0(X_0, X), \ldots, F_{d-1}(X_{d-1}, X) \text{ hold,}$$
$$F_0(Y_0, Y), \ldots, F_{d-1}(Y_{d-1}, Y) \text{ hold,}$$
$$F_0(Z_0, Z), \ldots, F_{d-1}(Z_{d-1}, Z) \text{ hold,}$$

and the tuple of the main components of $X_0, \ldots, X_{d-1}, Y_0, \ldots, Y_{d-1}, \ldots, Z_0, \ldots, Z_{d-1}$ satisfies P'. In other words, $E_d(X, Y)$ says that if X and Y are viewed as d-tuples of subsets of C then $E(X, Y)$ holds in $\hat{C}$. Similarly, $P_d(X, Y, Z)$ says that if X, Y and Z are viewed as d-tuples of subsets of C then $P'(X, Y, Z)$ holds in $\hat{C}$. It is easy to see that I_d respects C_d and the I_d-image of C_d is isomorphic to the I-image of $\hat{C}$. By Theorem 1.1, I_d is an interpretation of T in C_d.

Since the composition of one-dimensional universal interpretations is a one-dimensional universal interpretation, it remains to construct a one-dimensional universal interpretation of C_d in some short chain. In the rest of the proof we construct a short chain C^* and a one-dimensional universal interpretation J of C_d in C^*.

Let M be a chain of ordinal type INT $+ d +$ INT, where INT is the ordinal type of integers. M has a unique point without an immediate predecessor; let us call this point p_0. Let $p_1, \ldots, p_{d-1}$ be the $d - 1$ point following p_0; p_{d-1} is the only point of M without an immediate successor. For every point c of C, let $M(c)$ be an isomorphic copy of M, and let $c_0, \ldots, c_{d-1}$ be the points of $M(c)$ corresponding to the points $p_0, \ldots, p_{d-1}$ of M. Without loss of generality, the chains $M(c)$ are disjoint.

Let C^* be the chain $\sum\{M(c): c \in C\}$ obtained from C by replacing every point c by $M(c)$. For every $X \subseteq C^*$ and every $c \in C$, let $X' = \{c_i: c \in C \text{ and } i < d\}$ and $X'(c) = \{c_i: i < d\}$.

We now construct the desired one-dimensional universal interpretation J of C_d in C^*. The equality formula of J is the formula $X' = Y'$. The J-interpretation of the formula $X \subset Y$ is the formula $X' \subset Y'$. The J-interpretation $S'_0(X)$ of the formula $S_0(X)$ says that no element of X' has an immediate predecessor. In other words, $S'_0(X)$ expresses that for every $c \in C$, $X'(c) \subseteq \{c_0\}$. Call a subset X of C^* *standard* if it satisfies $S'_0(X)$. The J-translation $F'_i(X', Y')$ of the formula $F_i(X, Y)$ says that X' is standard and for every element z without immediate predecessor, $z \in X'$ if and only if the ith successor of z belongs to Y'. In other words, $F'_i(X', Y')$ expresses that $X' = \{c_0: c_i \in Y'\}$. The J-translation of the formula $X < Y$ says that X' and Y' are standard and $X' < Y'$. Finally, the J-translation of the formula Empty(X) says that X' is empty. It is easy to see that J respects C^* and the J-image of C^* is isomorphic to the I_d-image of C_d. By Theorem 1.1, J is an interpretation of T in C^*. □

§3. **The weak set theory.** The weak set theory was defined in the Introduction. It is a first-order theory whose signature consists of one binary predicate symbol P. The axioms of the weak set theory are as follows:

(a) $\forall x \exists y \forall z[P(z, y) \leftrightarrow z = x]$,

(b) $\forall x \forall y \exists u \forall z[P(z, u) \leftrightarrow (P(z, x) \text{ or } P(z, y))]$,

(c) $\exists x \forall y[\neg P(y, x)]$.

The intuitive meaning of axiom (a) is that for every element x there is a set y whose only element is x. Axiom (b) is the union axiom, and (c) is the empty set axiom.

LEMMA 3.1. *The weak set theory is interpretable in Peano arithmetic.*

PROOF. Here is one obvious interpretation I of the weak set theory in Peano arithmetic. I is one-dimensional and universal. The equality formula $E(x, y)$ of I is simply $x = y$. The I-interpretation of $P(x, y)$ says that there is a prime number p such that p^x divides y but p^{x+1} does not. □

COROLLARY. *The Main Theorem implies that Peano arithmetic is not interpretable in the monadic second-order theory of any colored short chain.*

PROOF. The relation of interpretability is transitive. □

LEMMA 3.2. *The weak theory does not have finite models.*

PROOF. Obvious. □

DEFINITION 3.1. Let M be a model for the weak set theory and y an element of M. The *extent* of y in M is $\{x: x \text{ is an element of } M \text{ and the statement } P(x, y) \text{ holds in } M\}$.

Notice that a model of the weak set theory may have different elements with the same extent.

LEMMA 3.3. *Let M be a model of the weak set theory. For every finite subset S of M there is an element y of M whose extent is S.*

The proof is easy. □

LEMMA 3.4. *There is a formula $\varphi(x, y, z)$ in the language of the weak set theory which satisfies the following requirement. Let M be a model of the weak set theory. For every finite nonempty subset S of $M \times M$ there is an element w of M such that $\{(a, b) \in M \times M: \varphi(a, b, w) \text{ holds in } M\} = S$.*

PROOF. With respect to the usual convention, we say that z is a *code* for an ordered pair (x, y) if there exist u and v such that the extent of u is $\{x\}$, the extent of v is $\{x, y\}$ and the extent of z is $\{u, v\}$. The desired $\varphi(x, y, z)$ says that z contains a code for (x, y).

Let M be a model for the weak set theory. For all a and b in M there is a code for (a, b) in M. The proof uses Lemma 3.3. There is some c whose extent is $\{a\}$, there is some d whose extent is $\{a, b\}$, and there is some e whose extent is $\{c, d\}$. Obviously, e is a code for (a, b).

