

# Absorbing Set Spectrum Approach for Practical Code Design

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**Abstract**—This paper focuses on controlling the absorbing set spectrum for a class of regular LDPC codes known as separable, circulant-based (SCB) codes. For a specified circulant matrix, SCB codes all share a common mother matrix, examples of which are array-based LDPC codes and many common quasi-cyclic codes. SCB codes retain the standard properties of quasi-cyclic LDPC codes such as girth, code structure, and compatibility with efficient decoder implementations. In this paper, we define a cycle consistency matrix (CCM) for each absorbing set of interest in an SCB LDPC code. For an absorbing set to be present in an SCB LDPC code, the associated CCM must not be full column-rank. Our approach selects rows and columns from the SCB mother matrix to systematically eliminate dominant absorbing sets by forcing the associated CCMs to be full column-rank. We use the CCM approach to select rows from the SCB mother matrix to design SCB codes of column weight 5 that avoid all low-weight absorbing sets (4, 8), (5, 9), and (6, 8). Simulation results demonstrate that the newly designed code has a steeper error-floor slope and provides at least one order of magnitude of improvement in the low error rate region as compared to an elementary array-based code.

## I. INTRODUCTION

It is well known that finite-length low-density parity-check (LDPC) codes suffer performance degradation in the high signal-to-noise ratio (SNR)/low frame error rate (FER) region. This degradation is commonly referred to as the error floor. Prior work indicated that certain sub-graphs called trapping sets [1], and, in particular, a subset of trapping sets called absorbing sets [2] are a primary cause of the error floor. Absorbing set is a particular type of a trapping set that is stable under bit-flipping decoding. This paper improves the performance by controlling absorbing sets for a class of regular LDPC codes, known as separable circulant-based (SCB) codes, that are constructed as an arrangement of circulant matrices.

Recent papers have proposed methods to lower the error floor by improving the absorbing set (or trapping set) spectrum. For example, small trapping sets can be avoided by introducing additional check nodes [3], or by increasing the girth [4]. The algorithm in [5] constructs quasi-cyclic codes from Latin squares so that the Tanner graph of the code does not contain certain trapping sets. Recent results [6], [7], [8] have also investigated the errors floor of certain practical codes in terms of their trapping/absorbing sets.

A recently proposed approach [9], [10] avoids certain dominant absorbing sets without compromising code properties by carefully selecting the rows/columns of the (SCB) mother matrix. This paper builds upon the cycle consistency matrix

(CCM) approach [11] to analyze SCB codes with column weight 5, i.e., codes with five rows of circulant submatrices.

For an absorbing set to be present in an SCB LDPC code, the associated CCM must not be full column-rank. Furthermore, analysis of the variable-node graph for a variety of absorbing sets reveals that the existence of certain absorbing sets is a necessary condition for other (larger) absorbing sets to exist. Using these two observations, this paper analyzes the smallest absorbing sets in  $r=5$  SCB codes, and systematically avoids these absorbing sets by selecting rows from the SCB mother matrix that force the associated CCMs to be full column-rank. FPGA simulation results confirm that the new codes have significantly steeper error-floor slopes.

As a representative instance of SCB codes, Tanner-construction codes [12] with moderate rates are analyzed. It is shown that these codes can have improved absorbing set spectra by carefully avoiding the smallest absorbing sets with suitable code parameter selection.

Section II describes separable circulant-based (SCB) codes and the cycle consistency matrix (CCM). Section III applies the CCM approach to analyze three groups of dominant absorbing sets in an example family of SCB codes. Section III then selects specific rows from the SCB mother matrix to eliminate all the dominant absorbing sets by forcing the associated CCMs to be full column-rank. Section III also analyzes the absorbing set spectrum of Tanner-construction codes and provides several good row-selection functions. Section IV provides hardware simulation results demonstrating the performance improvement obtained with the new codes. Section V delivers the conclusions.

## II. DEFINITION AND PRELIMINARIES

This section describes separable, circulant-based (SCB) codes and the cycle consistency matrix (CCM) associated with absorbing sets in SCB codes.

### A. Circulant-based LDPC codes

Circulant-based LDPC codes are a family of structured regular  $(r, c)$  codes where  $r$  is the variable-node degree and  $c$  is the check-node degree. They are constructed as  $r$  rows and  $c$  columns of circulant matrices. They are especially amenable to high-throughput hardware implementations [13].

