Abstract Interpretation (Non-Standard Semantics)

a.k.a. “Picking The Right Abstraction”

Why analyze programs statically?

The Problem

- It is extremely useful to predict program behavior *statically* (= without running the program)
  - For optimizing compilers, program analyses, software engineering tools, finding security flaws, etc.
- The semantics we studied so far give us the precise behavior of a program
- However, precise static predictions are impossible
  - The exact semantics is not computable
- We must settle for approximate, but correct, static analyses (e.g. VC vs. WP)

One-Slide Summary

- **Abstraction interpretation** is a static analysis for soundly approximating the semantics of a program.
- While the concrete semantics refers to what actually happens when you run the program (e.g., “x*x+1” may result in multiple integers), the abstract semantics tracks only certain information about that computation (e.g., “x*x+1” will be some *positive* number, but we don't know which one).
- Special functions transfer between the abstract domain (typically a lattice) and the concrete domain.

The Plan

- We will introduce abstract interpretation by example
- Starting with a miniscule language we will build up to a fairly realistic application
- Along the way we will see most of the ideas and difficulties that arise in a big class of applications

A Tiny Language

- Consider the following language of arithmetic ("shrIMP"?)

$$e ::= n \mid e_1 * e_2$$

- The operational semantics of this language

$$n \Downarrow n$$

$$e_1 * e_2 \Downarrow = e_1 \Downarrow \times e_2 \Downarrow$$

- We'll take opsem as the "ground truth"
- For this language the precise semantics is computable (but in general it's not)

An Abstraction

- Assume that we are interested not in the value of the expression, but only in its sign:
  - positive (+), negative (-), or zero (0)
- We can define an abstract semantics that computes ***only*** the sign of the result

$$\sigma: \text{Exp} \to \{-, 0, +\}$$

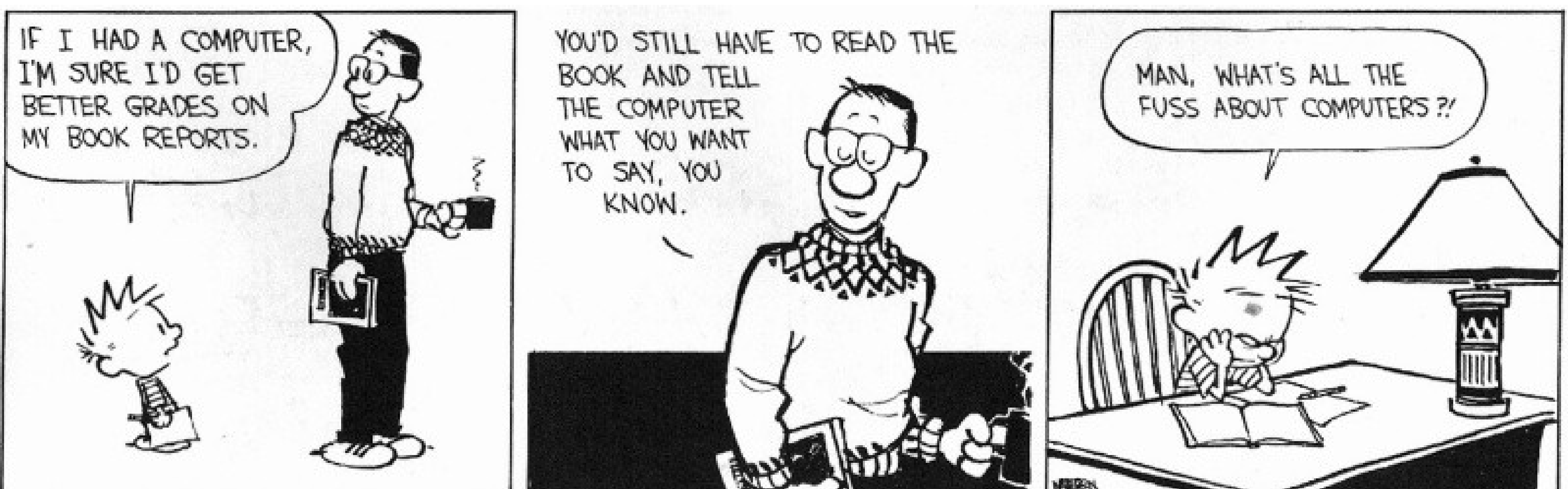
$\otimes$	-	0	+
-	+	0	-
0	0	0	0
+	-	0	+

$$\sigma(n) = \text{sign}(n)$$

$$\sigma(e_1 * e_2) = \sigma(e_1) \otimes \sigma(e_2)$$

I Saw the Sign

- Why did we want to compute the sign of an expression?
  - One reason: no one will believe you know abstract interpretation if you haven't seen the sign example :-)
- What could we be computing instead?



