Public-Key Cryptosystems from the Worst-Case Shortest Vector Problem

Chris Peikert
SRI → Georgia Tech

Impagliazzo's World Workshop

This Talk

1. State of Lattice-Based Cryptography
2. Main Result: Public-Key Encryption based on GapSVP
3. Proof & Future Work

Shortest Vector Problem(s)

A lattice $\mathcal{L} \subset \mathbb{R}^n$ having basis $\mathbf{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is:

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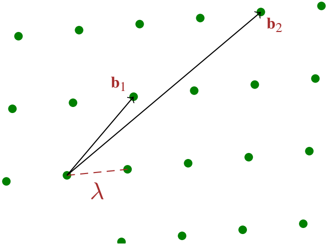

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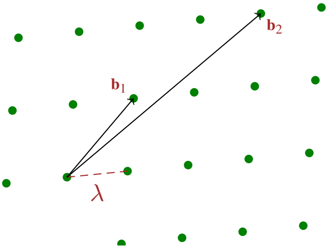

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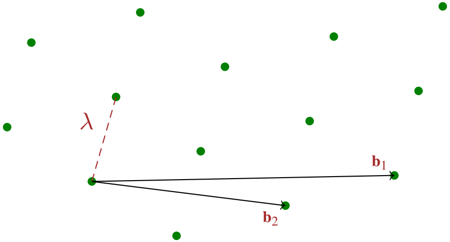

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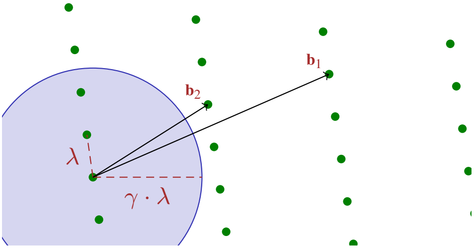

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Unique SVP (γ-uSVP)

- Given $\mathbf{B}$ with 'γ-unique' shortest vector, find it.

Worst-Case Complexity

GapSVP

$\gamma =$	$2^{(\log n)^{1-\epsilon}}$	$\sqrt{n}$	n	$2^{\sim n}$
	NP-hard*	$\in$ coNP	(some) crypto	$\in$ P
	[Ajt98,...,HR07]	[GG98,AR05]	[Ajt96,..., MR04,Reg05]	[LLL82,Sch87]

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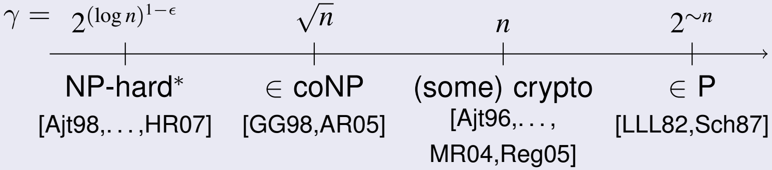

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Taxonomy of Lattice-Based Crypto

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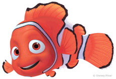

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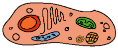

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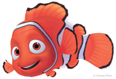

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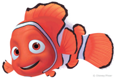

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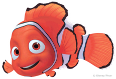

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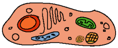

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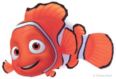

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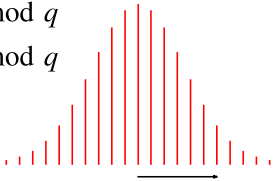

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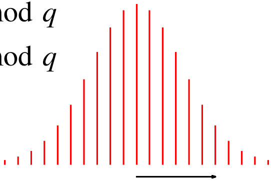

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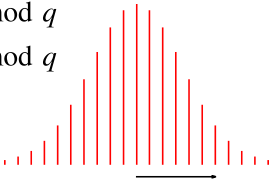

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State of the Art

(n/α)-GapSVP etc. $\leq$ search-LWE $\leq$ decision-LWE $\leq$ crypto

- **quantum** [Reg05]
- prime $q = \text{poly}(n)$ [BFKL94,R05]
- [R05,PW08,GPV08, PVW08,AGV09,ACPS09,...]