The parity-check matrix of circulant-based LDPC codes has

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the following general structure:

$$H_{p,f}^{r,c} = \begin{bmatrix} \sigma^{f(0,0)} & \sigma^{f(0,1)} & \sigma^{f(0,2)} & \dots & \sigma^{f(0,c-1)} \\ \sigma^{f(1,0)} & \sigma^{f(1,1)} & \sigma^{f(1,2)} & \dots & \sigma^{f(1,c-1)} \\ \sigma^{f(2,0)} & \sigma^{f(2,1)} & \sigma^{f(2,2)} & \dots & \sigma^{f(2,c-1)} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \sigma^{f(r-1,0)} & \sigma^{f(r-1,1)} & \sigma^{f(r-1,2)} & \dots & \sigma^{f(r-1,c-1)} \end{bmatrix},$$

where  $\sigma$  is a  $p \times p$  circulant matrix.

A column (row) group is a column (row) of circulant matrices. Each variable node has a label  $(j, k)$  with  $j \in \{0, \dots, c-1\}$  being the index of the corresponding column group and  $k \in \{0, \dots, p-1\}$  identifying the specific column within the group. Similarly, each check node has a label  $(i, l)$  where  $i \in \{0, \dots, r-1\}$  and  $l \in \{0, \dots, p-1\}$ .

Circulant-based LDPC codes include, for example, the constructions in [12], [14] and [15]. The girth can be guaranteed to be at least 6 by placing a constraint on the values of the submatrix exponent  $f(i, j)$  [2].

This paper focuses on separable, circulant-based (SCB) codes defined as follows:

*Definition 1 (Separable, Circulant-Based (SCB) Code):*

An SCB code is a circulant-based LDPC code with a parity-check matrix  $H_{p,f}^{r,c}$  in which  $f(i, j)$  is separable, i.e.,  $f(i, j) = a(i) \cdot b(j)$ . ■

Parity check matrices of SCB codes with the specified circulant matrix can be viewed as originating from a common SCB mother matrix  $H_{p,f_m}^{p,p}$  with  $f_m(i, j) = i \cdot j$ . The functions  $a(i)$  and  $b(j)$  effectively specify which rows and columns of the mother matrix are selected for the resultant SCB matrix. The ranges of  $a(i)$  and  $b(j)$  are  $\{0, \dots, p-1\}$ .

The SCB structure imposes four conditions [2] on the variable and check nodes: (1) bit consistency, (2) check consistency, (3) pattern consistency, and (4) cycle consistency. These conditions are essential to the CCM approach that is introduced next.

### B. Absorbing sets and the Cycle Consistency Matrix

An LDPC code with parity-check matrix  $H$  is often viewed as a bipartite (Tanner) graph  $G_H = (V, F, E)$ , where the set  $V$  represents the variable nodes, the set  $F$  represents the check nodes, and  $E$  corresponds to the edges between variable and check nodes.

For a variable node subset  $V_{as} \subset V$ , let  $G_{as} = (V_{as}, F_{as}, E_{as})$  be the bipartite graph of the edges  $E_{as}$  between the variable nodes  $V_{as}$  and their neighboring check nodes  $F_{as}$ . Let  $o(V_{as}) \subset F_{as}$  be the neighbors of  $V_{as}$  with odd degree (unsatisfied check nodes) in  $G_{as}$  and  $e(V_{as}) \subset F_{as}$  be the neighbors of  $V_{as}$  with even degree in  $G_{as}$  (satisfied check nodes). An  $(a, b)$  absorbing set [2]  $G_{as} = (V_{as}, F_{as}, E_{as})$  is a Tanner graph with  $a$  variable nodes,  $b$  odd-degree check nodes, and with each variable node having strictly fewer odd-degree neighbors than even-degree neighbors.

Suppose there are  $n$  variable nodes in the absorbing set. Let  $j_1, \dots, j_n$  be the column-group labels of these  $n$  nodes in the SCB mother matrix. Define  $u_m = j_m - j_1, m = 2, \dots, n$  and  $\mathbf{u} = [u_2, \dots, u_n]$ . For each cycle in the absorbing set, by

replacing the difference of  $j$ 's with the difference of  $u$ 's, we can rewrite the cycle consistency equation as

$$\sum_{m=2}^t (i_{m-1} - i_m) u_m = 0 \pmod{p}, \quad (1)$$

where  $2t$  is the cycle length. Note that  $i_m$  will be different for different cycles reflecting the particular cycle trajectories.