Let $S = \{(a_i, b_i): i < k\}$ be a finite nonempty subset of $M \times M$. For every $i < k$, choose a code c_i for (a_i, b_i). By Lemma 3.3, there is some w whose extent is $\{c_i: i < k\}$. It is easy to see that $\varphi(x, y, w)$ holds in M if and only if $(x, y) = (a_i, b_i)$ for some $i < k$. □

THEOREM 3.1. *Suppose that I is a one-dimensional universal interpretation of the weak set theory in the first-order theory of some structure N. Two elements a and b of N will be called* **equivalent** *if the pair (a, b) satisfies the equality formula of I.*

(a) *There is a formula $\varphi 1(x, y)$ in the first-order language of N such that for every finite $S \subseteq N$ there is some $w \in N$ such that for every $b \in N$, $\varphi 1(b, w)$ holds in N if and only if there is some $a \in S$ that is equivalent to b.*

(b) *There is a formula $\varphi 2(x, y, z)$ in the first-order language of N such that for every finite $S \subseteq N \times N$ there is some $w \in N$ such that for every $(c, d) \in N \times N$, $\varphi 2(c, d, w)$ holds in N iff there is some $(a, b) \in S$ with a equivalent to c and b equivalent to d.*

PROOF. Let M be the I-image of N. For every $x \in N$, let $[x]$ be the equivalence class $\{y: x \sim y\}$ of N.

(a) The desired $\varphi 1(x, y)$ is the I-translation of the formula $P(x, y)$. Given $S \subseteq N$, let $[S] = \{[x]: x \in S\}$. By Lemma 3.3, there is some $[w] \in M$ such that $[S]$ is the extent of $[w]$ in M. Now use Lemma 1.3.

(b) The desired $\varphi 2(x, y, z)$ is the I-translation of the formula $\varphi(x, y, z)$ of Lemma 3.4. Given $S \in N \times N$, let $[S] = \{([a], [b]): (a, b) \in S\}$. By Lemma 3.4, there is some $[w] \in M$ such that $\varphi([x], [y], [w])$ holds in M if and only if $([x], [y]) \in [S]$. Now use Lemma 1.3. □

§4. Finite subtheories and additive colorings. For the reader's convenience, we recall here some definitions and facts. We start with adapting the well-known notion of finite subtheories to the case of the monadic second-order theories of chains.

DEFINITION 4.1. Let C be a chain, and let $A_0, \ldots, A_{l-1}$ be subsets of C.

(a) The *0-theory* $\mathrm{Th}^0(C, A_0, \ldots, A_{l-1})$ of the system $(C, A_0, \ldots, A_{l-1})$ is the set of atomic formulas $\varphi(X_0, \ldots, X_{l-1})$ in the monadic language of order such that the statement $\varphi(A_0, \ldots, A_{l-1})$ holds in C.

(b) The *$(d+1)$-theory* $\mathrm{Th}^{d+1}(C, A_0, \ldots, A_{l-1})$ of the system $(C, A_0, \ldots, A_{l-1})$ is the set $\{\mathrm{Th}^d(C, A_0, \ldots, A_{l-1}, A_l): A_l \subseteq C\}$.

REMARK. An atomic formula $\varphi(X_0, \ldots, X_{l-1})$ may miss part of the variables $X_0, \ldots, X_{l-1}$.

LEMMA 4.1. *Suppose that* $\mathrm{Th}^d(C, A_0, \ldots, A_{l-1}) = \mathrm{Th}^d(C', A'_0, \ldots, A'_{l-1})$, *and let $\varphi(X_0, \ldots, X_{l-1})$ be an arbitrary formula in the monadic language of order without any free variables except for those shown. If the quantifier depth of $\varphi(X_0, \ldots, X_{l-1})$ is at most d, then $\varphi(A_0, \ldots, A_{l-1})$ holds in C if and only if $\varphi(A'_0, \ldots, A'_{l-1})$ holds in C'.*

PROOF. An easy induction on d. □

LEMMA 4.2. *Suppose that* $\mathrm{Th}^d(C, A_0, \dots, A_{l-1}) = \mathrm{Th}^d(C', A'_0, \dots, A'_{l-1})$, *and let* f *be a function from* $\{0, \dots, m-1\}$ *to* $\{0, \dots, l-1\}$. *Then*

$$\mathrm{Th}^d(C, A_{f(0)}, \dots, A_{f(m-1)}) = \mathrm{Th}^d(C', A'_{f(0)}, \dots, A'_{f(m-1)}).$$

PROOF. An easy induction on d. □

DEFINITION 4.2. If C and D are chains then $C + D$ is any chain that can be split into an initial segment isomorphic to C and a final segment isomorphic to D. If $\langle C_n : n < \omega \rangle$ is a sequence of chains then $\sum_{n<\omega} C_n$ is any chain D that is the concatenation of segments D_n, $n < \omega$, such that each D_n is isomorphic to C_n. It is supposed of course that D_0 is the initial segment of D, D_1 is the next segment, and so on.

THEOREM 4.1 (composition theorem). (1) *If*

$$\mathrm{Th}^d(C, A_0, \dots, A_{l-1}) = \mathrm{Th}^d(C', A'_0, \dots, A'_{l-1}$$

and

$$\mathrm{Th}^d(D, B_0, \dots, B_{l-1}) = \mathrm{Th}^d(D', B'_0, \dots, B'_{l-1}),$$

then

$$\begin{aligned}&\mathrm{Th}^d(C + D, A_0 \cup B_0, \dots, A_{l-1} \cup B_{l-1})\\ &\quad = \mathrm{Th}^d(C' + D', A'_0 \cup B'_0, \dots, A'_{l-1} \cup B'_{l-1}).\end{aligned}$$

(2) If $\mathrm{Th}^d(C_n, A_{n0}, \dots, A_{n(l-1)}) = \mathrm{Th}^d(C'_n, A'_{n0}, \dots, A'_{n(l-1)})$ *for each* n, *then*

$$\mathrm{Th}^d\left(\sum_{n<\omega} C_n, \bigcup_{n<\omega} A_{n0}, \dots, \bigcup_{n<\omega} A_{n(l-1)}\right) = \mathrm{Th}^d\left(\sum_{n<\omega} C'_n, \bigcup_{n<\omega} A'_{n0}, \dots, \bigcup_{n<\omega} A'_{n(l-1)}\right).$$

PROOF. This is a special case of much stronger composition theorems (see [Sh] and [Gu1]) which can easily be proved directly by induction on d. □

DEFINITION 4.3. (1)

$$\begin{aligned}&\mathrm{Th}^d(C, A_0, \dots, A_{l-1}) + \mathrm{Th}^d(D, B_0, \dots, B_{l-1})\\ &\quad = \mathrm{Th}^d(C + D, A_0 \cup B_0, \dots, A_{l-1} \cup B_{l-1}).\end{aligned}$$

(2) $\sum_{n<\omega} \mathrm{Th}^d(C_n, A_{n0}, \dots, A_{n(l-1)}) = \mathrm{Th}^d(\sum_{n<\omega} C_n, \bigcup_{n<\omega} A_{n0}, \dots, \bigcup_{n<\omega} A_{n(l-1)})$.