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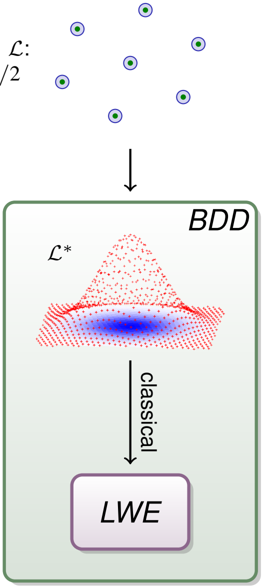
3. **New LWE-based chosen ciphertext-secure encryption**
   - ⋆ Much simpler, milder assumption than prior CCA [PW08]

[Regev05] Reduction to LWE

BDD on $\mathcal{L}$:
$d \ll \lambda/2$



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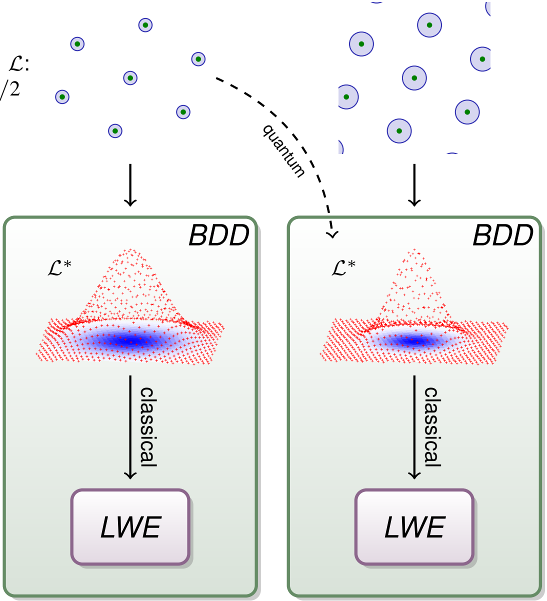

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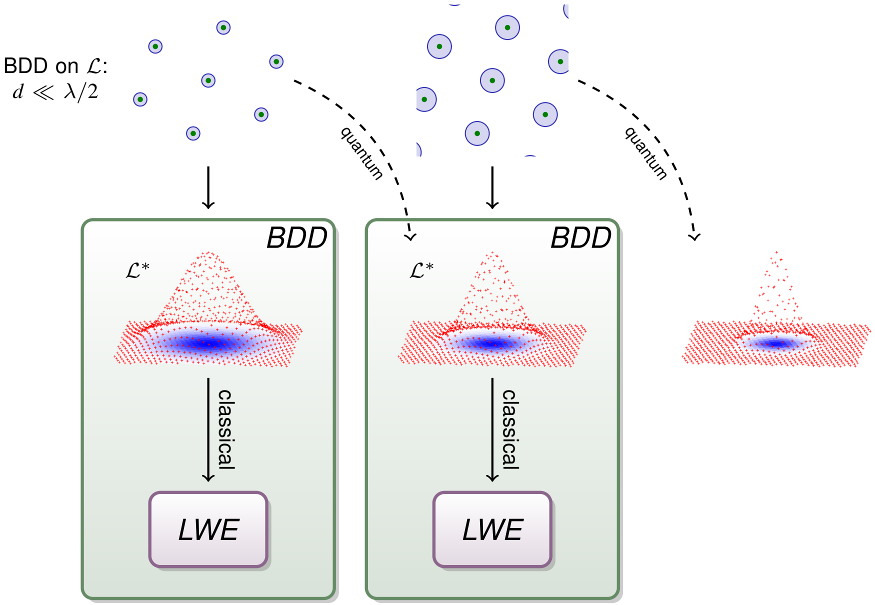

