DFT:DISCRETE FOURIER TRANSFORM

Professor Andrew E. Yagle, EECS 206 Instructor, Fall 2005

Dept. of EECS, The University of Michigan, Ann Arbor, MI 48109-2122

I. Abstract

The purpose of this document is to introduce EECS 206 students to the DFT (Discrete Fourier Transform), where it comes from, what it's for, and how to use it. It also derives the sampling theorem for periodic signals, Parseval's theorem, discusses orthogonality, and shows how to compute line spectra of sampled signals.

A. *Table of contents by sections:*

II. Summary of the DFT (How do I do the homework?)

I know, this is what you want to know *right now*, since it's Thursday night and you are having trouble with problem set #6. But you're missing the point of the DFT if this is all of these notes you read!

A. *Comparison of continuous and discrete time Fourier series*

One way to look at the DFT is as a discrete-time counterpart to the continuous-time Fourier series.

Let $x(t)$ be a real-valued continuous-time signal with period=T. Then $x(t)$ can be expanded as

$$\begin{aligned} x(t) = \quad x_0 \quad &+ x_1 e^{j\frac{2\pi}{T}t} + x_2 e^{j\frac{4\pi}{T}t} + x_3 e^{j\frac{6\pi}{T}t} + \ldots \\ &+ x_1^* e^{-j\frac{2\pi}{T}t} + x_2^* e^{-j\frac{4\pi}{T}t} + x_3^* e^{-j\frac{6\pi}{T}t} + \ldots \end{aligned} \tag{1}$$

where the coefficients x_k are computed using the following formula:

$$x_k = \frac{1}{T}\int_0^T x(t) e^{-j\frac{2\pi}{T}kt} dt \quad \text{for} \quad k = 0, \pm 1, \pm 2, \pm 3 \ldots \tag{2}$$

Now let $x[n]$ be a real-valued discrete-time signal with period=N. Then $x[n]$ can be expanded as:

$$\begin{aligned} x[n] = X_0 \quad &+ X_1 e^{j\frac{2\pi}{N}n} + X_2 e^{j\frac{4\pi}{N}n} + \ldots + X_{(N-1)/2} e^{j\pi\frac{N-1}{N}n} \\ &+ X_1^* e^{-j\frac{2\pi}{N}n} + X_2^* e^{-j\frac{4\pi}{N}n} + \ldots + X_{(N-1)/2}^* e^{-j\pi\frac{N-1}{N}n} \\ x[n] = X_0 \quad &+ X_1 e^{j\frac{2\pi}{N}n} + X_2 e^{j\frac{4\pi}{N}n} + \ldots + X_{N-1} e^{j2\pi\frac{N-1}{N}n} \end{aligned} \tag{3}$$

The 1^{st} series is written for odd N; if N is even, there is an additional term $X_{N/2}e^{j\pi n}$. The 2^{nd} series is clearly easier to use, but the analogy to the continuous-time Fourier series is easier to see using the 1^{st} series.

The coefficients X_k are computed using the following formula, which is the N-point DFT:

$$X_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}nk}, \quad k = 0, 1, 2 \ldots (N-1) \tag{4}$$

and $X_{-k} = X_{N-k} = X_k^*$ defines X_k for $k < 0$. We write $DFT\{x[n]\} = \{X_k\}$.

Parseval's theorem states that we can compute average power in either the time or frequency domains:

$$\begin{aligned} \frac{1}{T}\int_0^T |x(t)|^2 dt &= \sum_{k=-\infty}^{\infty} |x_k|^2 \\ \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 &= \sum_{k=0}^{N-1} |X_k|^2 \end{aligned} \tag{5}$$

since the average power of $x_k e^{j\omega t}$ is $|x_k|^2$, and the average power of $X_k e^{j\omega n}$ is $|X_k|^2$.

Comparing the continuous-time and discrete-time Fourier series reveals these *similarities*:

- Both expand the periodic signal $x(t)$ or $x[n]$ in terms of complex exponential functions of time $e^{j\omega t}$ or $e^{j\omega n}$ at frequencies ω that are integer multiples k of $(2\pi)/T$ or $(2\pi)/N$;
- In Hertz, these frequencies, called *harmonics*, are $\{0, \pm(1/T), \pm(2/T)\ldots\}$ or $\{0, \pm(1/N), \pm(2/N)\ldots\}$;
- x_k or X_k are computed using an integral over t, or a sum over n like a discretized version of the integral;

Comparing the continuous-time and discrete-time Fourier series reveals these *differences*:

- The continuous-time Fourier series has an infinite number of terms, while the discrete-time Fourier series has only N terms, since the fastest-oscillating discrete-time sinusoid is $\cos(\pi n) = (-1)^n$;
- The discrete-time Fourier series treats frequencies $-\pi < \omega < 0$ the same as frequencies $\pi < \omega < 2\pi$, since in discrete time these *are* the same frequencies: $\cos(-\omega n + \theta) = \cos([2\pi - \omega]n + \theta)$;
- The discrete-time Fourier series coefficients do not require evaluation of an integral, just a finite sum (the DFT), which in fact can be computed quickly in Matlab using `fft(X,N)/N:`.

B. Simple numerical example

Consider the discrete-time periodic signal

$$x[n] = \{\dots 24, 8, 12, 16, \underline{24}, 8, 12, 18, 24, 8, 12, 16 \dots\} \tag{6}$$

By inspection, the period=N=4. The DFT is computed using

$$X_k = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j\frac{2\pi}{4}nk}, \quad k = 0, 1, 2, 3 \tag{7}$$