ABBREVIATION. If D is a subchain of C and $X_0, \dots, X_{l-1}$ are subsets of C then $\mathrm{Th}^d(D, X_0, \dots, X_{l-1})$ abbreviates $\mathrm{Th}^d(D, X_0 \cap D, \dots, X_{l-1} \cap D)$.

THEOREM 4.2. *Let* T *be a finite semigroup, and let* f *be a function from* $\{(i,j): i < j < \omega\}$ *to* T. *Suppose that* f *satisfies the following additivity condition: for all* $i < j < k$, $f(i,k) = f(i,j) + f(j,k)$. *Then there are an element* t *of* T *and an infinite subset* S *of* ω *such that* $f(i,j) = t$ *for all* $i < j$ *in* S.

This is a special case of Theorem 1.1 in [Sh]. We will use the following corollary of Theorem 4.2.

THEOREM 4.3. *Let* C *be a chain,* $A_0, \dots, A_{m-1}$ *subsets of* C, *and* $\langle H_i : i < \omega \rangle$ *an increasing sequence of initial segments of* C. *For every* d *there is an infinite subset* S *of* ω *such that all* d*-theories* $\mathrm{Th}^d(H_j - H_i, A_0, \dots, A_{m-1})$, *where* i, j *belong to* S *and* $i < j$, *are the same.*

PROOF. Consider the set T of all d-theories $\mathrm{Th}^d(D, X_0, \dots, X_{m-1})$, where D is an arbitrary chain and $X_0, \dots, X_{m-1}$ are subsets of D. The set T is finite and forms a semigroup with respect to the addition operation defined above. For every pair $i < j$ of natural numbers, let $f(i,j)$ be the d-theory $\mathrm{Th}^d(H_j - H_i, A_0, \dots, A_{m-1})$. Obviously, f satisfies the additivity condition. Now apply Theorem 4.2. □

§5. Setting the stage for the proof by contradiction. The Main Theorem will be proved by contradiction. Suppose that the weak set theory is interpretable in the monadic second-order theory of some colored short chain. By Theorem 2.1, there is a one-dimensional universal interpretation η of the weak set theory in some noncolored short chain C. In the rest of the paper we deal with C and η. Elements of C will be called *points*. Subsets of C will be called *point sets*. The letters U, V, W, X, Y, Z with or without subscripts will be used to denote point sets.

Notice the slight change in our notation. The interpretation is called η rather than I. The letter I is reserved for other purposes.

Let $E(X, Y)$ be the equality formula of η. Point sets X and Y will be called *equivalent* if $E(X, Y)$ holds; to indicate the equivalence of X and Y we will write $X \sim Y$.

LEMMA 5.1. *The number of equivalence classes of point sets is infinite.*

PROOF. The equivalence classes form the universe of the η-image of $\hat{C}$. By Lemma 1.4, the η-image is a model of the weak set theory, but the weak set theory does not have finite models. □

LEMMA 5.2. *There are monadic second-order formulas $\varphi1(X, Y)$ and $\varphi2(X, Y, Z)$ such that*

(a) *for every finite nonempty collection S of point sets there is a point set W such that an arbitrary point set X satisfies the formula $\varphi1(X, W)$ in C if and only if X is equivalent to some $X' \in S$, and*

(b) *for every finite nonempty collection S of pairs of point sets there is a point set W such that an arbitrary pair (X, Y) of point sets satisfies the formula $\varphi2(X, Y, W)$ in C if and only if there is $(X', Y') \in S$ with $X \sim X'$ and $Y \sim Y'$.*

PROOF. Use Theorem 3.1. □

For future references, we fix appropriate formulas $\varphi1$ and $\varphi2$. Let $d0$ be the quantifier depth of the equality formula of η, and $d1$ and $d2$ the quantifier depths of $\varphi1$ and $\varphi2$ respectively. Without loss of generality, $d1 \geq d0$ and $d2 \geq d0$.

THEOREM 5.1. (a) *For all point sets U, V, X, Y, and every $d \geq d0$, if $\mathrm{Th}^d(C, U, V) = \mathrm{Th}^d(C, X, Y)$ then $U \sim V \leftrightarrow X \sim Y$.*

(b) *For every nonempty finite collection $\{U_i : i < m\}$ of point sets there is a point set W such that for every point set X, if $\mathrm{Th}^{d1}(C, X, W) = \mathrm{Th}^{d1}(C, U_i, W)$ for some $i < m$, then $X \sim U_j$ for some $j < m$.*

(c) *For every nonempty finite collection $\{(U_i, V_i) : i < m\}$ of pairs of point sets there is a point set W such that if $\mathrm{Th}^{d2}(C, X, Y, W) = \mathrm{Th}^{d2}(C, U_i, V_i, W)$ for some $i < m$, then there is some $j < m$ such that $X \sim U_j$ and $Y \sim V_j$.*

PROOF. Use Lemmas 5.2 and 4.1. □

The rest of this section is devoted to terminology and notation. A subchain D of a chain C' is a *segment* of C' if it is convex, i.e., for all points $x < y < z$, $x \in D$ and $z \in D$ imply $y \in D$. (It is supposed of course that subchains are not empty.) A segment D is *proper* if $D \neq C'$. A *proper Dedekind cut* of C' is a pair (L, R), where L is a proper initial segment of C' and R is the corresponding final segment $C' - L$; the pairs $(\varnothing, C')$ and $(C', \varnothing)$ also are *Dedekind cuts* of C' but they are not proper Dedekind cuts. In the case $C' = C$, we will often omit "of C"; segments will mean segments of C, and Dedekind cuts will mean Dedekind cuts of C.

The symbol $\subset$ will be used to denote proper inclusion; thus $A \subset B$ means that $A \subseteq B$ and $A \neq B$. As usual, the cardinality of a set or structure S will be denoted

$|S|$. We will say that point sets X and Y *coincide* on (respectively outside) a segment D if $X \cap D = Y \cap D$ (respectively $X - D = Y - D$).

§6. Minor segments. In this section, we define the bouquet size of a segment and prove that for every proper Dedekind cut (L, R) either the bouquet size of L or the bouquet size of R is bounded by an a priori fixed number.

DEFINITION 6.1. The *bouquet size* of a segment D is the supremum of cardinals $|S|$ where S ranges over collections of nonequivalent point sets coinciding outside D.

REMARK. The strange name "bouquet size" seems to require an explanation. We looked for a suggestive name that will distinguish the size of a segment introduced here from another size introduced in the next section. The bouquets in question are of course collections of nonequivalent point sets coinciding outside D.