Every cycle in the absorbing set satisfies an equation of the form (1). Taken together, these equations produce a matrix equation:  $\mathbf{M}\mathbf{u} = 0 \pmod{p}$ , where  $\mathbf{M}_{y,m}$  is the coefficient of  $u_m$  in (1) for the  $y$ th cycle.

A key property of  $\mathbf{M}$  is that  $\mathbf{M}\mathbf{u} = 0 \pmod{p}$  completely characterizes the requirement that every cycle in  $G_{as}$  satisfies (1). Even so, it is not necessary for  $\mathbf{M}$  to include a row for every cycle in the absorbing set.

A cycle need not be included in  $\mathbf{M}$  if it is a linear combination of cycles already included in  $\mathbf{M}$ . Thus the number of rows needed in  $\mathbf{M}$  is the number of linearly independent cycles in  $G_{as}$ . Two definitions [16] from graph theory are necessary to establish the number of linearly independent cycles in  $G_{as}$  and hence how many rows are needed for  $\mathbf{M}$ .

*Definition 2 (Incidence Matrix):* For a graph with  $n$  vertices and  $q$  edges, the (unoriented) incidence matrix is an  $n \times q$  matrix  $B$  with  $B_{ij} = 1$  if vertex  $v_i$  and edge  $x_j$  are incident and 0 otherwise. ■

*Definition 3 (Binary Cycle Space):* The binary cycle space of a graph is the null space of its incidence matrix over  $GF(2)$ . ■

Any absorbing-set bipartite graph  $G_{as}$  can be transformed into a graph whose only vertices are  $V_{as}$  and where two vertices are connected iff there is a check node that connects them. We call this graph the *variable-node (VN) graph* of the absorbing set. The incidence matrix of the VN graph provides a characterization of all the cycles in an absorbing set. The number of linearly independent cycles in an absorbing set, which is the dimension of its binary cycle space, is the size of the null space of the incidence matrix  $B_{as}$ :  $D_{bcs} = q - \text{rank}(B_{as})$ .

Having established the number of rows in  $\mathbf{M}$ , it can be formally defined as the Cycle Consistency Matrix:

*Definition 4 (Cycle Consistency Matrix):* The cycle consistency matrix  $\mathbf{M}$  of an absorbing-set graph  $G_{as}$  has  $|V_{as}| - 1$  columns and  $D_{bcs}$  rows. The rows of  $\mathbf{M}$  correspond to  $D_{bcs}$  linearly independent cycles in  $G_{as}$ . Each row has the coefficients of  $\mathbf{u}$  in (1) for each of these linearly independent cycles. ■

Note that  $\mathbf{M} \cdot \mathbf{u} = 0 \pmod{p}$  completely characterizes the requirement that every cycle in  $G_{as}$  satisfies (1).

The vector  $\mathbf{u}$  cannot be an all-zero vector because an all-zero  $\mathbf{u}$  indicates that all variable nodes have the same column group. This violates the check consistency condition, which requires that variable nodes sharing a check node have distinct column groups. Thus  $\mathbf{u} \neq \mathbf{0}$ , and a necessary condition for the existence of a given absorbing set is that its  $\mathbf{M}$  does not have full column-rank in  $GF(p)$ .

If the VN graph of the absorbing set  $G_{as}^A$  is a sub-graph of the VN graph of another absorbing set  $G_{as}^B$  with the same

number of variable nodes, then we say the VN graph of the absorbing set  $G_{as}^A$  is extensible.

**Theorem 1:** Given a proposed absorbing set graph  $G_{as} = (V_{as}, F_{as}, E_{as})$ , where every variable node is involved in at least one cycle<sup>1</sup>, specified column group labels of the variable nodes in  $V_{as}$  in the SCB mother matrix, and specified row-group labels of the check nodes in  $F_{as}$  in the SCB mother matrix, the following are necessary conditions for the proposed absorbing set to exist in each daughter SCB LDPC code (with a parity check matrix  $H$  that includes the specified row and column groups of that SCB mother matrix):

- 1) The CCM for  $G_{as}$  does not have full column-rank;
- 2) Variable nodes in  $V_{as}$  satisfy the Bit Consistency condition and can form a difference vector  $\mathbf{u}$  in the null space of the CCM; and
- 3) Each check node in  $F_{as}$  satisfies the Check Consistency condition.