Writing this out explicitly for $k = 0, 1, 2, 3$ yields

$$\begin{aligned}
X_0 = &\ \tfrac{1}{4}(x[0] + x[1]e^{-j0} + x[2]e^{-j0} + x[3]e^{-j0}) &&= \frac{1}{4}(24 + 8 + 12 + 16) = 15 \\
X_1 = &\ \tfrac{1}{4}(x[0] + x[1]e^{-j\frac{2\pi}{4}} + x[2]e^{-j\frac{4\pi}{4}} + x[3]e^{-j\frac{6\pi}{4}}) &&= \frac{1}{4}(24 - 8j - 12 + 16j) = 3 + 2j \\
X_2 = &\ \tfrac{1}{4}(x[0] + x[1]e^{-j\frac{4\pi}{4}} + x[2]e^{-j\frac{8\pi}{4}} + x[3]e^{-j\frac{12\pi}{4}}) &&= \frac{1}{4}(24 - 8 + 12 - 16) = 3 \\
X_3 = &\ \tfrac{1}{4}(x[0] + x[1]e^{-j\frac{6\pi}{4}} + x[2]e^{-j\frac{12\pi}{4}} + x[3]e^{-j\frac{18\pi}{4}}) &&= \frac{1}{4}(24 + 8j - 12 - 16j) = 3 - 2j
\end{aligned} \tag{8}$$

We just computed a 4-point DFT by hand. We could have used Matlab: `fft([24 8 12 16],4)/4`.

The discrete-time Fourier series is

$$x[n] = \sum_{k=0}^{3} X_k e^{j\frac{2\pi}{4}nk} = 15 + (3 + 2j)e^{j\frac{2\pi}{4}n} + (3)e^{j\frac{4\pi}{4}n} + (3 - 2j)e^{j\frac{6\pi}{4}n} \tag{9}$$

Plugging $n = 0, 1, 2, 3$ into this equation does indeed give $x[n] = 24, 8, 12, 16$, one period of $x[n]$ (try it!).

We can also write the discrete-time Fourier series in trigonometric form. Noting that

$$(3 + 2j) = 3.6e^{j34^o}(\text{rounding off}); \quad e^{j\pi n} = \cos(\pi n); \quad e^{j\frac{6\pi}{4}n} = e^{-j\frac{2\pi}{4}n} \tag{10}$$

we can rewrite the above expansion of $x[n]$ in complex exponentials in trigonometric form as

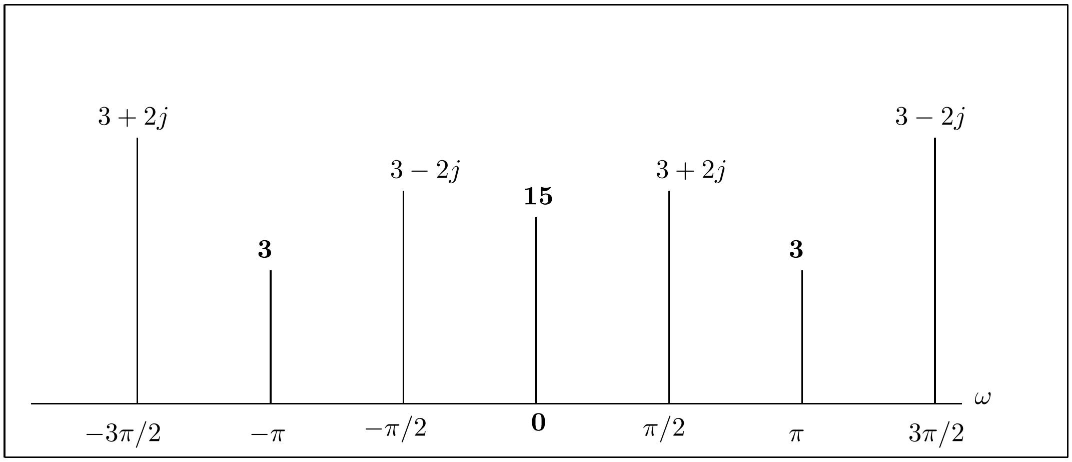
$$x[n] = 15 + 7.2\cos(\frac{\pi}{2}n + 34^o) + 3\cos(\pi n) \tag{11}$$

Note that in converting from complex exponential form to trigonometric form, *don't* double amplitudes of sinusoids at $\omega = 0$ and $\omega = \pi$, since there are not two contributing terms.

For this $x[n]$, *Parseval's theorem* states that the average power of $x[n]$ is

$$\frac{1}{4}((24)^2 + (8)^2 + (12)^2 + (16)^2) = 260 = |15|^2 + |3 + 2j|^2 + |3|^2 + |3 - 2j|^2 \tag{12}$$

The *line spectrum* of $x[n]$ looks like (remember, it's periodic with period 2π)



OK, now go do problem set #6. But what is all of this *for*? Read on...

III. Review of continuous-time Fourier series

A. *Summary of equations*

Let $x(t)$ be a real-valued periodic signal with period=T, so that $x(t) = x(t+T)$ for all t. Then $x(t)$ can be expanded in any of the following three Fourier series:

$$\begin{aligned}
x(t) &= x_0 + x_1 e^{j\frac{2\pi}{T}t} + x_2 e^{j\frac{4\pi}{T}t} + x_3 e^{j\frac{6\pi}{T}t} + \dots \\
&\quad + x_1^* e^{-j\frac{2\pi}{T}t} + x_2^* e^{-j\frac{4\pi}{T}t} + x_3^* e^{-j\frac{6\pi}{T}t} + \dots \\
x(t) &= a_0 + a_1 \cos(\tfrac{2\pi}{T}t) + a_2 \cos(\tfrac{4\pi}{T}t) + a_3 \cos(\tfrac{6\pi}{T}t) + \dots \\
&\quad + b_1 \sin(\tfrac{2\pi}{T}t) + b_2 \sin(\tfrac{4\pi}{T}t) + b_3 \sin(\tfrac{6\pi}{T}t) + \dots \\
x(t) &= c_0 + c_1 \cos(\tfrac{2\pi}{T}t - \theta_1) + c_2 \cos(\tfrac{4\pi}{T}t - \theta_2) + \dots
\end{aligned} \tag{13}$$

where the coefficients are computed using the following formulae:

$$\begin{aligned}
x_k &= \tfrac{1}{T}\int_0^T x(t) e^{-j\frac{2\pi}{T}kt} dt && \text{for} \quad k = 0, \pm1, \pm2, \pm3 \dots \\
a_0 &= \tfrac{1}{T}\int_0^T x(t) dt = M(x) && \text{for} \quad k = 0 \\
a_k &= \tfrac{2}{T}\int_0^T x(t) \cos(\tfrac{2\pi}{T}kt) dt && \text{for} \quad k = 1, 2, 3 \dots \\
b_k &= \tfrac{2}{T}\int_0^T x(t) \sin(\tfrac{2\pi}{T}kt) dt && \text{for} \quad k = 1, 2, 3 \dots \\
c_k &= \sqrt{a_k^2 + b_k^2} \quad \text{and} && \theta_k = \tan^{-1}(b_k/a_k)(+\pi?)
\end{aligned} \tag{14}$$