DEFINITION 6.2. A segment D of C is *minor* if the bouquet size of D is at most

$$N1 = |\{\mathrm{Th}^{d2}(D, A_0, A_1, A_2): D \text{ is a chain and } X, Y, Z \subseteq D\}|.$$

THEOREM 6.1. *For every proper Dedekind cut (L, R) of C, either L or R is minor.*

PROOF. To get a contradiction, suppose that neither L nor R is minor. Fix nonequivalent point sets $U_0, \dots, U_{N1}$ which coincide on R, and nonequivalent point sets $V_0, \dots, V_{N1}$ which coincide on L.

By Theorem 5.1, there is W such that if $\mathrm{Th}^{d2}(C, X, Y, W) = \mathrm{Th}^{d2}(C, U_i, V_i, W)$ for some $i \leq N1$, then there is $j < N1$ with $X \sim U_j$ and $Y \sim V_j$.

By the definition of $N1$, there are $i < j \leq N1$ such that

$$\mathrm{Th}^{d2}(R, U_i, V_i, W) = \mathrm{Th}^{d2}(R, U_j, V_j, W).$$

By the composition theorem,

$$\begin{aligned}
\mathrm{Th}^{d2}(C, U_i, V_j, W) &= \mathrm{Th}^{d2}(L, U_i, V_j, W) + \mathrm{Th}^{d2}(R, U_i, V_j, W) \\
&= \mathrm{Th}^{d2}(L, U_i, V_i, W) + \mathrm{Th}^{d2}(R, U_j, V_j, W) \\
&= \mathrm{Th}^{d2}(L, U_i, V_i, W) + \mathrm{Th}^{d2}(R, U_i, V_i, W) \\
&= \mathrm{Th}^{d2}(C, U_i, V_i, W).
\end{aligned}$$

Here the second equality holds since $V_j \cap L = V_i \cap L$ and $U_i \cap R = U_j \cap R$, and the third holds by the choice of $i < j$.

By the choice of W, $U_i \sim U_k$ and $V_j \sim V_k$ for some $k \leq N1$. By the choice of point sets $U_0, \dots, U_{N1}$, we have $i = k$. Similarly, $j = k$. Hence $i = j$, which is impossible. □

§7. Independence. In this section, we define the span of a segment and prove that the span of every minor segment is bounded by an a priori fixed number.

DEFINITION 7.1. Let D be a segment, and let S be a collection of point sets. A point set Y *depends on S with respect to D* if some point set $Y' \sim Y$ coincides with some $X \in S$ on D; otherwise Y is *independent from S with respect to D*.

DEFINITION 7.2. A sequence $\langle X_\alpha: \alpha < \lambda \rangle$ of point sets is *independent with respect to a segment D* if every X_α is independent from $\langle X_\beta: \beta < \alpha \rangle$ with respect to D. The *span* of a segment D is the supremum of ordinals λ such that there is a sequence $\langle X_\alpha: \alpha < \lambda \rangle$ of point sets which is independent with respect to D.

LEMMA 7.1 *Let ξ be the number-theoretical function such that $\xi(0, j) = 1$ and $\xi(i + 1, j) = j \times \xi(i, j) + 1$, and let m and n be positive integers. Let S be a set of cardinality at least $\xi(m, n)$, and let \$ be a binary operation on S such that for every*

$b \in S$, $|\{a \$ b: a \in S\}| \leq n$. Then there are elements $b_0, \ldots, b_{m-1}$ in S such that for every $i < m$ there is $c_i \in S$ with $b_j \$ b_i = c_i$ for all $j < i$.

PROOF. It suffices to construct elements $b_0, \ldots, b_{m-1}$ such that for every $i < m$ there is $c_i \in S$ with $b_j \$ b_i = c_i$ for all $j > i$.

Let $S_0 = S$. By induction on i, construct a sequence $\langle (b_i, c_i, S_{i+1}): i < m \rangle$. Suppose that $i = 0$ or $i > 0$ and the portion $\langle (b_j, c_j, S_{j+1}): j < i < m \rangle$ of the desired sequence is already constructed. Choose b_i arbitrarily in S_i. Choose c_j such that $|\{a \in S_i: a \$ b_i = c_i\}|$ is maximal possible. Set $S_{i+1} = \{a \in S_i: a \$ b_i = c_i\} - \{b_i\}$.

The construction fails if some S_i, $i < m$, turns out to be empty. To ensure that all the S_i are nonempty, we check that for every i, $|S_i| \geq \xi(m - i, n)$. For $i = 0$, this is one of the conditions of the lemma. Suppose that $i < m - 1$ and $|S_i| \geq \xi(m - i, n)$. By the definition of ξ, $|S_i| > n \times \xi(m - i - 1, n)$. Recall that for every $b \in S$, $|\{a \$ b: a \in S\}| \leq n$. Hence $|\{a \$ b_i: a \in S_i\}| \leq n$. By the choice of c_i, we obtain $|\{a \in S_i: a \$ b_i = c_i\}| > \xi(m - i - 1, n)$. Hence $S_{i+1} \geq \xi(m - i - 1, n)$.

Now suppose that $i < j < m$. Then $b_j \in S_j \subseteq S_{i+1}$ and therefore $b_j \$ b_i = c_i$. $\square$

DEFINITION 7.3. Let ξ be as in Lemma 7.1. Set

$$N2 = \xi(N1 + 2, N1) \times |\{\mathrm{Th}^{d1}(D, A_1, A_2): D \text{ is any chain and } A_1, A_2 \subseteq D\}|.$$

THEOREM 7.1. *The span of every minor initial or final segment is at most $N2$.*

PROOF. By virtue of symmetry, it suffices to prove the theorem for initial segments. For a contradiction, suppose that a sequence $S = \langle U_0, \ldots, U_{N2} \rangle$ of point sets is independent with respect to a minor initial segment L.

Using Theorem 5.1, fix point sets $W1$ and $W2$ such that

(a) for every X, if $\mathrm{Th}^{d1}(C, X, W1)$ equals some $\mathrm{Th}^{d1}(C, U_i, W1)$ then X is equivalent to some U_j, and

(b) for all X and Y, if $\mathrm{Th}^{d2}(C, X, Y, W2)$ equals some $\mathrm{Th}^{d2}(C, U_i, U_j, W2)$ with $i < j$, then there are $k < l$ such that $X \sim U_k$ and $Y \sim U_l$.