Taken together, these conditions are also sufficient if the VN graph of this absorbing set is not extensible.

*Proof:* The proof is provided in [17]. ■

**Corollary 1:** Given an  $(a_1, b_1)$  absorbing set graph  $G_{as}^1 = (V_{as}^1, F_{as}^1, E_{as}^1)$  and an  $(a_2, b_2)$  absorbing set graph  $G_{as}^2 = (V_{as}^2, F_{as}^2, E_{as}^2)$ , if  $a_1 \leq a_2$  and the VN graph of  $G_{as}^1$  is a sub-graph of the VN graph of  $G_{as}^2$ , then the existence of  $G_{as}^1$  is a necessary condition of the existence of  $G_{as}^2$ .

*Proof:* [17] provides the proof. ■

### III. ILLUSTRATIVE CASE STUDY WITH $r = 5$

The approach in [9] reveals that a careful selection of  $r$  row-groups from the SCB mother matrix can eliminate certain small absorbing sets to improve the error floor. This section provides an example with  $r = 5$  (five row groups) that shows how to use the new CCM approach to efficiently improve an SCB code by analytically avoiding low-weight absorbing sets with careful row selections.

Our example of SCB code design involves three classes of SCB codes: (1) Array-based codes [18]: the most elementary SCB codes in which the first  $r$  rows of the SCB mother matrix  $H_{p,f}^{p,p}, f(i,j) = i \cdot j$  comprise the parity-check matrix. We will refer to this class as the elementary array-based (EAB) codes. (2) Selected-row (SR) SCB codes: the parity-check matrix for these codes is  $H_{p,f}^{r,p}, f(i,j) = a(i) \cdot j$  where  $a(i)$  is called the row-selection function (RSF). (3) Shortened SR (SSR) SCB codes: the parity-check matrix for these codes is  $H_{p,f}^{r,c}, f(i,j) = a(i) \cdot b(j)$  where  $b(j)$  is called the column-selection function (CSF).

Theorem 1 shows that an absorbing set may be avoided either by forcing the associated CCM to be full column rank or by precluding  $\mathbf{u}$  from being in the null space of  $\mathbf{M}$ . Corollary 1 shows that if an absorbing set does not exist, then all absorbing sets whose VN graphs contain the VN graph of this absorbing set also do not exist. The CCM approach carefully selects the

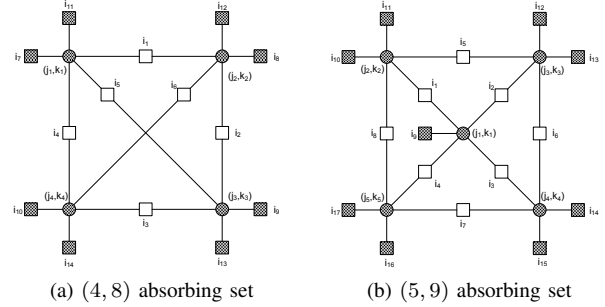


Fig. 1: Depiction of absorbing sets.

RSF and CSF to systematically eliminate small absorbing sets, in the order of the size of the VN graph of the absorbing sets.

Prior results [19] prove (4, 8) absorbing sets to be the smallest possible for a general  $r = 5$  SCB code family and to dominate the low BER region [13]. Hardware simulation results in [9] show that the next two dominant absorbing sets in  $r = 5$  EAB codes or SR SCB codes are (5, 9) and (6, 8).

Careful selection of the RSF via the CCM approach successfully avoids all (4, 8), (5, 9), and (6, 8) absorbing sets. Sections III-A, III-B, and III-C, respectively, show that (4, 8), (5, 9), and (6, 8) absorbing sets exist in the EAB SCB code with  $r = 5$  but also that SR SCB codes can systematically eliminate these configurations with a carefully selected RSF. Section III-D provides example RSFs that eliminate small absorbing sets with  $r = 5$ . Section III-E explores the absorbing set spectrum of the existing quasi-cyclic LDPC codes with the Tanner construction [12].