These formulae are derived by noting the *orthogonality* equations

$$\begin{aligned}
0 &= \int_0^T e^{j\frac{2\pi}{T}mt} e^{-j\frac{2\pi}{T}nt} dt && \text{unless} \quad m = n \\
0 &= \int_0^T \cos(\tfrac{2\pi}{T}mt) \cos(\tfrac{2\pi}{T}nt) dt && \text{unless} \quad m = n
\end{aligned}$$

$$0 = \int_0^T \sin(\tfrac{2\pi}{T}mt)\sin(\tfrac{2\pi}{T}nt)dt \quad \text{unless} \quad m=n$$
$$0 = \int_0^T \sin(\tfrac{2\pi}{T}mt)\cos(\tfrac{2\pi}{T}nt)dt \quad \text{even if} \quad m=n \tag{15}$$

For example, multiply the first equation of (1) by $e^{-j\frac{2\pi}{T}kt}$ and integrate from 0 to T. By the first orthogonality equation, all terms except the k^{th} term are zero. This gives us the formula for x_k.

The formulae for $\{c_k, \theta_k\}$ are derived using phasors:

$$\begin{aligned} a\cos(\omega t) + b\sin(\omega t) \quad &\rightarrow \quad a - jb = \sqrt{a^2+b^2}e^{-j\tan^{-1}(b/a)} \\ &\rightarrow \quad \sqrt{a^2+b^2}\cos\left(\omega t - j\tan^{-1}(b/a)\right) \end{aligned} \tag{16}$$

with the usual caveat that the phase will be off by π if $a < 0$.

B. So who needs the DFT?

A mathematician would regard this as the end of the matter, as far as computing the Fourier coefficients is concerned: "What's the problem? Just plug into any of those integrals in (2)." He would be more interested in pathological functions which do not have Fourier series expansions, since they don't satisfy "Dirichlet conditions." There was an argument between Fourier and Lagrange at the Paris Academy in 1807 over this.

But an engineer would say, "I don't have some function $x(t)$. I have a continuous-time recording of Elvis Presley singing. How am I supposed to compute $x_k = \frac{1}{T}\int_0^T \text{elvis}(t)e^{-j\frac{2\pi}{T}kt}dt$?"

She might consider discretizing the integral into a sum. Choosing a small number Δ, she would discretize $t = n\Delta$, so that $0 = 0\Delta$ and $T = N\Delta$. The integral then becomes the finite sum

$$x_k = \frac{1}{T}\int_0^T \text{elvis}(t)e^{-j\frac{2\pi}{T}kt}dt \approx \frac{1}{T}\sum_{n=0}^{N-1}\text{elvis}(n\Delta)e^{-j\frac{2\pi}{T}kn\Delta}\Delta = \frac{1}{N}\sum_{n=0}^{N-1}\text{elvis}(n\Delta)e^{-j\frac{2\pi}{N}kn} \tag{17}$$

since $\frac{\Delta}{T} = \frac{1}{N}$. So she could *approximate* x_k by a finite weighted sum of *samples* elvis$(n\Delta)$ of elvis(t). But that's only an approximation, and that's not good enough for Elvis! This has us singing "Heartbreak Hotel."

But we have a "Good Luck Charm": the DFT. Amazingly, the finite sum above will be *exact*, **not** an approximation, if elvis(t) is *bandlimited* to $1/(2\Delta)$ Hertz, i.e., elvis(t) has no frequencies at or above $1/(2\Delta)$ Hertz. You should be "All Shook Up" by this; it's what makes DSP possible. And it was discovered by a UM alumnus, Claude Shannon (that's a bust of him outside the west entrance to the EECS building).

How can this work? Read on. . .

IV. Bandlimited signals and finite Fourier series

A. Bandlimited signals

Now suppose that $x(t)$ is not only *periodic* with period=T, but also *bandlimited* to B Hertz, so that $x(t)$ has no frequencies at or above B Hertz. What does this do for us?

Since $x(t)$ is periodic with period=T, its Fourier series consists of sinusoids or complex exponentials at frequencies $f = 0, \frac{1}{T}, \frac{2}{T} \ldots$ Hertz. $f = 0$ is the DC term, $f = \frac{1}{T}$ is the fundamental, and $f = \frac{k}{T}$ is the k^{th} harmonic (some would say it is the $(k-1)^{th}$ harmonic). There is some integer N such that

$$(N/T) < B < (N+1)/T, \quad B = \text{bandwidth of signal} \tag{18}$$

so that B can actually be replaced with any number between these limits. For example, if a signal has a period of T=50 seconds, then The Following Are Equivalent (TFAE):

- The signal is bandlimited to 100.005 Hz (N=5000)
- The signal is bandlimited to 100.015 Hz (N=5000)

In fact, B can be any value between 100 Hz and 100.02 Hz, since the signal has no frequency components between those two harmonics. Without loss of generality (WLOG), we will henceforth split the difference and assume that (note this implies that what counts for N is the dimensionless product BT)

$$B = (N + 0.5)/T = \text{bandwidth of signal} \rightarrow 2N + 1 = 2BT \tag{19}$$