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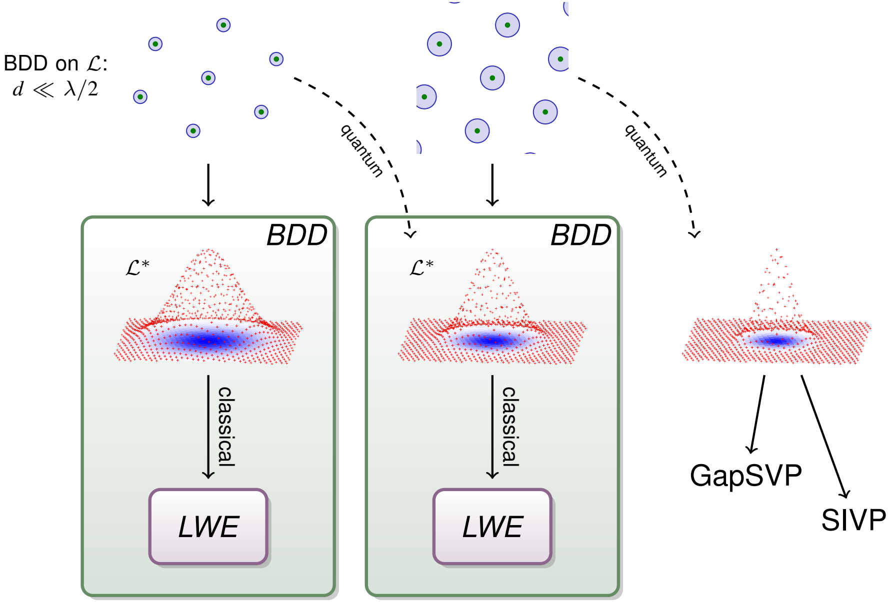

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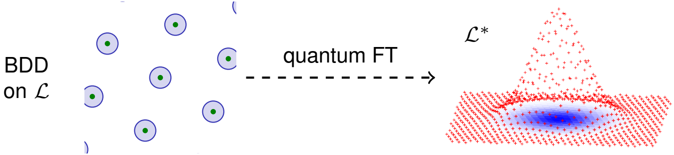

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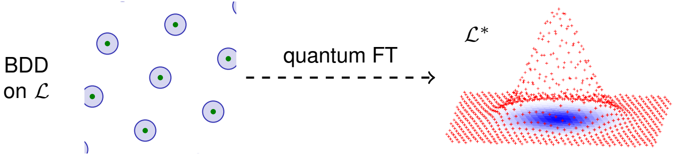

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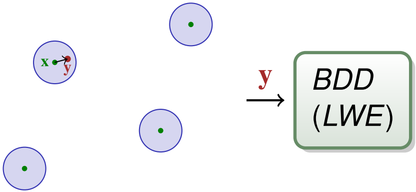

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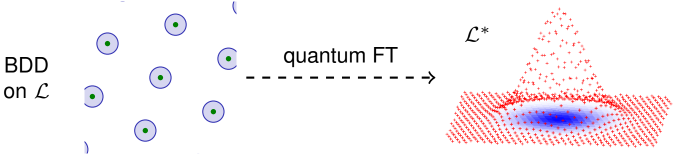

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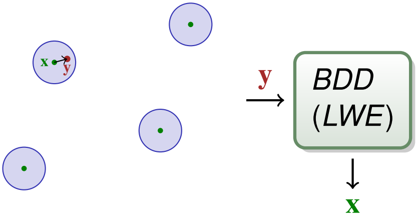

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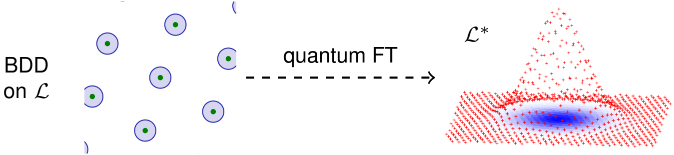

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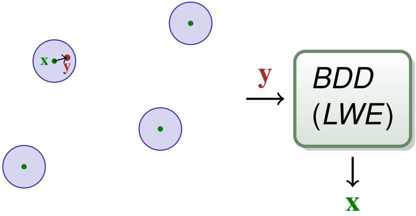
✔ *Quantum can "uncompute"* $\mathbf{x}$



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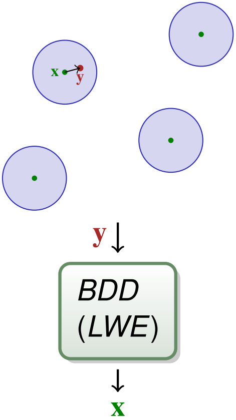

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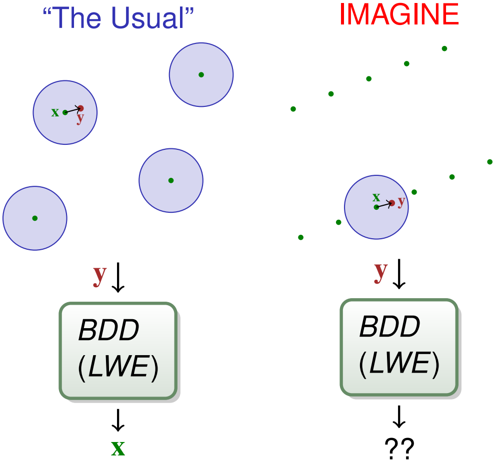

