1 Derivation of Band-Limited Expressions

Given a shader function $f(x)$ and a band-limiting kernel $k(x, w)$ where w describes the width of the kernel, we wish to determine the convolution function

$$\hat{f}(x, w) = \int_{-\infty}^{\infty} f(x')k(x - x', w)\, dx', \tag{1}$$

or equivalently,

$$\hat{f}(x, w) = \int_{-\infty}^{\infty} f(x - x')k(x', w)\, dx'. \tag{2}$$

Except as noted below, we will use a normalized Gaussian kernel with standard deviation w as our band-limiting kernel:

$$k(x, w) = \frac{1}{w\sqrt{2\pi}} e^{-\frac{x^2}{2w^2}}. \tag{3}$$

1.1 Identity Function

Let $f(x) = x$. Then,

$$\begin{aligned}
\hat{f}(x, w) &= \frac{1}{w\sqrt{2\pi}} \int_{-\infty}^{\infty} (x - x')e^{-\frac{x'^2}{2w^2}}\, dx' \\
&= \frac{x}{w\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x'^2}{2w^2}}\, dx' - \frac{1}{w\sqrt{2\pi}} \int_{-\infty}^{\infty} x' e^{-\frac{x'^2}{2w^2}}\, dx'
\end{aligned}$$

Note that the second integral evaluates to 0, since the integral from $-\infty$ to 0 has the same magnitude but opposite sign as the integral from 0 to ∞. Thus, we are left with

$$\begin{aligned}
\hat{f}(x, w) &= \frac{x}{w\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x'^2}{2w^2}}\, dx' \\
&= \frac{x}{w\sqrt{2\pi}} w\sqrt{2\pi} \\
&= x
\end{aligned}$$

1.2 Absolute Value

Let $f(x) = |x|$. Note that

$$|x - x'| = \begin{cases} x - x' & \text{if } x' < x \\ x' - x & \text{otherwise.} \end{cases}$$

This allows us to partition the integral as follows,

$$\begin{aligned}
\hat{f}(x,w) &= \frac{1}{w\sqrt{2\pi}} \int_{-\infty}^{\infty} |x-x'| e^{-\frac{x'^2}{2w^2}}\, dx' \\
&= \frac{1}{w\sqrt{2\pi}} \int_{-\infty}^{x} (x-x') e^{-\frac{x'^2}{2w^2}}\, dx' + \frac{1}{w\sqrt{2\pi}} \int_{x}^{\infty} (x'-x) e^{-\frac{x'^2}{2w^2}}\, dx' \\
&= \frac{x}{w\sqrt{2\pi}} \left(\int_{-\infty}^{x} e^{-\frac{x'^2}{2w^2}}\, dx' - \int_{x}^{\infty} e^{-\frac{x'^2}{2w^2}}\, dx' \right) \\
&\quad + \frac{1}{w\sqrt{2\pi}} \left(\int_{x}^{\infty} x' e^{-\frac{x'^2}{2w^2}}\, dx' - \int_{-\infty}^{x} x' e^{-\frac{x'^2}{2w^2}}\, dx' \right) \\
&= \frac{x}{w\sqrt{2\pi}} \left(w\sqrt{\frac{\pi}{2}} \left(\left(1 + \operatorname{erf} \frac{x}{w\sqrt{2}} \right) - \left(1 - \operatorname{erf} \frac{x}{w\sqrt{2}} \right) \right) \right) \\
&\quad + \frac{1}{w\sqrt{2\pi}} \left(w^2 e^{-\frac{x^2}{2w^2}} + w^2 e^{-\frac{x^2}{2w^2}} \right) \\
&= \frac{x}{w\sqrt{2\pi}} \left(\frac{w\sqrt{2\pi}}{2} 2 \operatorname{erf} \frac{x}{w\sqrt{2}} \right) + \frac{\sqrt{2}}{2w\sqrt{\pi}} 2w^2 e^{-\frac{x^2}{2w^2}} \\
&= x \operatorname{erf} \frac{x}{w\sqrt{2}} + w\sqrt{\frac{2}{\pi}}\, e^{-\frac{x^2}{2w^2}},
\end{aligned}$$

where erf is the Gauss error function.