B. Finite Fourier series

The significance of $x(t)$ being bandlimited is that its Fourier series is *finite*:

$$\begin{aligned}
x(t) = x_0 \quad &+ x_1 e^{j\frac{2\pi}{T}t} + x_2 e^{j\frac{4\pi}{T}t} + \ldots + x_N e^{j\frac{N\pi}{T}t} \\
&+ x_1^* e^{-j\frac{2\pi}{T}t} + x_2^* e^{-j\frac{4\pi}{T}t} + \ldots + x_N^* e^{-j\frac{N\pi}{T}t} \\
x(t) = a_0 \quad &+ a_1 \cos(\tfrac{2\pi}{T}t) + a_2 \cos(\tfrac{4\pi}{T}t) + \ldots + a_N \cos(\tfrac{N\pi}{T}t) \\
&+ b_1 \sin(\tfrac{2\pi}{T}t) + b_2 \sin(\tfrac{4\pi}{T}t) + \ldots + b_N \sin(\tfrac{N\pi}{T}t) \\
x(t) = c_0 \quad &+ c_1 \cos(\tfrac{2\pi}{T}t - \theta_1) + \ldots + c_N \cos(\tfrac{N\pi}{T}t - \theta_N)
\end{aligned} \tag{20}$$

No longer are there $\ldots$ at the ends–each series has a finite number of terms.

This means that $x(t)$ is completely specified by $(2N + 1)$ real numbers:

$$\{x_0, Re[x_1] \ldots Re[x_N], Im[x_1] \ldots Im[x_N]\} \text{ or } \{a_0, a_1 \ldots a_N, b_1 \ldots b_N\} \text{ or } \{c_0, c_1 \ldots c_N, \theta_1 \ldots \theta_N\} \tag{21}$$

If we can somehow come up with those $(2N + 1)$ numbers, we can plug into any of (20) and compute $x(t)$ *exactly* for *any* value of t. That is, we have reduced the dimensionality of $x(t)$ from ∞ to $(2N + 1)$.

But that still doesn't tell us how to come up with those $(2N+1)$ numbers without computing an integral. What do we do? Read on...

V. Sampling theorem for periodic signals

A. Sampling

Since we only need $(2N+1)$ numbers, one thing we could do is *sample* $x(t)$ at $(2N+1)$ different times in a period (note that sampling $x(t)$ at $t = 0, T, 2T \ldots$ will only give us a single number!). Let's sample at $(2N+1)$ equally-spaced times within a period. That is, we sample $x(t)$ at

$$t = (nT)/(2N+1), \quad n = 0, 1, 2 \ldots 2N \tag{22}$$

Note that $n = 2N+1$ would give us $t = \frac{2N+1}{2N+1}T = T$, which would give the same sample as $t = 0$; there are $(2N+1)$ integers between 0 and $(2N)$, inclusive.

Setting $t = \frac{n}{2N+1}T$ in the first Fourier expansion of (20) gives:

(**Warning:** Although this is simply plugging in $t = \frac{n}{2N+1}T$ in a previous equation, get a cup of coffee.)

$$\begin{aligned}
x(0) = \quad & x_0 \quad +x_1 e^{j\frac{2\pi}{T}0} + x_2 e^{j\frac{4\pi}{T}0} + \ldots + x_N e^{j\frac{N\pi}{T}0} \\
& \qquad +x_1^* e^{-j\frac{2\pi}{T}0} + x_2^* e^{-j\frac{4\pi}{T}0} + \ldots + x_N^* e^{-j\frac{N\pi}{T}0} \\
x(T/(2N+1)) = \quad & x_0 \quad +x_1 e^{j\frac{2\pi}{T}\frac{T}{2N+1}} + x_2 e^{j\frac{4\pi}{T}\frac{T}{2N+1}} + \ldots + x_N e^{j\frac{N\pi}{T}\frac{T}{2N+1}} \\
& \qquad +x_1^* e^{-j\frac{2\pi}{T}\frac{T}{2N+1}} + x_2^* e^{-j\frac{4\pi}{T}\frac{T}{2N+1}} + \ldots + x_N^* e^{-j\frac{N\pi}{T}\frac{T}{2N+1}} \\
x((2T)/(2N+1)) = \quad & x_0 \quad +x_1 e^{j\frac{2\pi}{T}\frac{2T}{2N+1}} + x_2 e^{j\frac{4\pi}{T}\frac{2T}{2N+1}} + \ldots + x_N e^{j\frac{N\pi}{T}\frac{2T}{2N+1}} \\
& \qquad +x_1^* e^{-j\frac{2\pi}{T}\frac{2T}{2N+1}} + x_2^* e^{-j\frac{4\pi}{T}\frac{2T}{2N+1}} + \ldots + x_N^* e^{-j\frac{N\pi}{T}\frac{2T}{2N+1}} \\
x((3T)/(2N+1)) = \quad & x_0 \quad +x_1 e^{j\frac{2\pi}{T}\frac{3T}{2N+1}} + x_2 e^{j\frac{4\pi}{T}\frac{3T}{2N+1}} + \ldots + x_N e^{j\frac{N\pi}{T}\frac{3T}{2N+1}} \\
& \qquad +x_1^* e^{-j\frac{2\pi}{T}\frac{3T}{2N+1}} + x_2^* e^{-j\frac{4\pi}{T}\frac{3T}{2N+1}} + \ldots + x_N^* e^{-j\frac{N\pi}{T}\frac{3T}{2N+1}} \\
\vdots \quad & \vdots \quad \vdots \\
x((2NT)/(2N+1)) = \quad & x_0 \quad +x_1 e^{j\frac{2\pi}{T}\frac{2NT}{2N+1}} + x_2 e^{j\frac{4\pi}{T}\frac{2NT}{2N+1}} + \ldots + x_N e^{j\frac{N\pi}{T}\frac{2NT}{2N+1}} \\
& \qquad +x_1^* e^{-j\frac{2\pi}{T}\frac{2NT}{2N+1}} + x_2^* e^{-j\frac{4\pi}{T}\frac{2NT}{2N+1}} + \ldots + x_N^* e^{-j\frac{N\pi}{T}\frac{2NT}{2N+1}}
\end{aligned} \tag{23}$$