#### A. (4, 8) absorbing sets

A result in [19] shows that (4, 8) absorbing set in Fig. 1a is the smallest absorbing set in the  $H_{p,f(i,j)}^{5,p}$  code family, and [9] shows that certain RSFs eliminate the (4, 8) absorbing sets in SR SCB codes. This section uses the CCM approach to efficiently establish necessary and sufficient conditions for the existence of (4, 8) absorbing sets.

**Lemma 1:**  $\det \mathbf{M} = 0 \pmod p$  is a necessary and sufficient condition for the existence of (4, 8) absorbing sets in Fig. 1a.

*Proof:* [17] provides the proof. ■

Corollaries 2 and 3 restate Lemmas 1 and 2 of [9]. The CCM approach concisely proves these corollaries in [17].

**Corollary 2:** (4, 8) absorbing sets exist in EAB codes described by the parity check matrix  $H_{p,i \cdot j}^{5,p}$ , and their number scales as  $O(p^3)$ .

**Corollary 3:** There are no (4, 8) absorbing sets in the SR SCB codes described by the parity check matrix  $H_{p,a(i) \cdot j}^{5,p}$ , for prime  $p$  large enough with a proper choice of RSF.

**Remark 1:** The SR SCB codes avoid (4, 8) absorbing sets by carefully choosing RSF such that  $\det \mathbf{M} \neq 0 \pmod p$  for  $p$  large enough. One such RSF is  $[0, 1, 2, 4, 6]$ , and the resulting SR SCB codes avoids (4, 8) absorbing sets for prime  $p > 23$ .

#### B. (5, 9) absorbing sets

Assuming an RSF that avoids the (4, 8) absorbing sets, this section proves that the (5, 9) absorbing sets are the smallest

<sup>1</sup> If the variable-node degree is at least 2, then each variable node in a given absorbing set must be a part of at least one cycle.

remaining. The CCM approach shows that the (5, 9) absorbing sets always exist in the EAB SCB codes, but are avoided for SR SCB codes by some of the RCFs that precluded the (4, 8) absorbing sets.

*Lemma 2:* Assuming that (4, 8) absorbing sets do not exist, (5,  $b$ ) absorbing sets also do not exist for  $b < 9$ .

*Corollary 4:* Assuming that (4, 8) absorbing sets do not exist, the (5, 9) absorbing set is the smallest absorbing set.

*Lemma 3:*  $\det \mathbf{M} = 0 \pmod p$  is necessary and sufficient for the existence of (5, 9) absorbing sets in Fig. 1b.

*Proof:* The proof of these statements is provided in the extended version [17]. ■

Enumeration result in [17] shows that the number of (5, 9) absorbing sets in the EAB codes scales as  $O(p^3)$ . However, with a proper choice of RSF,  $\det \mathbf{M} \neq 0 \pmod p$  for  $p$  large enough. Therefore we can conclude with the following two corollaries.

*Corollary 5:* (5, 9) absorbing sets exist in EAB codes described by the parity check matrix  $H_{p,i,j}^{5,p}$ , and their number scales as  $O(p^3)$ .

*Corollary 6:* There are no (5, 9) absorbing sets in the SR SCB codes described by the parity check matrix  $H_{p,a(i),j}^{5,p}$ , for prime  $p$  large enough with a proper choice of RSF.

*Proof:* For SR SCB codes, it is sufficient to select RSF such that (7) in [17] does not evaluate to zero. One such example is  $[0, 1, 2, 4, 7]$ , where it is sufficient for the prime  $p > 89$  and not be in the set  $\{101, 103, 131, 179\}$ . Therefore in the SR SCB code there is no (5, 9) absorbing set if  $p$  is sufficiently large and an appropriate RCF is chosen. ■

### C. (6, 8) absorbing sets

This section considers the (6, 8) absorbing sets, which are the smallest remaining after the (4, 8) and (5, 9) absorbing sets. We will investigate the (6, 8) absorbing sets both for EAB codes and for SR-SCB codes that preclude the (4, 8) and (5, 9) absorbing sets. For the six candidate configurations of (6, 8) absorbing sets, all satisfied checks have degree 2 and all unsatisfied checks have degree 1 in the absorbing set graph. Combinatorial and consistency arguments show that four of these six configurations are not present for  $p$  sufficiently large in either the EAB code or in SR-SCB codes that preclude the (4, 8) and (5, 9) absorbing sets. The remaining two configurations have the cardinality  $O(p^3)$  in the EAB code. However, both of these configurations contain a (4, 8) absorbing set as a subset and thus cannot be present in SR-SCB codes that preclude the (4, 8) absorbing sets.