1.3 ceiling

Note that when x is not an integer, $\lceil x \rceil = \lfloor x \rfloor + 1$. The case when x is an integer does not change the value of the integral, since its support has measure zero. We can therefore treat the ceiling function in terms of the floor function. Thus,

$$\widehat{ceil}(x,w) = \widehat{floor}(x,w) + 1$$

1.4 cos

Let $f(x) = \cos x$. Then,

$$\hat{f}(x,w) = \frac{1}{w\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos(x-x') e^{-\frac{x'^2}{2w^2}}\, dx'.$$

Using the identity

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta,$$

we substitute to get,

$$\begin{aligned}
\hat{f}(x,w) &= \frac{1}{w\sqrt{2\pi}} \int_{-\infty}^{\infty} (\cos x \cos x' + \sin x \sin x') e^{-\frac{x'^2}{2w^2}}\, dx' \\
&= \frac{1}{w\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos x \cos x' e^{-\frac{x'^2}{2w^2}}\, dx' + \frac{1}{w\sqrt{2\pi}} \int_{-\infty}^{\infty} \sin x \sin x' e^{-\frac{x'^2}{2w^2}}\, dx'
\end{aligned}$$

Note that the second integral evaluates to 0, since the portion from $-\infty$ to 0 has the same magnitude but opposite sign as the portion from 0 to ∞. Thus, we are left with

$$\begin{aligned}
\hat{f}(x,w) &= \frac{1}{w\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos x \cos x' e^{-\frac{x'^2}{2w^2}}\, dx' \\
&= \frac{\cos x}{w\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos x' e^{-\frac{x'^2}{2w^2}}\, dx'
\end{aligned}$$

Letting $\beta = \frac{1}{2w^2}$ and $b = 1$, and using the fact that $\cos x'$ and $e^{-\beta x'^2}$ are both even functions this becomes

$$\begin{aligned}
\hat{f}(x,w) &= \frac{\cos x}{w\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos bx'\, e^{-\beta x'^2}\, dx' \\
&= \frac{\cos x}{w\sqrt{2\pi}} 2 \int_{0}^{\infty} e^{-\beta x'^2}\, \cos bx'\, dx'
\end{aligned}$$

Gradshteyn and Ryzhik [1] include a solution for the integral (equation 3.896-4), which we substitute and simplify:

$$\begin{aligned}
\hat{f}(x,w) &= \frac{\cos x}{w\sqrt{2\pi}} 2 \left[\frac{1}{2} \sqrt{\frac{\pi}{\beta}}\, e^{-\frac{b^2}{4\beta}} \right] \\
&= \frac{\cos x}{w\sqrt{2\pi}} \sqrt{\frac{\pi}{\frac{1}{2w^2}}}\, e^{-\frac{1}{4\frac{1}{2w^2}}} \\
&= \frac{\cos x}{w\sqrt{2\pi}} \sqrt{2\pi w^2}\, e^{-\frac{w^2}{2}} \\
&= \cos x\, e^{-\frac{w^2}{2}}
\end{aligned}$$

1.5 floor

Note that by definition, $\mathit{fract}(x) = x - \lfloor x \rfloor$. Therefore, $\lfloor x \rfloor = x - \mathit{fract}(x)$. We band-limit the floor function as a linear combination of the identity function and *fract*. As shown in Section 1.1, the identity function is unchanged under band-limiting. Thus,

$$\widehat{\mathit{floor}}(x, w) = x - \widehat{\mathit{fract}}(x, w)$$

1.6 fract

We define $\mathit{fract}(x) = x - \lfloor x \rfloor$.

1.6.1 Gaussian Kernel (fract_1)

We apply the convolution theorem to compute the convolution of $\mathit{fract}(x)$ with a Gaussian kernel.