(I warned you about the coffee. But these are the worst equations.) Defining the samples $x[n]$ of $x(t)$ as

$$x[n] = x(t = (nT)/(2N+1)) \tag{24}$$

these equations can be rewritten in the much-easier-to-read form

$$\begin{aligned}
x[0] = \quad & x_0 \quad +x_1 + \ldots + x_N + x_1^* + \ldots + x_N^* \\
x[1] = \quad & x_0 \quad +x_1 e^{j\frac{2\pi}{2N+1}} + x_2 e^{j\frac{4\pi}{2N+1}} + \ldots + x_N e^{j\frac{N\pi}{2N+1}} \\
& \qquad +x_1^* e^{-j\frac{2\pi}{2N+1}} + x_2^* e^{-j\frac{4\pi}{2N+1}} + \ldots + x_N^* e^{-j\frac{N\pi}{2N+1}} \\
x[2] = \quad & x_0 \quad +x_1 e^{j\frac{4\pi}{2N+1}} + x_2 e^{j\frac{8\pi}{2N+1}} + \ldots + x_N e^{j\frac{2N\pi}{2N+1}}
\end{aligned}$$

$$
\begin{aligned}
& & & +x_1^* e^{-j\frac{4\pi}{2N+1}} + x_2^* e^{-j\frac{8\pi}{2N+1}} + \ldots + x_N^* e^{-j\frac{2N\pi}{2N+1}} \\
&\vdots & \vdots & \vdots \\
x[2N] = & & x_0 & +x_1 e^{j\frac{4N\pi}{2N+1}} + x_2 e^{j\frac{8N\pi}{2N+1}} + \ldots + x_N e^{j\frac{2N^2\pi}{2N+1}} \\
& & & +x_1^* e^{-j\frac{4N\pi}{2N+1}} + x_2^* e^{-j\frac{8N\pi}{2N+1}} + \ldots + x_N^* e^{-j\frac{2N^2\pi}{2N+1}} \qquad (25)
\end{aligned}
$$

which in turn can be written as the $(2N+1)$ equations (recall $x_{-k} = x_k^*$ if $x(t)$ is real-valued)

$$
x[n] = \sum_{k=-N}^{N} x_k e^{j\frac{2\pi}{2N+1}nk}, \quad n = 0, 1, 2 \ldots 2N \qquad (26)
$$

While these equations can give you a concussion, they also give you a system of $(2N+1)$ linear equations in $(2N+1)$ unknowns. If this system is nonsingular, we should be able to reconstruct the $(2N+1)$ Fourier coefficients $\{x_k, |k| \leq N\}$ from the $(2N+1)$ samples $\{x[n] = x(\frac{nT}{2N+1}), n = 0, 1 \ldots 2N\}$.

B. Sampling theorem for periodic signals

The significance of this linear system of equations is that we can compute the $(2N+1)$ Fourier coefficients $\{x_k, |k| \leq N\}$ from the $(2N+1)$ samples $\{x[n] = x(\frac{nT}{2N+1}), n = 0, 1 \ldots 2N\}$. That is, **we no longer need to compute the integral** $\frac{1}{T}\int_0^T \text{elvis}(t) e^{-j\frac{2\pi}{T}kt} dt$, even approximately–we have avoided it altogether!

Assume this system of $(2N+1)$ linear equations in $(2N+1)$ unknowns is nonsingular. We have:

THEOREM: Let $x(t)$ be periodic with period=T seconds and bandlimited to B Hertz, where B has been chosen so that $B = (N + \frac{1}{2})/T$ (we already know we can do this). Then $x(t)$ can be completely reconstructed from its samples $\{x(\frac{nT}{2N+1}), n = 0, 1 \ldots 2N\}$. Sampling $x(t)$ every $\frac{1}{2B}$ seconds (a sampling rate of $2B$ Hertz) is sufficient to reconstruct $x(t)$. Note T is irrelevant!

This is quite remarkable, in three different ways:

- Sampling a signal faster than twice its bandwidth (i.e., twice its maximum frequency) allows us to reconstruct the signal *perfectly* from its samples! (Recall from (8) that $2N+1 = 2BT \rightarrow \frac{T}{2N+1} = \frac{1}{2B}$.)
- The period of the signal is irrelevant; we can let the period T=1 century, if we wish!
- We can compute the Fourier coefficients x_k directly from the samples of $x(t)$–no integrals needed!

Claude Shannon actually derived this result for non-periodic signals using a different approach, which you will see in EECS 306. But the above argument works for a periodic signal of arbitrarily large period T.

This still leaves us with the problem of solving the system of $(2N+1)$ linear equations in $(2N+1)$ unknowns. In fact, we can not only solve it, but solve it in closed form. How? Read on…

VI. Review of quirks of discrete-time frequency

We now make a brief side trip to review quirks of discrete-time frequency. These can be summarized as:

Discrete-time frequency is itself periodic with period=2π: If you learn nothing else in EECS 206, learn this! Electric shocks will be given every 2π lectures until this sinks in. What it means is that there is no difference between a frequency of $\omega = \pi/4$ and $\omega = \pi/4 + 2\pi$ (or $\pi/4 + 4\pi$, etc.) This follows since

$$A\cos(\omega_o n + \theta) = A\cos([\omega_o + 2\pi]n + \theta) = A\cos([\omega_o + 4\pi]n + \theta) = \ldots \tag{27}$$

In particular, line spectra of discrete-time signals are periodic with period=2π.

The fastest possible discrete-time frequency is $\omega = \pi$: Note *fastest* is not the same as *largest*. In continuous time, the higher the frequency, the faster the oscillation. In discrete time, $\cos(\pi n) = (-1)^n$ is the fastest possible oscillation. Increasing frequency above π makes the oscillation *slow down*, until at frequency 2π it stops altogether (recall that $\omega = 2\pi$ is the same as $\omega = 0$). To see this, try increasing frequency to greater than π by an amount $\delta\omega$, and note that the oscillation slows down to frequency $(\pi - \delta\omega)$:

$$A\cos([\pi + \delta\omega]n + \theta) = A\cos([\pi - \delta\omega]n - \theta) \tag{28}$$

A frequency of $\omega = \pi + \delta\omega$ is equivalent to $\omega = -(\pi - \delta\omega)$: This is also a special case of discrete-time frequency being periodic with period 2π. But it will be useful below.