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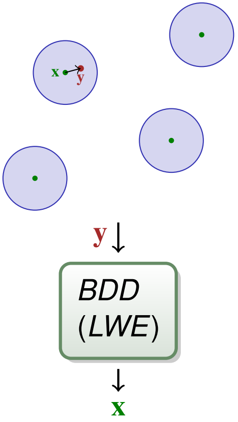

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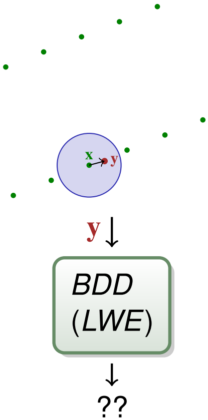

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⇓

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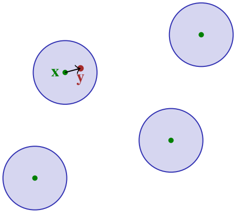

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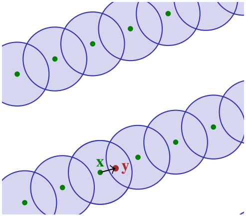

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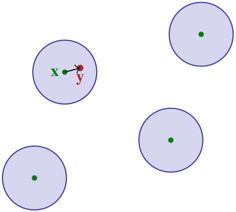

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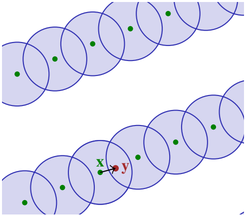

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- View as [GoldGold98] AM proof between reduction and oracle

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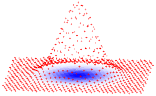
1. What about  in $BDD \rightarrow LWE$ reduction?

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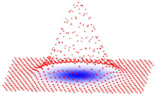
1. What about  in *BDD* $\rightarrow$ *LWE* reduction?

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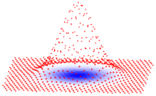
1. What about  in *BDD* $\longrightarrow$ *LWE* reduction?

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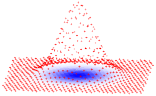
1. What about  in *BDD* $\longrightarrow$ *LWE* reduction?

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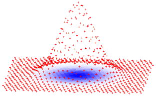
1. What about  in *BDD* $\longrightarrow$ *LWE* reduction?

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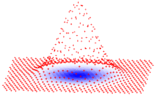
2. LWE search / decision equivalence?

   (Normally requires prime $q = \text{poly}(n) \dots$)

Technical Obstacles

1. What about  in *BDD* $\to$ *LWE* reduction?

(No quantum allowed!)

- ⋆ Use [GPV08] sampling algorithm with 'best available' basis for $\mathcal{L}^*$.

'ζ-good' basis $\Rightarrow$ LWE modulus $q \approx \zeta$.

(LLL-reduced basis is 2^n-good.)

- ⋆ 'One shot' (non-iterative) reduction

2. LWE search / decision equivalence?

(Normally requires prime $q = \text{poly}(n) \ldots$)

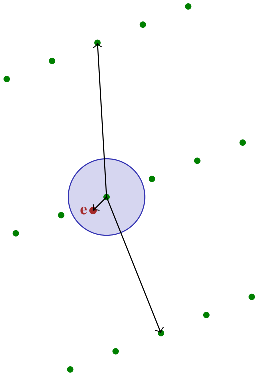
Option 1: crypto directly based on search-LWE

Option 2: search = decision for 'smooth' q and Gaussian error

Details of Reduction

Given any ("ζ-good") $\mathbf{B}$:

1. Choose $\mathbf{e} \leftarrow \sqrt{n} \cdot \mathcal{B}_n$



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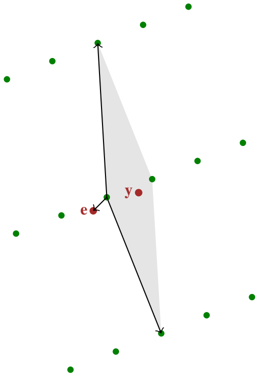