We start with the Fourier series expansion for $\mathit{fract}(x)$:

$$\mathit{fract}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi n x + b_n \sin 2\pi n x)$$

where

$$\begin{aligned} a_0 &= 2 \int_0^1 x \, dx \\ a_n &= 2 \int_0^1 x \cos(2\pi n x) \, dx \\ b_n &= 2 \int_0^1 x \sin(2\pi n x) \, dx \end{aligned}$$

The first coefficient, then is simply 1. To solve for a_n, let $u = x$ and $dv = \cos(2\pi n x) \, dx$. Then $du = dx$ and $v = \frac{1}{2\pi n} \sin(2\pi n x)$. Using integration by parts, we have

$$\begin{aligned} a_n &= 2 \left[\frac{x}{2\pi n} \sin(2\pi n x) \right]_0^1 - \frac{2}{2\pi n} \int_0^1 \sin(2\pi n x) \, dx \\ &= 0 + \left[\frac{\cos(2\pi n x)}{2\pi^2 n^2} \right]_0^1 \\ &= 0 \end{aligned}$$

To solve for b_n, let $u = x$ and $dv = \sin(2\pi nx)\,dx$, so that $du = dx$ and $v = -\frac{1}{2\pi n}\cos(2\pi nx)$. Using integration by parts again,

$$\begin{aligned}
b_n &= 2\left[-\frac{x}{2\pi n}\cos(2\pi nx)\right]_0^1 + \frac{2}{2\pi n}\int_0^1 \cos(2\pi nx)\,dx \\
&= -\frac{1}{\pi n} + \left[\frac{\sin(2\pi nx)}{\pi n}\right]_0^1 \\
&= -\frac{1}{\pi n}.
\end{aligned}$$

Substituting these coefficients back into the Fourier series expansion formula, we find

$$\mathit{fract}(x) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{1}{\pi n}\sin(2\pi nx).$$

From this, we can compute the Fourier transforms of *fract* and our Gaussian kernel:

$$\begin{aligned}
\mathcal{F}\left[\mathit{fract}(x)\right](k) &= \frac{1}{2}\delta(k) - \sum_{n=1}^{\infty} \frac{1}{2\pi n} i\left(\delta(k+n) - \delta(k-n)\right) \\
\mathcal{F}\left[\frac{1}{w\sqrt{2\pi}}\,e^{-\frac{x^2}{2w^2}}\right](k) &= \frac{1}{w\sqrt{2\pi}} w\sqrt{2\pi} e^{-2w^2\pi^2k^2} = e^{-2w^2\pi^2k^2}
\end{aligned}$$

Observing that the coefficient of $\delta(k)$ is only relevant when $k = 0$, we multiply these together to get

$$\frac{1}{2}\delta(k) - \sum_{n=1}^{\infty} \frac{e^{-2w^2\pi^2n^2}}{2\pi n} i\left(\delta(k+n) - \delta(k-n)\right)$$

Finally, we take the inverse Fourier transform, which, by the convolution theorem, results in the convolution of our original functions:

$$\hat{f}(x, w) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{e^{-2w^2\pi^2n^2}}{\pi n}\sin(2\pi nx)$$

1.6.2 Box Kernel (fract_2)

We use Heckbert's technique of repeated integration [2] to derive the convolution of $\mathit{fract}(x)$ with a box kernel. This technique requires the computation of the first integral of *fract*:

$$F(x) = \int \mathit{fract}(x')\, dx'$$

We treat $\mathit{fract}(x)$ as a continuous function with a slope of 1 almost everywhere and arbitrarily large slope in the neighborhood of integer values of x. Note that this is consistent with the use of the Fourier series in the previous derivation. We use the first fundamental theorem of calculus to find F:

$$F(b) - F(a) = \int_a^b \mathit{fract}(x')\, dx'.$$

We relate this integral to the integral from $\lfloor a \rfloor$ to $\lfloor b \rfloor$ as follows,

$$\int_a^b \mathit{fract}(x')\, dx' = \int_{\lfloor a \rfloor}^{\lfloor b \rfloor} \mathit{fract}(x')\, dx' - \int_{\lfloor a \rfloor}^{a} \mathit{fract}(x')\, dx' + \int_{\lfloor b \rfloor}^{b} \mathit{fract}(x')\, dx'.$$

The first term may be partitioned at integer values of x' as

$$\int_{\lfloor a \rfloor}^{\lfloor b \rfloor} \mathit{fract}(x')\, dx' = \sum_{n=\lfloor a \rfloor}^{\lfloor b \rfloor} \int_n^{n+1} \mathit{fract}(x')\, dx' - \int_{\lfloor b \rfloor}^{\lfloor b \rfloor + 1} \mathit{fract}(x')\, dx'.$$