#### 1) (6, 8) configuration 1 - check nodes with degree > 2:

*Lemma 4:* For  $H_{p,i,j}^{5,p}$  and  $p$  sufficiently large, there are no (6, 8) absorbing sets for which a check node connects to more than two variable nodes in the absorbing set graph.

*Proof:* Again, [17] provides the proof. ■

Therefore attention is restricted to the case where all check nodes in the absorbing set graph have degree at most 2.

In a candidate (6, 8) absorbing set, variable nodes can have 3, 4 or 5 satisfied checks. By the girth constraint, there can be at most 2 variable nodes with 5 satisfied checks. Suppose

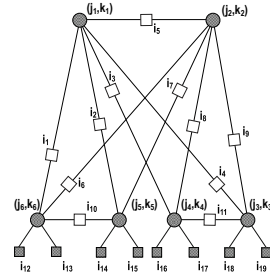


Fig. 2: (6, 8) configuration candidate 2.

there are two such variable nodes. Since there are a total of 8 unsatisfied checks, the other 4 variable nodes must each have 3 satisfied and 2 unsatisfied checks. This necessarily implies the configuration shown in Fig. 2 which we discuss next.

2) (6, 8) configuration candidate 2 - Fig. 2: Since the configuration in Fig. 2 has two overlapping (4, 8) absorbing sets, the existence of this (6, 8) absorbing set relies on both CCM determinants of these two (4, 8) absorbing sets that share an edge. Therefore with a proof similar to that of Lemma 1, we can show the following result:

*Lemma 5:* The necessary and sufficient condition for the existence of (6, 8) absorbing sets in Fig. 2 is  $\det \mathbf{M}_1 = 0 \pmod p$  and  $\det \mathbf{M}_2 = 0 \pmod p$ , where  $\mathbf{M}_1$  and  $\mathbf{M}_2$  are CCMs of the two internal (4, 8) absorbing sets.

We can also prove that there are  $p^2(p - 1)$  such absorbing sets in the EAB codes with any prime  $p$ . The proof is shown in [17].

Similarly, the following corollaries are consequences of Corollary 2 and Corollary 3.

*Corollary 7:* The (6, 8) absorbing sets with the configuration shown in Fig. 2 exist in EAB codes described by the parity check matrix  $H_{p,i,j}^{5,p}$ , and their number scales as  $O(p^3)$ .

*Corollary 8:* There are no (6, 8) absorbing sets with the configuration shown in Fig. 2 in the SR SCB codes described by the parity check matrix  $H_{p,a(i),j}^{5,p}$ , for prime  $p$  large enough with proper choice of RSF.

The analogous analysis of the remaining four (6, 8) configurations is in [17].

### D. Non-existence of (4, 8), (5, 9) or (6, 8) absorbing sets in SR SCB codes with a well-chosen RSF

The following is a consequence of Lemma 1 to Lemma 10, and Corollary 3 to Corollary 11 in [17].

*Theorem 2:* In the EAB SCB code the number of (4, 8), (5, 9) and (6, 8) absorbing sets scales as  $O(p^3)$  whereas in an SR SCB code with a well-chosen RSF there are no (4, 8), (5, 9) or (6, 8) absorbing sets for sufficiently large  $p$ .

*Remark 2:* For small  $p$ 's, (4, 8), (5, 9) and (6, 8) absorbing sets cannot be eliminated simultaneously for the SR SCB codes. Since there are 3 equivalence classes of SR codes [10], we only need to consider the RSF that contains 0 and 1. Here are a few good RSFs for  $p > 61$  that eliminate all (4, 8), (5, 9) and (6, 8) absorbing sets: (1)  $[0, 1, 2, 4, 17]$  for  $p = 67$ , (2)  $[0, 1, 2, 3, 11]$  for  $p = 73$ , (3)  $[0, 1, 2, 6, 7]$  for  $p = 79$ , (4)  $[0, 1, 2, 3, 7]$  for  $p = 83, 97, 101, 103, 107, 109, 113, 127$ , (5)  $[0, 1, 2, 4, 11]$  for  $p = 89$ , and (6)  $[0, 1, 2, 4, 7]$  for  $p > 179$ .