Let ξ be as in the definition of $N2$. Let $R = C - L$. By the definition of $N2$, there is a subset I of $\{0, \ldots, N2\}$ such that $|I| \geq \xi(N1 + 2, N1)$ and $\mathrm{Th}^{d1}(R, U_i, W1)$ has the same value for all $i \in I$. If $i, j \in I$ and $i < j$, let $U_{ij} = (U_i \cap L) \cup (U_j \cap R)$. Then

$$\begin{aligned}\mathrm{Th}^{d1}(C, U_{ij}, W1) &= \mathrm{Th}^{d1}(L, U_i, W1) + \mathrm{Th}^{d1}(R, U_j, W1)\\ &= \mathrm{Th}^{d1}(L, U_i, W1) + \mathrm{Th}^{d1}(R, U_i, W1) = \mathrm{Th}^{d1}(C, U_i, W1)\end{aligned}$$

and, by the choice of $W1$, U_{ij} is equivalent to some U_k. Fix a function f that, given $i < j$ in I, produces k with $U_k \sim U_{ij}$. Since L is minor, for every j, the number of different k's such that some U_{ij} is equivalent to U_k is at most $N1$.

By Lemma 7.1 with $m = N1 + 2$ and $n = N1$, there are a subset $J \subseteq I$ and a function $g: J \to I$ such that $|J| \geq N1$, and for all $i < j$ in J, $fij = gj$. Notice that if $U_{ij} \sim U_k$ then $k \leq i$; otherwise U_k depends on $\{U_0, \ldots, U_{k-1}\}$ with respect to L, which is impossible. Let $i = \min(J)$. Then for every $j > i$, $g(j) \leq i$.

As $|J| \geq N1 + 2$, there are j, k in J such that $i < j < k$ and $\mathrm{Th}^{d2}(R, U_i, U_j, W2) = \mathrm{Th}^{d2}(R, U_i, U_k, W2)$. Then

$$\begin{aligned}\mathrm{Th}^{d2}(C, U_i, U_j, W2) &= \mathrm{Th}^{d2}(L, U_i, U_j, W2) + \mathrm{Th}^{d2}(R, U_i, U_j, W2)\\ &= \mathrm{Th}^{d2}(L, U_i, U_j, W2) + \mathrm{Th}^{d2}(R, U_i, U_k, W2) = \mathrm{Th}^{d2}(C, U_i, U_{jk}, W2).\end{aligned}$$

By the choice of $W2$, there are $a < b \leq N2$ such that $U_i \sim U_a$ and $U_{jk} \sim U_b$. By the independence of $\langle U_0, \ldots, U_{N2} \rangle$, $a = i$. Let $n = gk$, so that $n \leq i = a < b$ and $fjk = n$. Then $U_n \perp U_{jk} \sim U_b$, which contradicts the independence of $\langle U_0, \ldots, U_{N2} \rangle$. □

§8. Major segments.

DEFINITION 8.1. A segment is *major* if its bouquet size is infinite.

LEMMA 8.1. *C is major, and for every proper Dedekind cut (L, R), either L or R is major.*

PROOF. By Lemma 5.1, the bouquet size of C is infinite. Let (L, R) be a proper Dedekind cut (L, R). By Theorem 6.1, either L or R is minor. Without loss of generality, L is minor. By Theorem 7.1, the span of L is finite. Choose a longest possible sequence $\langle U_0, \ldots, U_{m-1} \rangle$ of point sets which is independent with respect to L. For a contradiction, suppose that the bouquet size of R is finite too. For every U_i, choose a maximal (with respect to the cardinality) collection S_i of nonequivalent point sets coinciding with U_i on L. Since the bouquet size of R is finite, every S_i is finite.

Consider an arbitrary point set X. By the maximality of $\langle U_0, \ldots, U_{m-1} \rangle$, there are $Y \sim X$ and $i < m$ such that Y coincides with U_i on L. By the maximality of S_i, Y is equivalent to some $Z \in S_i$. Thus an arbitrary X is equivalent to a member of $\bigcup \{S_i : i < m\}$, which contradicts the fact that there are infinite many equivalence classes (Lemma 5.1). □

LEMMA 8.2. *There is a segment D of C satisfying one of the following two conditions*:

(a) *D is a major initial segment, and all proper initial segments of D are minor, or*

(b) *D is a major final segment, and all proper final segments of D are minor.*

PROOF. Let L be the union of all minor initial segments, and let $R = C - L$. If L is major then L is the desired segment D. Suppose that L is minor. By Lemma 8.1, R is major. Suppose that R' is a proper final segment of R, and $L' = C - R'$. Then $L' \supset L$ and therefore L' is not minor; by Theorem 6.1, R' is minor. Thus R is the desired segment D. □

By virtue of symmetry we can assume that C has a major initial segment all of whose proper initial segments are minor.

DEFINITION 8.2. D is the major initial segment all of whose proper initial segments are minor.

Since the bouquet size of D is infinite, there is an infinite collection of nonequivalent point sets coinciding outside D.

DEFINITION 8.3. We choose and fix an arbitrary subset $D' \subseteq C - D$ such that there are infinitely many nonequivalent point sets coinciding with D' outside D. A point set X is *normal* if it coincides with D' on $C - D$.

LEMMA 8.3. *For all normal point sets U, V, X, Y, and every $d \geq d0$, if $\mathrm{Th}^d(D, U, V) = \mathrm{Th}^d(D, X, Y)$ then $U \sim V \leftrightarrow X \sim Y$.*

PROOF. Use Theorem 5.1 and the composition theorem. □

DEFINITION 8.4. Any proper initial segment of D will be called a *head of C*, or simply a *head*.

LEMMA 8.4. *There is a strictly increasing sequence $\langle H_n : n < \omega \rangle$ of heads that converges to D.*

PROOF. If D does not have a last point then the statement follows from the fact that C is short. Thus it suffices to show that D does not have a last point. For a contradiction, suppose that b is the last point of D. Let $L = D - \{b\}$ and

$R = C - D$, so that both L and R are minor. An easy alteration of the proof of Lemma 8.1 establishes that there are only finitely many equivalence classes of point sets, which is impossible. □

§9. Vicinities and shufflings.

DEFINITION 9.1. The *vicinity* $[X]$ of a point set X is the collection $\{Y$: some point set $Z \sim Y$ coincides with X outside some head $H\}$.