Substituting, we get

$$F(b) - F(a) = \sum_{n=\lfloor a \rfloor}^{\lfloor b \rfloor} \int_n^{n+1} \mathit{fract}(x')\, dx' - \int_{\lfloor a \rfloor}^{a} \mathit{fract}(x')\, dx' - \int_{b}^{\lfloor b \rfloor + 1} \mathit{fract}(x')\, dx'$$

Note that, due to the periodicity of $\mathit{fract}(x)$, we can subtract any integer value from both bounds of integration without changing the value of the integral. In particular, since n, $\lfloor a \rfloor$, and $\lfloor b \rfloor$ are all integers, the integral is equivalent to

$$F(b) - F(a) = \sum_{n=\lfloor a \rfloor}^{\lfloor b \rfloor} \int_0^{1} \mathit{fract}(x')\, dx' - \int_{0}^{a - \lfloor a \rfloor} \mathit{fract}(x')\, dx' - \int_{b - \lfloor b \rfloor}^{1} \mathit{fract}(x')\, dx'.$$

Note that the integral in the first term no longer depends on n. In addition, since the bounds of the integrals in each term span a subset of the range $[0,1]$, we can replace $\mathit{fract}(x')$ with x'. Thus,

$$
\begin{aligned}
F(b) - F(a) &= (\lfloor b \rfloor - \lfloor a \rfloor + 1) \int_0^1 x'\,dx' - \int_0^{a-\lfloor a \rfloor} x'\,dx' - \int_{b-\lfloor b \rfloor}^1 x'\,dx' \\
&= \frac{\lfloor b \rfloor - \lfloor a \rfloor + 1}{2} - \frac{(a - \lfloor a \rfloor)^2}{2} - \frac{1 - (b - \lfloor b \rfloor)^2}{2} \\
&= \frac{\lfloor b \rfloor - \lfloor a \rfloor + 1 - \mathit{fract}^2(a) - 1 + \mathit{fract}^2(b)}{2} \\
&= \frac{\mathit{fract}^2(b) + \lfloor b \rfloor}{2} - \frac{\mathit{fract}^2(a) - \lfloor a \rfloor}{2}.
\end{aligned}
$$

Thus, we conclude that

$$
F(x) = \frac{\mathit{fract}^2(x) - \lfloor x \rfloor}{2}.
$$

Using Heckbert's result, the convolution of $\mathit{fract}(x)$ with a box kernel with width w is given by,

$$
\begin{aligned}
\hat{f}(x, w) &= \frac{1}{w}\left(F\left(x + \frac{w}{2}\right) - F\left(x - \frac{w}{2}\right)\right) \\
&= \frac{\mathit{fract}^2\left(x + \frac{w}{2}\right) - \mathit{fract}^2\left(x - \frac{w}{2}\right) + \left\lfloor x + \frac{w}{2} \right\rfloor - \left\lfloor x - \frac{w}{2} \right\rfloor}{2w}
\end{aligned}
$$

1.6.3 Tent Kernel (fract_3)

We again use Heckbert's technique of repeated integration to derive the convolution of $\mathit{fract}(x)$ with a tent kernel. Since the second derivative of the tent kernel consists of three impulses, we must compute the second integral of the $\mathit{fract}(x)$. Starting from the first integral of $\mathit{fract}(x)$, derived above, we wish to find

$$
F(x) = \int \frac{\mathit{fract}^2(x) - \lfloor x \rfloor}{2}\,dx = \frac{1}{2}\int \mathit{fract}^2(x)\,dx - \frac{1}{2}\int \lfloor x \rfloor\,dx.
$$

We compute the two integrals separately. As above, we compute the definite integral and use the first fundamental theorem of calculus to determine the indefinite integral. Starting with the left term, we relate the bounds of integration to $\lfloor a \rfloor$ and $\lfloor b \rfloor$:

$$
\frac{1}{2}\int_{\lfloor a \rfloor}^{\lfloor b \rfloor} \mathit{fract}^2(x')\,dx' - \frac{1}{2}\int_{\lfloor a \rfloor}^{a} \mathit{fract}^2(x')\,dx' + \frac{1}{2}\int_{\lfloor b \rfloor}^{b} \mathit{fract}^2(x')\,dx'.
$$

