On Ideal Lattices and Learning With Errors Over Rings

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Eurocrypt 2010

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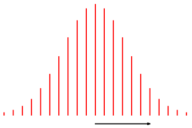

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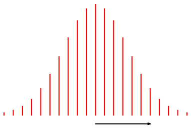

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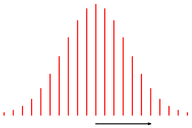

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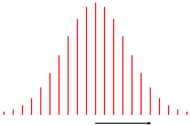

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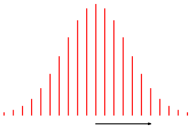

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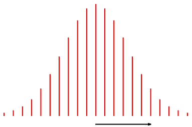

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- (Also some *classical* hardness for search-LWE [P'09])

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  UC Oblivious Transfer [PVW'08]

  Leakage Resilience [AGV'09,DGKPV'10,GKPV'10,ADNSWW'10,...]

  Circular/KDM-Secure Encryption [ACPS'09,BHHI'10]

  Quadratic-Formula Homomorphic Encryption [GHV'10]

  Bi-Deniable Encryption [OP'10]

  and more...

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- Can fix $\mathbf{A}$ for all users, but at best, still $\tilde{\Omega}(n^2)$ work to encrypt & decrypt an n-bit message

Wishful Thinking...

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- Similar ring structures appear in heuristic NTRU scheme [HPS'98], in compact one-way / CR hash functions [Mic'02,PR'06,LM'06,...], and in fully homomorphic encryption [Gen'09].

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2. A 'cookbook' for porting LWE-based schemes to Ring-LWE, plus an entirely new & even more efficient PKE scheme.

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Proof of CPA Security

1. Public key $(\mathbf{a}, \mathbf{b}) \approx_c (\mathbf{a}, \mathbf{b})$ by decision Ring-LWE
   (even for 'short' $\mathbf{s}$ [ACPS'09])
2. Ciphertext $(\mathbf{c}_1, \mathbf{c}_2) \approx_c (\mathbf{c}_1, \mathbf{c}_2)$, again by decision Ring-LWE

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- New reduction technique for 'clearing the ideal' ($\mathcal{I}/q\mathcal{I} \mapsto R/qR$), in an 'algebra-preserving' way.

  Uses Chinese remainder theorem and theory of duality for ideals.

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- [HPS'98,M'02,PR'06,LM'06,G'09,...]: 'coefficient embedding'

$$a(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} \quad \mapsto \quad (a_0, \ldots, a_{n-1}) \in \mathbb{Z}^n$$

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- An ideal $\mathcal{I} \subseteq R$ is closed under $+$ and $-$, and under $\times$ with R.

To get ideal lattices, embed R and its ideals into $\mathbb{R}^n$. How?

- [HPS'98,M'02,PR'06,LM'06,G'09,...]: 'coefficient embedding'

$$a(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} \quad \mapsto \quad (a_0, \ldots, a_{n-1}) \in \mathbb{Z}^n$$

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- Modulo any prime $q = 1 \bmod 2n$, $(x^n + 1)$ has n roots $\omega^{2i-1} \in \mathbb{Z}_q$.
  For Ring-LWE schemes, this gives an embedding into $\mathbb{Z}_q^n$.

Pseudorandomness of Ring-LWE

Theorem 2

Solving decision Ring-LWE in $R_q = \mathbb{Z}_q[x]/(x^n + 1)$

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is as hard as solving search Ring-LWE.

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Proof Outline

Given: $\mathcal{O}$ distinguishes samples $(\mathbf{a}, \mathbf{b} \approx \mathbf{a} \times \mathbf{s})$ from uniform $(\mathbf{a}, \mathbf{b})$.

Goal: Find $\mathbf{s} \in R_q$. Equivalent to finding $\mathbf{s}(\omega^{2j-1}) \in \mathbb{Z}_q$ for $j = 1, \ldots, n$.

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1. Hybrid argument: randomize $\mathbf{b}(\omega^1) \in \mathbb{Z}_q$, then $(\mathbf{b}(\omega^1), \mathbf{b}(\omega^3)), \ldots$
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(Math jargon: use the automorphism (Galois) group of the cyclotomic number field.)

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- Questions? More details? Find me here: