Homework 3

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Student: YOUR NAME HERE

This homework is due by the **start of class on November 4** via the course Canvas page. Start early!

Instructions. Solutions must be typeset in LaTeX (a template for this homework is available on the course web page). Your work will be graded on *correctness*, *clarity*, and *conciseness*. You should only submit work that you believe to be correct; if you cannot solve a problem completely, you will get significantly more partial credit if you clearly identify the gap(s) in your solution. It is good practice to start any long solution with an informal (but accurate) "proof summary" that describes the main idea.

You may collaborate with others on this problem set and consult external sources. However, you must ***write your own solutions*** and ***list your collaborators/sources*** for each problem.

1. Using an oracle for SVP_γ for some $\gamma \geq 1$, show that one can efficiently solve "almost all" subset-sum instances of density as large as $C/\log\gamma$, where $C > 0$ is some universal constant. *Hint:* describe how to adapt Frieze's attack and analysis. To handle the case $\gamma = o(\sqrt{n})$, you will need a tighter analysis of the number of integer vectors of length at most $\gamma\sqrt{n}$. Consider centering unit cubes at all such points, and use a volume argument.

2. Let $\mathbf{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ be a set of linearly independent vectors and let $\mathbf{D} = (\mathbf{d}_1, \ldots, \mathbf{d}_n) = \mathbf{B}^{-t}$ be its dual set. Let $(\widetilde{\mathbf{b}}_1, \ldots, \widetilde{\mathbf{b}}_n)$ be the Gram-Schmidt orthogonalization of $\mathbf{B}$, and $(\widetilde{\mathbf{d}}_1, \ldots, \widetilde{\mathbf{d}}_n)$ be the GS orthogonalization of $\mathbf{D}$ *in reverse order*, i.e., $\widetilde{\mathbf{d}}_n = \mathbf{d}_n$, $\widetilde{\mathbf{d}}_{n-1}$ is the component of $\mathbf{d}_{n-1}$ orthogonal to $\mathbf{d}_n$, etc.

   Prove that $\widetilde{\mathbf{d}}_i = \widetilde{\mathbf{b}}_i / \|\widetilde{\mathbf{b}}_i\|^2$.

3. Define the lattice parameter $\widetilde{bl}(\mathcal{L}) := \min_{\mathbf{B}} \max_i \|\widetilde{\mathbf{b}}_i\|$, where the minimum is taken over all bases $\mathbf{B}$ of $\mathcal{L}$, and $\{\widetilde{\mathbf{b}}_i\}$ denotes the Gram-Schmidt orthogonalization of $\mathbf{B}$.

   (a) Prove that $\eta_\varepsilon(\mathcal{L}) \leq \widetilde{bl}(\mathcal{L}) \cdot \sqrt{\ln(2n(1+1/\varepsilon))/\pi}$. (*Hint:* use question 2.)

   (b) In class we proved the above bound with λ_n in place of $\widetilde{bl}$; one can show that $\widetilde{bl}(\mathcal{L}) \leq \lambda_n(\mathcal{L})$, so the bound here is at least as good. Show that there exist n-dimensional lattices for which

   $$\lambda_n(\mathcal{L}) / \widetilde{bl}(\mathcal{L}) \geq \Omega(\sqrt{n}).$$

   Hint: consider the lattice generated by the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}$, and a "random" $\mathbf{r}$ such that $r_n = 1$, and show that this works with good probability.