We partition the first term at integer values of x' leaving us with

$$\frac{1}{2}\sum_{n=\lfloor a\rfloor}^{\lfloor b\rfloor}\int_{n}^{n+1} fract^2(x')\,dx' - \frac{1}{2}\int_{\lfloor a\rfloor}^{a} fract^2(x')\,dx' - \frac{1}{2}\int_{b}^{\lfloor b+1\rfloor} fract^2(x')\,dx'.$$

As above, the periodicity of $fract(x')$ means that subtracting any integer from the bounds of integration does not affect the value of the integral. Thus, this expression is equivalent to

$$\frac{1}{2}\sum_{n=\lfloor a\rfloor}^{\lfloor b\rfloor}\int_{0}^{1} fract^2(x')\,dx' - \frac{1}{2}\int_{0}^{a-\lfloor a\rfloor} fract^2(x')\,dx' - \frac{1}{2}\int_{b-\lfloor b\rfloor}^{1} fract^2(x')\,dx'.$$

Noting that the bounds of integration are all subsets of the range $[0,1]$, we replace $fract^2(x')$ with the equivalent expression x'^2:

$$\frac{\lfloor b\rfloor - \lfloor a\rfloor + 1}{2}\int_{0}^{1} x'^2\,dx' - \frac{1}{2}\int_{0}^{a-\lfloor a\rfloor} x'^2\,dx' - \frac{1}{2}\int_{b-\lfloor b\rfloor}^{1} x'^2\,dx'.$$

Substituting for the integrals, we have

$$\frac{\lfloor b\rfloor - \lfloor a\rfloor + 1}{6} - \frac{(a-\lfloor a\rfloor)^3}{6} - \frac{1-(b-\lfloor b\rfloor)^3}{6},$$

or equivalently,

$$\frac{\lfloor b\rfloor - \lfloor a\rfloor}{6} - \frac{fract^3(a)}{6} + \frac{fract^3(b)}{6}.$$

Thus, the indefinite integral of $\frac{1}{2}fract^2(x)$ is

$$\frac{fract^3(x) + \lfloor x\rfloor}{6}.$$

We now turn our attention to $\frac{1}{2}\int_a^b \lfloor x'\rfloor\, dx'$. By definition, this is equivalent to

$$\frac{1}{2}\int_{a}^{b}(x' - fract(x'))\,dx'.$$

Substituting the known definitions for $\int_a^b x'\,dx'$ and $\int_a^b fract(x')\,dx'$, we have

$$\frac{b^2-a^2}{4} - \left(\frac{fract^2(b)+\lfloor b\rfloor}{4} - \frac{fract^2(a)+\lfloor a\rfloor}{4}\right),$$

which we can regroup as

$$\frac{b^2 - \mathit{fract}^2(b) - \lfloor b \rfloor}{4} - \frac{a^2 - \mathit{fract}^2(a) - \lfloor a \rfloor}{4}.$$

Thus, the indefinite integral of $\frac{1}{2} \lfloor x \rfloor$ is

$$\frac{x^2 - \mathit{fract}^2(x) - \lfloor x \rfloor}{4}.$$

Putting these together, we have

$$\begin{aligned} F(x) &= \frac{\mathit{fract}^3(x) + \lfloor x \rfloor}{6} + \frac{x^2 - \mathit{fract}^2(x) - \lfloor x \rfloor}{4} \\ &= \frac{2\mathit{fract}^3(x) + 2\lfloor x \rfloor}{12} + \frac{3x^2 - 3\mathit{fract}^2(x) - 3\lfloor x \rfloor}{12} \\ &= \frac{3x^2 + 2\mathit{fract}^3(x) - 3\mathit{fract}^2(x) - \lfloor x \rfloor}{12} \\ &= \frac{3x^2 + 2\mathit{fract}^3(x) - 3\mathit{fract}^2(x) - x + \mathit{fract}(x)}{12}. \end{aligned}$$