LEMMA 9.1. *Every vicinity $[X]$ is the union of at most $N1$ different equivalence classes.*

PROOF. By contradiction, suppose that some $[X]$ contains nonequivalent point sets $Y_0, \ldots, Y_{N1}$. For each $i \leq N1$, choose a point set Z_i and a head H_i witnessing that $Y_i \in [X]$. Then $Z_0, \ldots, Z_{N1}$ witness that the bouquet size of $H = \bigcup\{H_i: i \leq N1\}$ exceeds $N1$, which contradicts the fact H is minor. □

THEOREM 9.1. *There is a sequence $\langle U_n: n < \omega\rangle$ of nonequivalent normal point sets such that for all $i < j$, $[U_i] \cap [U_j] = [U_0] \cap [U_1] \neq [U_i]$.*

PROOF. By the definition of normal point sets in §8, there is an infinite sequence $\langle X_n: n < \omega\rangle$ of nonequivalent normal point sets. Choose any collection S of nonequivalent point sets such that $\{n: S \subseteq [X_n]\}$ is infinite and the cardinality of S is maximal possible. (S can be empty. By Lemma 9.1, the cardinality of S is bounded by $N1$. This guarantees the existence of the desired S.) Without loss of generality, every $[X_n]$ includes S. Let $S^* = \{Z: S$ contains a point set equivalent to $Z\}$. S^* contains at most $N1$ different point sets X_n. Let $\langle X_{0n}: n < \omega\rangle$ be an infinite subsequence of $\langle X_n: n < \omega\rangle$ such that no X_{0n} belongs to S^*.

Choose $U_0 = X_{00}$, and let $\langle X_{1n}: n < \omega\rangle$ be the sequence obtained from $\langle X_{0n}: n < \omega\rangle$ by throwing away all members whose vicinities meet $[U_0] - S^*$; by the maximality of S, only finitely many members should be thrown away. Choose $U_1 = X_{10}$, and let $\langle X_{2n}: n < \omega\rangle$ be the sequence obtained from $\langle X_{1n}: n < \omega\rangle$ by throwing away all members whose vicinities meet $[U_1] - S^*$. Choose $U_2 = X_{20}$, etc. Then for all $i < j$, $[U_i] \cap [U_j] = S^* \neq [U_i]$. □

DEFINITION 9.2. Given $\langle H_n: n < \omega\rangle$, a strictly increasing sequence of heads converging to D, define $D_0 = H_0$ and $D_{n+1} = H_{n+1} - H_n$. The sequence $\langle D_n: n < \omega\rangle$ is the *partition of D corresponding to* $\langle H_n: n < \omega\rangle$. A *legal partition* of D is the partition of D corresponding to some strictly increasing sequence of heads converging to D.

DEFINITION 9.3. Let $\langle D_n: n < \omega\rangle$ be a legal partition of D, and $A \subseteq \omega$. The *shuffling* of point sets X and Y with respect to $\langle D_n: n < \omega\rangle$ and A is a normal point set Z such that:

(a) if $n \in \omega - A$ then Z coincides with X on D_n, and

(b) if $n \in A$ then Z coincides with Y on D_n.

LEMMA 9.2. *Let $\langle H_n: n < \omega\rangle$ be a strictly increasing sequence of heads converging to D, let $\langle D_n: n < \omega\rangle$ be the partition of D corresponding to $\langle H_n: n < \omega\rangle$, let U_0, U_1, V be normal point sets, and let A and B be infinite and coinfinite subsets of ω such that $0 \in A \leftrightarrow 0 \in B$. Let X be the shuffling of U_0, U_1 with respect to $\langle D_n: n < \omega\rangle$ and A, and Y the shuffling of U_0, U_1 with respect to $\langle D_n: n < \omega\rangle$ and B. Suppose that*

(a) *all d-theories* $\mathrm{Th}^d(H_n, U_i, V)$, *where $n < \omega$ and $i \leq 1$, are the same, and*

(b) *all d-theories* $\mathrm{Th}^d(H_n - H_m, U_i, V)$, *where $m < n < \omega$ and $i \leq 1$, are the same. Then $X \sim V \leftrightarrow Y \sim V$.*

PROOF. We consider only the case when 0 does not belong to A; the case $0 \in A$

is similar. Without loss of generality, B is the set ODD of odd natural numbers. Let $f0 = 0$; let $f(2l+1)$ be the least $j > f(2l)$ in A, and $f(2l+2)$ the least $j > f(2l+1)$ in $\omega - A$. Let $D_0^* = \sum\{D_n : n < f1\}$, $D_{2l+1}^* = \sum\{D_n : f(2l+1) \leq n < f(2l+2)\}$, and $D_{2l+2}^* = \sum\{D_n : f(2l+2) \leq n < f(2l+3)\}$. Notice that X coincides with U_0 on each D_{2l}^* and with U_1 on each D_{2l+1}^*. Thus, $\mathrm{Th}^d(D_n^*, X, V) = \mathrm{Th}^d(D_n, Y, V)$ for each n.

By the composition theorem,

$$\mathrm{Th}^d(D, X, V) = \sum_{n<\omega} \mathrm{Th}^d(D_n^*, X, V) = \sum_{n<\omega} \mathrm{Th}^d(D_n, Y, V) = \mathrm{Th}^d(D, Y, V).$$

Now use Lemma 8.3. □

§10. Finishing the proof of the Main Theorem. On the grounds of Theorem 9.1, choose a sequence $\langle U_n : n < \omega \rangle$ of nonequivalent normal point sets such that for all $i < j$, $[U_i] \cap [U_j] = [U_0] \cap [U_1] \neq [U_i]$.

DEFINITION 10.1. $N3$ is 1 plus the square of the number

$$|\{\mathrm{Th}^{d1}(M, A_1, A_2) : M \text{ is any chain and } A_1, A_2 \subseteq M\}|.$$

$N4$ is the least number such that for every partition of the set $\{(i,j,k) : i < j < k < N4\}$ into 32 colors, there is a subset I of $\{i : i < N4\}$ such that $|I| \geq N3$ and all triples in $\{(i,j,k) : i \in I,\ j \in I,\ k \in I \text{ and } i < j < k < N4\}$ have the same color. (The existence of $N4$ follows from Ramsey's theorem [Ra].)

Applying Theorem 4.3, choose a strictly increasing sequence $\langle H_n' : n < \omega \rangle$ of heads converging to D such that:

(a) all $d1$-theories $\mathrm{Th}^{d1}(H_n', U_i, U_j)$, where $n < \omega$ and $i < j < N4$, are equal, and

(b) all $d1$-theories $\mathrm{Th}^{d1}(H_n' - H_m', U_i, U_j)$, where $m < n < \omega$ and $i < j < N4$, are equal.