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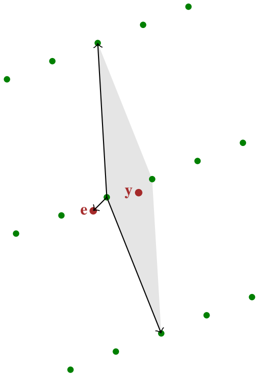

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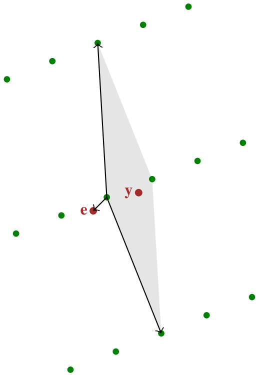

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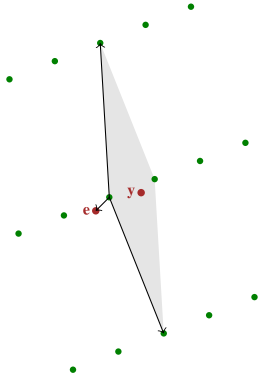

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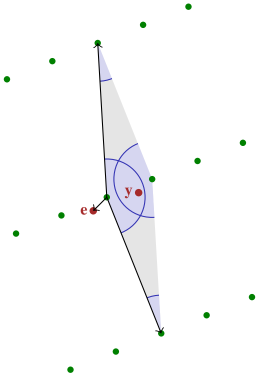

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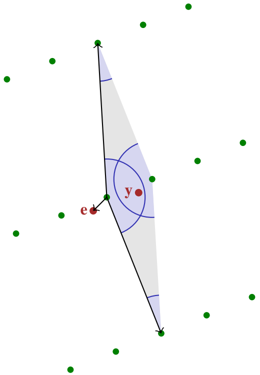

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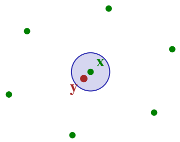
$\Rightarrow$ Step 3 (no matter what it is!) can't guess original $\mathbf{e}$.

Reduction: Step 3

Given "ζ-good" $\mathbf{B}$ and $\mathbf{y} = \mathbf{x} + \mathbf{e}$ for $\mathbf{x} = \mathbf{B}\mathbf{c} \in \mathcal{L}$ and $\|\mathbf{e}\| \leq \sqrt{n}$.



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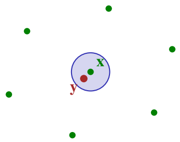

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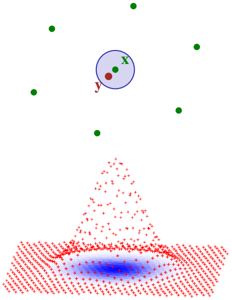

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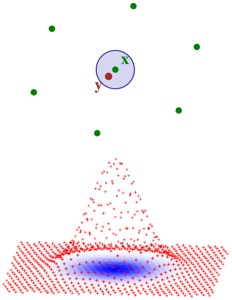

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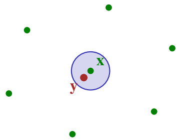

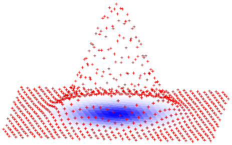

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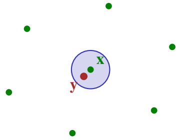

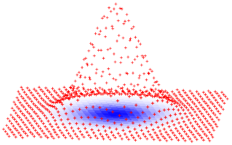

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$$= \langle \mathbf{B}^*\mathbf{z}, \mathbf{Bc} \rangle + \langle \mathbf{v}, \mathbf{e} \rangle \simeq \langle \mathbf{a}, \mathbf{s} \rangle + D_{\zeta \cdot \|\mathbf{e}\|} \bmod q.$$

Finally, $\zeta \cdot \|\mathbf{e}\| \le \alpha \cdot q$.

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- (NB: for general error dists, hybrid argument over q_i's fails.)

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- Correlation-secure injective TDF $\Rightarrow$ CCA-secure encryption
  But much care needed to make $g_{\mathbf{A}}$ "chosen-output secure."

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3. **Open**: complexity of 'Improve ζ to γ'-GapSVP?

   NP-hard for nontrivial ζ? Better algorithms?