Therefore, using Heckbert's result, the convolution of $\mathit{fract}(x)$ with a tent kernel with width w is given by

$$\hat{f}(x, w) = \frac{1}{w^2}(F(x + w) - 2F(x) + F(x + w)).$$

1.7 saturate

Let $f(x) = \mathit{saturate}(x) = \max(0, \min(1, x))$. Then,

$$\begin{aligned} \hat{f}(x, w) &= \frac{1}{w\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathit{saturate}(x - x') e^{-\frac{x'^2}{2w^2}}\, dx' \\ &= \frac{1}{w\sqrt{2\pi}} \int_{-\infty}^{\infty} \max(0, \min(1, x - x')) e^{-\frac{x'^2}{2w^2}}\, dx'. \end{aligned}$$

Note that $\min(1, x - x') \leq 0$ when $x \leq x'$. Thus, $\max(0, \min(1, x - x')) = 0$ when $x \leq x'$. Also note that $\min(1, x - x') > 0$ when $x' < x$. Therefore, $\max(0, \min(1, x - x')) = \min(1, x - x')$ when $x' < x$. Thus, we can simplify the above integral without reference to the max function:

$$\hat{f}(x, w) = \frac{1}{w\sqrt{2\pi}} \int_{-\infty}^{x} \min(1, x - x') e^{-\frac{x'^2}{2w^2}}\, dx'.$$

Now note that when $x' \leq x-1$, $\min(1, x-x') = 1$ and when $x-1 < x' \leq x$, $\min(1, x-x') = x - x'$. Thus, we can partition the integral into two terms without reference to the min function:

$$\begin{aligned}
\hat{f}(x,w) &= \frac{1}{w\sqrt{2\pi}} \int_{x-1}^{x} (x-x') e^{-\frac{x'^2}{2w^2}}\, dx' + \frac{1}{w\sqrt{2\pi}} \int_{-\infty}^{x-1} e^{-\frac{x'^2}{2w^2}}\, dx' \\
&= \frac{x}{w\sqrt{2\pi}} \int_{x-1}^{x} e^{-\frac{x'^2}{2w^2}}\, dx' - \frac{1}{w\sqrt{2\pi}} \int_{x-1}^{x} x' e^{-\frac{x'^2}{2w^2}}\, dx' \\
&\quad + \frac{1}{w\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{x'^2}{2w^2}}\, dx' + \frac{1}{w\sqrt{2\pi}} \int_{0}^{x-1} e^{-\frac{x'^2}{2w^2}}\, dx' \\
&= \frac{x}{w\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{x'^2}{2w^2}}\, dx' - \frac{x}{w\sqrt{2\pi}} \int_{-\infty}^{x-1} e^{-\frac{x'^2}{2w^2}}\, dx' \\
&\quad - \frac{1}{w\sqrt{2\pi}} \int_{-\infty}^{x} x' e^{-\frac{x'^2}{2w^2}}\, dx' + \frac{1}{w\sqrt{2\pi}} \int_{-\infty}^{x-1} x' e^{-\frac{x'^2}{2w^2}}\, dx' \\
&\quad + \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{x-1}{w\sqrt{2}}\right) \\
&= \frac{x}{2} \operatorname{erf}\left(\frac{x}{w\sqrt{2}}\right) - \frac{x}{2} \operatorname{erf}\left(\frac{x-1}{w\sqrt{2}}\right) + \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{x-1}{w\sqrt{2}}\right) \\
&\quad + \frac{w}{\sqrt{2\pi}} \left(1 - e^{-\frac{(x-1)^2}{2w^2}}\right) - \frac{w}{\sqrt{2\pi}} \left(1 - e^{-\frac{x^2}{2w^2}}\right) \\
&= \frac{x}{2} \operatorname{erf}\left(\frac{x}{w\sqrt{2}}\right) - \frac{x-1}{2} \operatorname{erf}\left(\frac{x-1}{w\sqrt{2}}\right) + \frac{w}{\sqrt{2\pi}} \left(e^{-\frac{x^2}{2w^2}} - e^{-\frac{(x-1)^2}{2w^2}}\right) + \frac{1}{2} \\
&= \frac{1}{2} \left(x \operatorname{erf}\left(\frac{x}{w\sqrt{2}}\right) - (x-1) \operatorname{erf}\left(\frac{x-1}{w\sqrt{2}}\right) + w\sqrt{\frac{2}{\pi}} \left(e^{-\frac{x^2}{2w^2}} - e^{-\frac{(x-1)^2}{2w^2}}\right) + 1\right)
\end{aligned}$$