By Lemma 4.2, (a) and (b) imply

(a′) all $d1$-theories $\mathrm{Th}^{d1}(H_n', U_i, U_i)$, where $n < \omega$ and $i < N4$, are equal, and

(b′) all $d1$-theories $\mathrm{Th}^{d1}(H_n' - H_m', U_i, U_i)$, where $m < n < \omega$ and $i < N4$, are equal.

Let $\langle D_n' : n < \omega \rangle$ be the partition of D corresponding to $\langle H_n' : n < \omega \rangle$, and for each $A \subseteq \omega$, let $S'(i,j,A)$ be the shuffling of U_i and U_j with respect to $\langle D_n' : n < \omega \rangle$ and A. Set

$$\kappa(i,j,A) = \min\{k : \text{either } S'(i,j,A) \sim U_k \text{ or } k = N4\}.$$

LEMMA 10.1. *If A and B are infinite and coinfinite subsets of $\omega - \{0\}$, then $\kappa(i,j,A) = \kappa(i,j,B)$.*

PROOF. If $k = \kappa(i,j,A) < N4$ or $k = \kappa(i,j,B) < N4$ then use Lemma 9.2. Otherwise $\kappa(i,j,A) = \kappa(i,j,B) = N4$. □

Let $\kappa(i,j) = \kappa(i,j,A)$, where A is any subset of $\omega - \{0\}$ which is infinite and coinfinite.

LEMMA 10.2. *There is a subset I of $\{0, \ldots, N4 - 1\}$ of cardinality at least $N3$ such that the following five statements have the same truth values for all triples $i < j < k$ in I: $\kappa(j,k) = i$, $\kappa(i,k) = j$, $\kappa(i,j) = k$, $\kappa(i,j) = i$, and $\kappa(i,j) = j$. Further, if $\kappa(i,j) \in I$ for some pair $i < j$ in I, then either $\kappa(i,j) = i$ for all $i < j$ in I or $\kappa(i,j) = j$ for all $i < j$ in I.*

PROOF. The first statement follows from the definition of $N4$. Suppose that $\kappa(\alpha, \beta) = \gamma \in I$ for some $\alpha < \beta$ in I. If $\gamma < \alpha$ then $\kappa(j,k) = i$ for all $i < j < k$ in I, which contradicts the fact that κ is a function (with unique $\kappa(j,k)$). Similarly, the cases $\alpha < \gamma < \beta$ and $\beta < \gamma$ are impossible. It remains that $\gamma = \alpha$ or $\gamma = \beta$. If $\gamma = \alpha$ then $\kappa(i,j) = i$ for all $i < j$ in I. If $\gamma = \beta$ then $\kappa(i,j) = j$ for all $i < j$ in I. □

On the grounds of Theorem 5.1, choose W such that for every X, if $\mathrm{Th}^{d1}(C, X, W) = \mathrm{Th}^{d1}(C, U_i, W)$ for some $i \in I$, then $X \sim U_j$ for some $j \in I$. Use Theorem 4.3 to choose a subsequence $\langle H_n : n < \omega\rangle$ of $\langle H'_n : n < \omega\rangle$ such that:

(c) all $d1$-theories $\mathrm{Th}^{d1}(H_n, U_i, U_j, W)$, where $n < \omega$ and $i < j < N4$, are equal, and

(d) all $d1$-theories $\mathrm{Th}^{d1}(H_n - H_m, U_i, U_j, W)$, where $m < n < \omega$ and $i < j < N4$, are equal.

By Lemma 4.2, (c) and (d) imply

(c′) all $d1$-theories $\mathrm{Th}^{d1}(H_n, U_i, U_i, W)$, where $n < \omega$ and $i < N4$, are equal, and

(d′) all $d1$-theories $\mathrm{Th}^{d1}(H_n - H_m, U_i, U_i, W)$, where $m < n < \omega$ and $i < N4$, are equal.

Let $\langle D_n : n < \omega\rangle$ be the partition of D corresponding to $\langle H_n : n < \omega\rangle$. For all $i < j < N4$ and every $A \subseteq \omega$, let $S(i,j,A)$ be the shuffling of U_i and U_j with respect to $\langle D_n : n < \omega\rangle$ and A.

LEMMA 10.3. *For all $i < j < N4$, every $k < N4$ and every $A \subseteq \omega$,*

$$S(i,j,A) \sim U_k \leftrightarrow S'(i,j,A) \sim U_k.$$

Hence, $\min\{k\text{: } \textit{either } S(i,j,A) \sim U_k \textit{ or } k = N4\} = \kappa(i,j,A)$.

PROOF. The second statement follows from the first statement and the definition of $\kappa(i,j,A)$. We prove the first statement. For each n there is $l \in \{i,j\}$ such that $S'(i,j,A)$ coincides with U_l on D'_n and $S(i,j,A)$ coincides with U_l on D_n, so that

$$\mathrm{Th}^{d1}(D'_n, S'(i,j,A), U_k) = \mathrm{Th}^{d1}(D_n, S(i,j,A), U_k).$$

Thus,

$$\begin{aligned}\mathrm{Th}^{d1}(D, S'(i,j,A), U_k) &= \sum_{n<\omega} \mathrm{Th}^{d1}(D'_n, S'(i,j,A), U_k)\\ &= \sum_{n<\omega} \mathrm{Th}^{d1}(D_n, S(i,j,A), U_k) = \mathrm{Th}^{d1}(D, S(i,j,A), U_k).\end{aligned}$$

By Lemma 8.3, $S(i,j,A) \sim U_k \leftrightarrow S'(i,j,A) \sim U_k$. □

By the definition of $N3$, there is a pair $i < j$ in I such that if $n = 0$ or $n = 1$ then $\mathrm{Th}^{d1}(D_n, U_i, W)$ equals $\mathrm{Th}^{d1}(D_n, U_j, W)$. Let $U = U_i$ and $V = U_j$. By the choice of $\langle H_n : n < \omega\rangle$, $\mathrm{Th}^{d1}(D_n, U, W)$ equals $\mathrm{Th}^{d1}(D_n, V, W)$ for all n. For each $A \subseteq \omega$, let $S(U, V, A)$ be the shuffling $S(i,j,A)$ of U and V with respect to $\langle D_n : n < \omega\rangle$ and A.