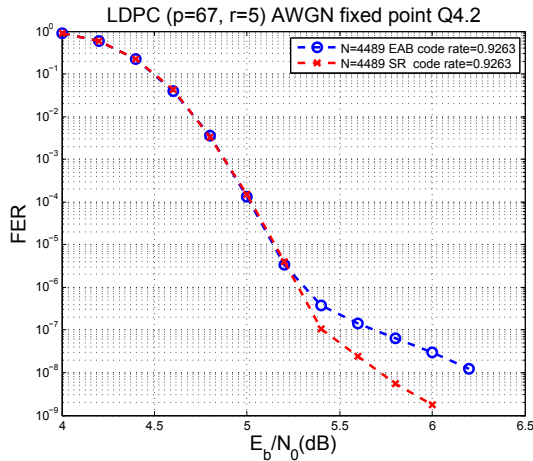


Fig. 3: Performance comparison of the (4489, 4158) EAB SCB and SR SCB LDPC codes.

### E. Absorbing set spectrum in the Tanner construction

We can easily extend our analysis to the Tanner construction in [12], which is an example of SSR SCB codes.

*Lemma 6:* In the Tanner graph corresponding to quasi-cyclic LDPC  $H_{p,\bar{f}(i,j)}^{5,p}$  in [12], no (4, 8) or (6, 8) absorbing set exists with parameters selected in Table I of [12].

*Proof:* The proof is presented in [17]. ■

*Remark 3:* This code has a good absorbing set spectrum and thus will have low error floor as expected. Moreover, the analysis can be easily extended to the quasi-cyclic codes constructed in [14], which have the first sub-row and first sub-column as identity matrices.

Note that the codes listed in Table I of [12] are mostly moderate-rate codes. However, for higher rates, the Tanner construction may introduce smaller absorbing sets [11]. These absorbing sets can also be avoided by a carefully chosen CSF that precludes the  $\mathbf{u}$  from the null space of  $\mathbf{M}$ .

## IV. RESULTS

In this section we experimentally demonstrate performance improvement with well-designed SR codes. In simulations, we use 200 iterations and a  $Q4.2$  fixed-point quantization with 4, resp. 2, bits to represent integer, resp. fractional, values. We simulate sum-product algorithm [13] on an FPGA platform.

We simulated a pair of longer block length codes contrasted in Fig. 3. The performance improvement of the SR-SCB code is due to the elimination of the (4, 8), (5, 9) and (6, 8) absorbing sets with a proper choice of the RSF. (Here the SR-SCB code's RSF is  $[0, 1, 2, 4, 17]$ .) We observe that in the error profile as shown in Table I the (4, 8), (5, 9) and (6, 8) absorbing sets are completely eliminated in the SR SCB code.

Row selection alone cannot avoid the next smallest absorbing sets, which are the (6, 10). Each of these (6, 10) absorbing sets exists if and only if the corresponding subset (6, 4) absorbing set studied in [10] exists. As shown in [10], these (6, 4) absorbing sets cannot be eliminated only with row selection. However, [10] also shows that these absorbing sets can be precluded by column selection.

TABLE I: Error profiles for the EAB SCB (4489, 4158), code (top), and the SR SCB code (bottom). n.e. is the number of collected errors.

SNR	n.e.	(4,8)	(5,9)	(6,8)	(6,10)	(7,9)	(7,11)	(8,6)	(8,8)	(8,10)
5.6dB	150	67	17	22	7	6	5	6	6	3
5.8dB	139	83	18	16	6	5	1	3	3	1
6.0dB	131	77	18	22	5	1	1	2	1	1
6.2dB	107	85	10	5	4	2	0	0	0	0

SNR	n.e.	(4,8)	(5,9)	(6,8)	(6,10)	(7,9)	(7,11)	(8,6)	(8,8)	(8,10)
5.6dB	106	0	0	0	25	15	6	15	13	6
5.8dB	140	0	0	0	35	29	14	16	6	8
6.0dB	60	0	0	0	25	7	5	9	5	3

## V. CONCLUSION

This paper presents a detailed analysis of the absorbing set spectrum of a class of LDPC codes based on circulant matrices. Using the cycle consistency matrix description of the dominant absorbing sets we characterized code performance and provided tools for a systematic code design. Simulation results in low FER region support the proposed methodology.

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