1.8 sin

Let $f(x) = \sin x$. Then,

$$\hat{f}(x,w) = \frac{1}{w\sqrt{2\pi}} \int_{-\infty}^{\infty} \sin(x-x') e^{-\frac{x'^2}{2w^2}}\, dx'.$$

Using the identity

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta,$$

we substitute to get

$$\begin{aligned}\hat{f}(x,w) &= \frac{1}{w\sqrt{2\pi}}\int_{-\infty}^{\infty}(\sin x\cos x' - \cos x\sin x')e^{-\frac{x'^2}{2w^2}}\,dx' \\ &= \frac{1}{w\sqrt{2\pi}}\int_{-\infty}^{\infty}\sin x\cos x' e^{-\frac{x'^2}{2w^2}}\,dx' - \frac{1}{w\sqrt{2\pi}}\int_{-\infty}^{\infty}\cos x\sin x' e^{-\frac{x'^2}{2w^2}}\,dx'\end{aligned}$$

Note that the second integral evaluates to 0, since the portion from $infty$ to 0 has the same magnitude but opposite sign as the portion from 0 to ∞. Thus, we are left with

$$\begin{aligned}\hat{f}(x,w) &= \frac{1}{w\sqrt{2\pi}}\int_{-\infty}^{\infty}\sin x\cos x' e^{-\frac{x'^2}{2w^2}}\,dx' \\ &= \frac{\sin x}{w\sqrt{2\pi}}\int_{-\infty}^{\infty}\cos x' e^{-\frac{x'^2}{2w^2}}\,dx'\end{aligned}$$

Letting $\beta = \frac{1}{2w^2}$ and $b = 1$, and using the fact that $\cos x'$ and $e^{-\beta x'^2}$ are both even functions, this becomes

$$\begin{aligned}\hat{f}(x,w) &= \frac{\sin x}{w\sqrt{2\pi}}\int_{-\infty}^{\infty}\cos bx'\, e^{-\beta x'^2}\,dx' \\ &= \frac{\sin x}{w\sqrt{2\pi}}2\int_{0}^{\infty}e^{-\beta x'^2}\cos bx'\,dx'\end{aligned}$$

Gradshteyn and Ryzhik [1] include a solution for the integral (equation 3.896-4), which we substitute and simplify:

$$\begin{aligned}\hat{f}(x,w) &= \frac{2\sin x}{w\sqrt{2\pi}}\left[\frac{1}{2}\sqrt{\frac{\pi}{\beta}}\,e^{-\frac{b^2}{4\beta}}\right] \\ &= \frac{\sin x}{w\sqrt{2\pi}}\sqrt{\frac{\pi}{\frac{1}{2w^2}}}\,e^{-\frac{1}{4\frac{1}{2w^2}}} \\ &= \frac{\sin x}{w\sqrt{2\pi}}\sqrt{2\pi w^2}\,e^{-\frac{w^2}{2}} \\ &= \sin x\, e^{-\frac{w^2}{2}}\end{aligned}$$

1.9 x^2

Let $f(x) = x^2$. Then,

$$\begin{aligned}\hat{f}(x,w) &= \frac{1}{w\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-x')^2 e^{-\frac{x'^2}{2w^2}}\,dx' \\ &= \frac{1}{w\sqrt{2\pi}}\int_{-\infty}^{\infty}(x^2-2xx'+x'^2) e^{-\frac{x'^2}{2w^2}}\,dx' \\ &= x^2 - \frac{2x}{w\sqrt{2\pi}}\int_{-\infty}^{\infty} x' e^{-\frac{x'^2}{2w^2}}\,dx' + \frac{1}{w\sqrt{2\pi}}\int_{-\infty}^{\infty} x'^2 e^{-\frac{x'^2}{2w^2}}\,dx'\end{aligned}$$

Note that the second integral evaluates to 0, since the portion from $-\infty$ to 0 has the same magnitude but opposite sign as the portion from 0 to ∞. Thus, we are left with

$$\begin{aligned}\hat{f}(x,w) &= x^2 + \frac{1}{w\sqrt{2\pi}}\int_{-\infty}^{\infty} x'^2 e^{-\frac{x'^2}{2w^2}}\,dx' \\ &= x^2 + \frac{1}{w\sqrt{2\pi}} w^3\sqrt{2\pi} \\ &= x^2 + w^2\end{aligned}$$

1.10 step

Let $f(x) = \mathit{step}(x) = H(x)$, where H is the Heaviside step function. Thus,

$$\hat{f}(x,w) = \frac{1}{w\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathit{step}(x-x') e^{-\frac{x'^2}{2w^2}}\,dx'$$

By definition,

$$H(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{otherwise} \end{cases}$$

Therefore $\mathit{step}(x-x') = 0$ when $x < x'$ and $\mathit{step}(x-x') = 1$ when $x > x'$.