LEMMA 10.4. *For every $A \subseteq \omega$ there is $k \in I$ with $U_k \sim S(U, V, A)$.*

PROOF. If $n \in \omega - A$ then $S(U, V, A)$ coincides with U on D_n and therefore

$$\mathrm{Th}^{d1}(D_n, S(U, V, A), W) = \mathrm{Th}^{d1}(D_n, U, W) = \mathrm{Th}^{d1}(D_n, V, W).$$

If $n \in A$ then $S(U, V, A)$ coincides with V on D_n and therefore

$$\mathrm{Th}^{d1}(D_n, S(U, V, A), W) = \mathrm{Th}^{d1}(D_n, V, W).$$

Thus

$$\mathrm{Th}^{d1}(D, S(U, V, A), W) = \sum_{n<\omega} \mathrm{Th}^{d1}(D_n, S(U, V, A), W)$$
$$= \sum_{n<\omega} \mathrm{Th}^{d1}(D_n, V, W) = \mathrm{Th}^{d1}(D, V, W).$$

By the choice of W, $S(U, V, A)$ is equivalent to some U_k with $k \in I$. □

Let $\mathrm{POS} = \omega - \{0\}$, $\mathrm{EVEN} = \{2n + 2 : n < \omega\}$ and $\mathrm{ODD} = \{2n + 1 : n < \omega\}$. Notice that EVEN does not contain 0.

LEMMA 10.5. *$S(U, V, \mathrm{POS}) \sim V$, $S(U, V, \{0\}) \sim U$, and either $S(U, V, \mathrm{ODD}) \sim U$ or $S(U, V, \mathrm{ODD}) \sim V$.*

PROOF. By Lemma 10.4, there is $k \in I$ with $U_k \sim S(U, V, \mathrm{POS})$. Notice that $S(U, V, \mathrm{POS})$ belongs to the vicinity of V; hence U_k belongs to the vicinity of V. By the choice of $\langle U_n : n < \omega \rangle$, $U_k = V$. The second statement is proved in a similar way.

We prove the third statement. Recall that $U = U_i$ and $V = U_j$. By Lemma 10.4, $S(U_i, U_j, \mathrm{ODD})$ is equivalent to some U_k with $k \in I$. By Lemma 10.3, $S'(U_i, U_j, \mathrm{ODD}) \sim U_k$. Hence $\kappa(i, j) \in I$. By Lemma 10.2, either $\kappa(i, j) = i$ or $\kappa(i, j) = j$. If $\kappa(i, j) = i$ then $S(U, V, \mathrm{ODD}) \sim U$; if $\kappa(i, j) = j$ then $S(U, V, \mathrm{ODD}) \sim V$. □

LEMMA 10.6. *If $S(U, V, \mathrm{ODD}) \sim U$ then $S(U, V, \mathrm{EVEN}) \sim S(U, V, \mathrm{POS})$.*

PROOF. If n is positive and even then

$$\mathrm{Th}^{d1}(D_n, S(U, V, \mathrm{ODD}), U) = \mathrm{Th}^{d1}(D_n, U, U) = \mathrm{Th}^{d1}(D_n, V, V)$$
$$= \mathrm{Th}^{d1}(D_n, S(U, V, \mathrm{POS}), S(U, V, \mathrm{EVEN})).$$

If n is odd then

$$\mathrm{Th}^{d1}(D_n, S(U, V, \mathrm{ODD}), U) = \mathrm{Th}^{d1}(D_n, V, U)$$
$$= \mathrm{Th}^{d1}(D_n, S(U, V, \mathrm{POS}), S(U, V, \mathrm{EVEN})).$$

Also,

$$\mathrm{Th}^{d1}(D_0, S(U, V, \mathrm{ODD}), U) = \mathrm{Th}^{d1}(D_n, U, U)$$
$$= \mathrm{Th}^{d1}(D_n, S(U, V, \mathrm{POS}), S(U, V, \mathrm{EVEN})).$$

By the composition theorem,

$$\mathrm{Th}^{d1}(D, S(U, V, \mathrm{ODD}), U) = \mathrm{Th}^{d1}(D, S(U, V, \mathrm{POS}), S(U, V, \mathrm{EVEN})).$$

Now use Lemma 8.3. □

LEMMA 10.7. *If $S(U, V, \mathrm{ODD}) \sim V$ then $S(U, V, \mathrm{EVEN}) \sim U$.*

PROOF. If n is positive and even then

$$\mathrm{Th}^{d1}(D_n, U, S(U, V, \mathrm{EVEN})) = \mathrm{Th}^{d1}(D_n, U, V)$$
$$= \mathrm{Th}^{d1}(D_n, S(U, V, \mathrm{ODD}), S(U, V, \mathrm{POS})).$$

If n is odd then

$$\mathrm{Th}^{d1}(D_n, U, S(U, V, \mathrm{EVEN})) = \mathrm{Th}^{d1}(D_n, U, U) = \mathrm{Th}^{d1}(D_n, V, V)$$
$$= \mathrm{Th}^{d1}(D_n, S(U, V, \mathrm{ODD}), S(U, V, \mathrm{POS})).$$

Finally,

$$\begin{aligned}\mathrm{Th}^{d1}(D_0, U, S(U, V, \mathrm{EVEN})) &= \mathrm{Th}^{d1}(D_0, U, U) = \mathrm{Th}^{d1}(D_0, V, V)\\ &= \mathrm{Th}^{d1}(D_n, S(U, V, \mathrm{ODD}), S(U, V, \mathrm{POS})).\end{aligned}$$

By the composition theorem,

$$\mathrm{Th}^{d1}(D, U, S(U, V, \mathrm{EVEN})) = \mathrm{Th}^{d1}(D, S(U, V, \mathrm{ODD}), S(U, V, \mathrm{POS})).$$

By Lemma 8.3,

$$U \sim S(U, V, \mathrm{EVEN}) \leftrightarrow S(U, V, \mathrm{ODD}) \sim S(U, V, \mathrm{POS}).$$

Suppose $S(U, V, \mathrm{ODD}) \sim V$. Then, by the proof of Lemma 10.5, $S(U, V, \mathrm{ODD}) \sim S(U, V, \mathrm{POS})$ and therefore $U \sim S(U, V, \mathrm{EVEN})$. □

Now we are ready to finish the proof of the Main Theorem. First suppose that $S(U, V, \mathrm{ODD}) \sim U$. Then, by Lemmas 10.1 and 10.3, U is equivalent to $S(U, V, \mathrm{EVEN})$. Then, by Lemma 10.6, U is equivalent to $S(U, V, \mathrm{POS})$. Then, by Lemma 10.5, U is equivalent to V, which is impossible.

In virtue of Lemma 10.5, $S(U, V, \mathrm{ODD}) \sim V$. Then, by Lemmas 10.1 and 10.3, V is equivalent to $S(U, V, \mathrm{EVEN})$. But $S(U, V, \mathrm{EVEN})$ is, by Lemma 10.7, equivalent to U. Thus V is equivalent to U, which is impossible.

The Main Theorem is proved.

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