VII. Orthogonality and its significance

What's the big deal about orthogonality? Two big deals, actually:

- Orthogonality is why there are explicit formulae for computing the coefficients of Fourier series, in both continuous time and discrete time. Otherwise we wouldn't even have the integral formulae.
- If two signals are orthogonal, the average power of their sum is the sum of their average powers. This *only* works for orthogonal signals! It also is why Parseval's theorem exists.

The orthogonality equations (3) are used to derive the formulae (2) for the Fourier coefficients in (1). We will use similar equations to derive the DFT. Specifically,

$$\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}mk} e^{-j\frac{2\pi}{N}nk} = 0 \quad \text{unless} \quad m = n \tag{29}$$

since

$$\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}mk} e^{-j\frac{2\pi}{N}nk} = \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}(m-n)k} = \frac{e^{j\frac{2\pi}{N}(m-n)N} - 1}{e^{j\frac{2\pi}{N}(m-n)} - 1} = 0 \tag{30}$$

using the formulae

$$1 + r + r^2 + \ldots + r^{N-1} = \frac{r^N - 1}{r - 1} \quad \text{with} \quad r = e^{j\frac{2\pi}{N}(m-n)} \quad \text{and} \quad e^{j2\pi(m-n)} = 1 \tag{31}$$

If $m = n$ the previous equation gives $\frac{0}{0}$, but in that case we have simply $\sum_{k=0}^{N-1} 1 = N$. (Whenever you get an indeterminate form like $\frac{0}{0}$, go back earlier in the problem–usually the answer can be seen directly.)

As for average power, note that **if** $M[xy^*] = 0$, i.e., x and y are orthogonal, then

$$MS[x+y] = M[|x+y|^2] = M[(x+y)(x+y)^*] = M[xx^*]+M[yy^*]+M[xy^*]+M[yx^*] = MS[x]+MS[y] \tag{32}$$

Hence the average power of the sum of orthogonal signals is the sum of their average powers.

This leads directly to *Parseval's theorem*: we can compute the average power of $x(t)$ by summing the average powers of each term of its Fourier series, since these terms are all orthogonal to each other. Thus,

$$\frac{1}{T}\int_0^T |x(t)|^2\,dt = \sum |x_k|^2 = a_0^2 + \frac{1}{2}\sum(a_k^2 + b_k^2) = c_0^2 + \frac{1}{2}\sum c_k^2 \tag{33}$$

since the average power of $x_k e^{j\frac{2\pi}{T}kt}$ is $|x_k|^2$, while the average power of the sinusoid $c_k \cos(\frac{2\pi}{T}kt - \theta_k)$ is $\frac{1}{2}c_k^2$ (remember the rms value of a sinusoid is its amplitude/$\sqrt{2}$, and average power of a sinusoid=(rms)^2).

VIII. Discrete Fourier Transform (DFT)

A. *Derivation*

Now let's return to (26), the system of $(2N+1)$ linear equations in $(2N+1)$ unknowns

$$x[n] = \sum_{k=-N}^{N} x_k e^{j\frac{2\pi}{2N+1}nk}, \quad n = 0, 1, 2 \ldots 2N \tag{34}$$

Recall from the previous section that negative frequencies $-\pi < \omega < 0$ are equivalent to positive frequencies $\pi < \omega < 2\pi$, since frequency ω is periodic with period=2π. If we define

$$x_k = x^*_{2N+1-k} = x_{k-2N-1} \quad \text{for} \quad (N+1) \le k \le (2N) \tag{35}$$

which amounts to taking a periodic extension of $\{x_k\}$, then the linear system of equations becomes

$$x[n] = \sum_{k=0}^{2N} x_k e^{j\frac{2\pi}{2N+1}nk}, \quad n = 0, 1, 2 \ldots 2N \tag{36}$$

This follows since both $e^{j\frac{2\pi}{2N+1}nk}$ and the now-periodically-extended $\{x_k\}$ are periodic in k with period $(2N+1)$. So changing the range of summation from $[-N, N]$ to $[0, 2N]$ still sums the periodic summand over one complete period. We are merely summing the terms in a different order; if you believe addition is commutative, then the two sums are the same for all n.

Why does changing the index range of k matter? Because if we multiply this latter equation by $e^{-j\frac{2\pi}{2N+1}nk}$, sum from $n=0$ to $2N$, and use orthogonality equation (29) with N replaced with $(2N+1)$, we obtain

$$x_k = \frac{1}{2N+1}\sum_{n=0}^{2N} x[n]e^{-j\frac{2\pi}{2N+1}nk}, \quad k = 0,1,2\ldots 2N \tag{37}$$

This is done algebraically in the Official Lecture Notes, but it is easier to see if you don't get bogged down in the algebra. The sum on the right side becomes

$$0+0+\ldots+0+(2N+1)x_k+0+\ldots+0=(2N+1)x_k \tag{38}$$

from which (37) follows. Equation (37) is the $(2N+1)$-point DFT of $x[n]$. Compare it to the N-point DFT and the discretized integral formula from Section 2 (with N replaced with $(2N+1)$). The difference is that the DFT is an *exact* computation of the Fourier series coefficients x_k from *samples* $x[n]$ of $x(t)$.

B. Discrete-time Fourier series

We can also regard the DFT as an explicit computation of the Fourier coefficients of a periodic discrete-time signal $x[n]$, even if this signal did not come from sampling a continuous-time signal.