Thus, we can simplify the integral without reference to the step function:

$$\begin{aligned}
\hat{f}(x,w) &= \frac{1}{w\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{x'^2}{2w^2}}\, dx' \\
&= \frac{1}{w\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{x'^2}{2w^2}}\, dx' + \frac{1}{w\sqrt{2\pi}} \int_{0}^{x} e^{-\frac{x'^2}{2w^2}}\, dx' \\
&= \frac{1}{2} + \frac{1}{w\sqrt{2\pi}} \frac{w\sqrt{2\pi}}{2} \operatorname{erf} \frac{x}{w\sqrt{2}} \\
&= \frac{1}{2} \left(1 + \operatorname{erf} \frac{x}{w\sqrt{2}} \right)
\end{aligned}$$

1.11 trunc

We define a function, $trunc(x)$, that rounds x toward zero. That is,

$$trunc(x) = \begin{cases} \lfloor x \rfloor & \text{if } x \geq 0 \\ \lceil x \rceil & \text{otherwise} \end{cases}$$

Note that, when x is not an integer, $\lceil x \rceil = \lfloor x \rfloor + 1$. Since the case when x is an integer has measure zero, it does not effect the result of the integral. Thus, we can define *trunc* as

$$trunc(x) = \begin{cases} \lfloor x \rfloor + 0 & \text{if } x \geq 0 \\ \lfloor x \rfloor + 1 & \text{otherwise} \end{cases}$$

Noting the similarity to $step(x)$, we arrive at our final definition of $trunc(x)$:

$$trunc(x) = \lfloor x \rfloor - step(x) + 1$$

We band-limit $trunc(x)$ as a linear combination of the floor function and step function. That is,

$$\widehat{trunc}(x,w) = \widehat{floor}(x,w) - \widehat{step}(x,w) + 1$$

2 Summary of Sampling Functions

$f(x)$	$\hat{f}(x,w)$
x	x
x^2	$x^2 + w^2$
$\mathit{fract}(x)$	$\frac{1}{2} - \sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{\pi n} e^{-2c^2\pi^2 n^2}$
$\lvert x \rvert$	$x \operatorname{erf} \frac{x}{w\sqrt{2}} + w\sqrt{\frac{2}{\pi}}\, e^{-\frac{x^2}{2w^2}}$
$\lfloor x \rfloor$	$x - \widehat{\mathit{fract}}(x,w)$
$\lceil x \rceil$	$\widehat{\mathit{floor}}(x,w) + 1$
$\cos x$	$\cos x\, e^{-\frac{w^2}{2}}$
$\sin x$	$\sin x\, e^{-\frac{w^2}{2}}$
$\mathit{saturate}(x)$	$\frac{1}{2}\left(x \operatorname{erf}\left(\frac{x}{w\sqrt{2}}\right) - (x-1) \operatorname{erf}\left(\frac{x-1}{w\sqrt{2}}\right) + w\sqrt{\frac{2}{\pi}}\left(e^{-\frac{x^2}{2w^2}} - e^{-\frac{(x-1)^2}{2w^2}}\right) + 1\right)$
$\mathit{step}(\alpha, x)$	$\frac{1}{2}\left(1 + \operatorname{erf} \frac{x-\alpha}{w\sqrt{2}}\right)$
$\mathit{trunc}(x)$	$\widehat{\mathit{floor}}(x,w) - \widehat{\mathit{step}}(x,w) + 1$

References

[1] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products.* Academic Press, Amsterdam, fifth edition, 1994.

[2] P. S. Heckbert. Filtering by repeated integration. *SIGGRAPH Computer Graphics*, 20(4), Aug 1986.