Let $x[n]$ be a periodic signal having period=N (note we have changed from period=2N+1 to period=N; Fourier coefficients from x_k to X_k). Then $x[n]$ can be expanded in either of the discrete-time Fourier series

$$\begin{aligned} x[n] = X_0 &+ X_1 e^{j\frac{2\pi}{N}n} + X_2 e^{j\frac{4\pi}{N}n} + \ldots + X_{(N-1)/2}e^{j\pi\frac{N-1}{N}n} \\ &+ X_1^* e^{-j\frac{2\pi}{N}n} + X_2^* e^{-j\frac{4\pi}{N}n} + \ldots + X_{(N-1)/2}^* e^{-j\pi\frac{N-1}{N}n} \\ x[n] = &\sum_{k=0}^{N-1} X_k e^{j\frac{2\pi}{N}nk}, \quad n=0,1,2\ldots(N-1) \end{aligned} \tag{39}$$

where the X_k are computed using the N-point DFT of one period of $x[n]$. For negative indices use

$$X_{-k} = X_{N-k} = X_k^* \tag{40}$$

The first Fourier series is more physical: we expand the periodic signal $x[n]$ into harmonics having frequencies $\omega = 2\pi\frac{k}{N}$, or $f=\frac{k}{N}$, for $|k| \le (N-1)/2$. These formulae are for odd N; if N is even, there is one more term $X_{N/2}e^{j\pi n}$ in the series. The fastest discrete-time frequency is $\omega=\pi$, so we don't need harmonics higher in frequency than $\omega=\pi$; they are equivalent to harmonics at lower frequencies.

The N-point DFT of a signal contained in a row vector X can be computed in Matlab by `fft(X,N)/N`.

- $X_0 = \frac{1}{N}(x[0]+x[1]+x[2]+x[3]+\ldots+x[N-1])$;
- $X_{N/2} = \frac{1}{N}(x[0]-x[1]+x[2]-x[3]+\ldots-x[N-1])$ if N is even;
- $X_{N-k} = X_k^*$ if $x[n]$ is real-valued.

The second half of the $\{X_k\}$ is the complex conjugate of the first half, in reverse order and *excluding* X_0.

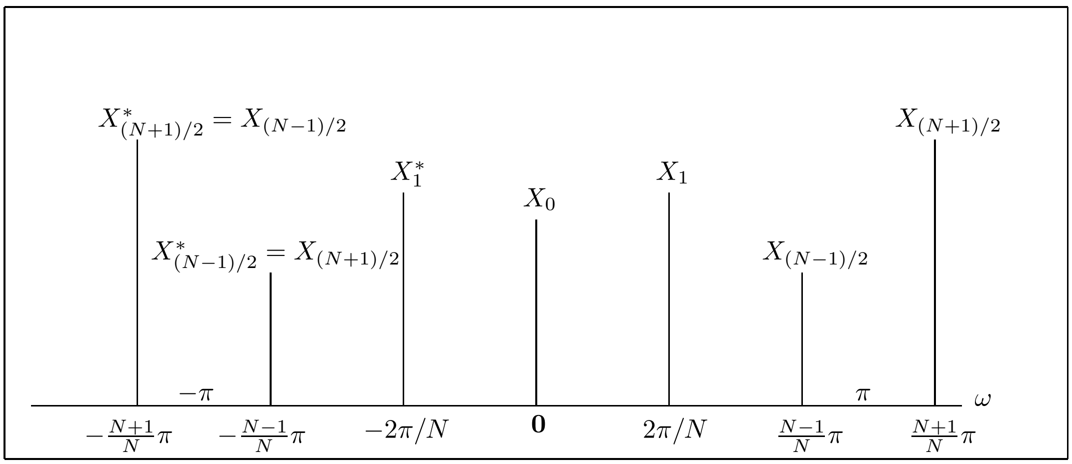
For example, the 4-point DFT of $\{24, 8, 12, 16\}$ is $\{15, 3+2j, 3, 3-2j\}$ (we computed this in Section 2).

- Use Matlab: `fft([24 8 12 16],4)/4`. Output: `15,3+2j,3,3-2j`;
- The 4^{th} element is conjugate of the 2^{nd} (exclude the 1^{st});
- The 1^{st} element (DC) is mean of the signal;
- The 3^{rd} element $X_{4/2} = \frac{1}{4}(24 - 8 + 12 - 16)$

Matlab indexing starts at 1, while DSP indexing starts at 0. This can drive you crazy if you let it!

C. Line spectra of discrete-time signals

It should always be remembered that line spectra of discrete-time signals are periodic with period=2π. For an N-point DFT, X_k is the line spectrum component at $\omega = \frac{2\pi}{N}k$. That is really the whole story.



IX. Use of DFT to compute line spectra

A. Example: From signal specs to DFT specs

Suppose we are given the following signal specs:

- **Duration:** elvis(t) is 4 minutes=240 seconds long
- **Period:** Use T=240 seconds (take periodic extension)
- **Bandwidth:** elvis(t) has bandwidth 4000 Hz (say)

From these signal specs, we can determine these DFT specs:

- **Sampling rate:** Need to sample at 2(4000)=8000 Hz (or faster)
- **Interval:** Need to sample every $\frac{1}{8000} = 125\mu sec$ (or more often)
- **Samples:** elvis[n]=elvis($t = n/8000$)
- **DFT length:** $2N + 1 = 2BT = 2(4000)(240) = 1,920,000$ (!)

In fact, we are usually only in a snippet of elvis(t) a few seconds long

Now compute the 1,920,000-point DFT

$$X_k = \frac{1}{1920000} \sum_{n=0}^{1919999} x[n] e^{-j\frac{2\pi nk}{1920000}}, \quad k = 0, 1, 2 \ldots 1919999 \tag{41}$$

and plot the line spectrum of elvis[n], which has component X_k at

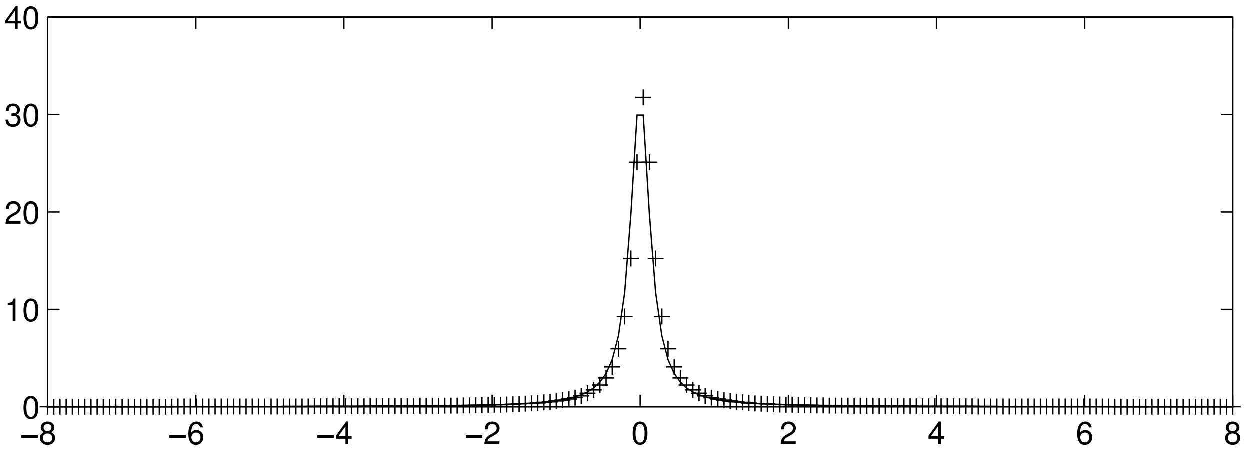
$$\omega = \frac{2\pi k}{1920000} \quad \text{for} \quad \text{elvis}[n]; \qquad f = \frac{8000k}{1920000} \quad \text{for} \quad \text{elvis}(t) \tag{42}$$

Using Matlab, `plot(abs(fftshift(fft(X,1920000))))`. `fftshift` swaps the uppper and lower halves of X_k, so the plot has negative frequencies to the left of positive frequencies. This is really just looking at a different period of the spectrum, which remember (one last time!) is periodic with period=2π.

B. Example: Spectrum of analytic signal

Consider the two-sided decaying exponential $x(t) = e^{-|t|}$, which decays rapidly to zero as $|t| \to \infty$. You will learn in EECS 306 that the actual continuous-frequency spectrum of $x(t)$ is $2/(\omega^2 + 1)$, i.e., the continuous Fourier transform of $e^{-|t|}$ is $2/(\omega^2 + 1)$. But we can compute that now, numerically, using the DFT.

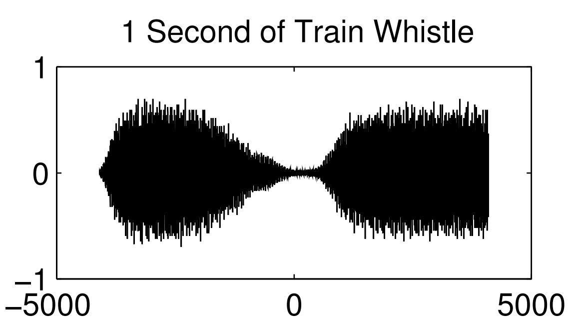
- **Duration:** $e^{-6} = 0.0025 \approx 0 \to$ support $x(t) \approx [-6, 6] \to$ duration = 12.
- **Period:** Use T=12 seconds (take periodic extension).
- **Bandwidth:** Assume (unknown) spectrum of $e^{-|t|}$ is bandlimited to 8 Hertz.
- In fact, 8 Hertz$\to \omega \approx 50 \to |\text{spectrum}| = 0.0008 \approx 0$. But we don't know this;
- **DFT length:** Use DFT order $2N + 1 = 2BT = 2(8)(12) = 192$-point DFT.
- 192 sample points in time support $[-6, 6] \to$ sample every $12/192 = 1/16$ second.
- **Sampling rate:** Note this is Nyquist sampling: $T_s = \Delta = 1/(2B) = 1/(2[8\,\text{Hertz}]) = 1/16$ second.
- 192 sample points in frequency support $[-8, 8] \to$ output line spectrum every $16/192 = 1/12$ Hertz.
- DFT output samples continuous spectrum of $x(t)$ every 1/12 Hertz.
- Also have to scale DFT output for EECS 306 reasons. Matlab:
- `T=linspace(-6,6,192);X=exp(-abs(T));XK=fftshift(abs(fft(X,192)));`
- `F=linspace(-8,8,192);XF=32./(4*pi^2*F.^2+1);plot(F,XF,F,XK,'+')`

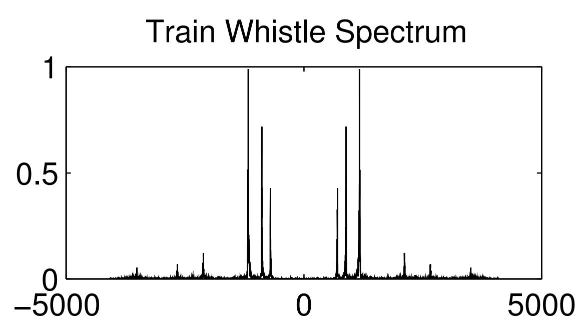


C. Example: Spectrum of real-world signal

The line spectrum of a train whistle can be computed as follows. Matlab includes the file `train.mat`, which is a train whistle sampled at 8192 Hertz. The signal was anti-alias filtered to remove all frequencies above 4096 Hertz before sampling, so there's no aliasing. Using a 1-second snippet of it, we have:

- **Duration:** Use a 1-second snippet of `train.mat` (8192 samples).
- **Period:** Use T=1 second (take periodic extension).
- **Bandwidth:** 4096 Hertz, assuming that 8192 Hz is the Nyquist sampling rate;
- **DFT length:** Use DFT order $2N+1 = 2BT = 2(4096)(1) = 8192$-point DFT.
- **Sampling rate:** Nyquist sampling: $T_s = \Delta = 1/(2B) = 1/(2[4096\,\text{Hertz}]) = 1/8192$ second.
- 8192 sample points in frequency support $[-4096, 4096] \rightarrow$ output line spectrum every 1 Hertz.
- `load train;X=y(1:8192);plot(-4095:4096,fftshift(abs(fft(X))))`





It sure beats computing $x_k = \frac{1}{T}\int_0^T \text{elvis}(t)e^{-j\frac{2\pi}{T}kt}dt$!