Correctness of Sign Abstraction

- We can show that the abstraction is correct in the sense that it predicts the sign

  $e\Downarrow > 0 \Leftrightarrow \sigma(e) = +$

  $e\Downarrow = 0 \Leftrightarrow \sigma(e) = 0$

  $e\Downarrow < 0 \Leftrightarrow \sigma(e) = -$

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  $e\Downarrow < 0 \Leftrightarrow \sigma(e) = -$

- Our semantics is abstract but precise
- Proof is by structural induction on the expression e
  - Each case repeats similar reasoning

Another View of Soundness

- Link each concrete value to an abstract one:

$$\beta : \mathbb{Z} \to \{ -, 0, + \}$$

- This is called the abstraction function (β)
  - This three-element set is the abstract domain
- Also define the concretization function (γ):

$$\gamma : \{-, 0, +\} \to \mathcal{P}(\mathbb{Z})$$

$$\gamma(+) = \{ n \in \mathbb{Z} \mid n > 0 \}$$

$$\gamma(0) = \{ 0 \}$$

$$\gamma(-) = \{ n \in \mathbb{Z} \mid n < 0 \}$$

Another View of Soundness 2

- Soundness can be stated succinctly

$$\forall e \in \text{Exp}.\ e\Downarrow \in \gamma(\sigma(e))$$

(the real value of the expression is among the concrete values represented by the abstract value of the expression)

- Let C be the concrete domain (e.g. $\mathbb{Z}$) and A be the abstract domain (e.g. {-, 0, +})
- Commutative diagram:

$$\begin{tikzcd}
\text{Exp} \arrow[r, "\sigma"] \arrow[d, "\Downarrow"'] & A \arrow[d, "\gamma"] \\
C \arrow[r, "\in"] & \mathcal{P}(C)
\end{tikzcd}$$

Another View of Soundness 3

- Consider the generic abstraction of an operator

$$\sigma(e_1 \text{ op } e_2) = \sigma(e_1) \ \underline{\text{op}} \ \sigma(e_2)$$

- This is sound iff

$$\forall a_1 \forall a_2.\ \gamma(a_1 \ \underline{\text{op}} \ a_2) \supseteq \{n_1 \text{ op } n_2 \mid n_1 \in \gamma(a_1),\ n_2 \in \gamma(a_2)\}$$

- e.g. $\gamma(a_1 \otimes a_2) \supseteq \{ n_1 * n_2 \mid n_1 \in \gamma(a_1),\ n_2 \in \gamma(a_2) \}$
- This reduces the proof of correctness to one proof for each operator

Abstract Interpretation

- This is our first example of an abstract interpretation
- We carry out computation in an abstract domain
- The abstract semantics is a sound approximation of the standard semantics
- The concretization and abstraction functions establish the connection between the two domains

Adding Unary Minus and Addition

- We extend the language to

  $e ::= n \mid e_1 * e_2 \mid - e$

- We define $\sigma(- e) = \ominus\, \sigma(e)$

	-	0	+
$\ominus$	+	0	-

$\oplus$	-	0	+
-	-	-	?
0	-	0	+
+	?	+	+

- Now we add addition:

  $e ::= n \mid e_1 * e_2 \mid - e \mid e_1 + e_2$

- We define $\sigma(e_1 + e_2) = \sigma(e_1) \oplus \sigma(e_2)$

Adding Addition

- The sign values are not closed under addition
- What should be the value of "$+ \oplus -$"?
- Start from the soundness condition:

$$\gamma(+ \oplus -) \supseteq \{ n_1 + n_2 \mid n_1 > 0, n_2 < 0\} = \mathbb{Z}$$

- We don't have an abstract value whose concretization includes $\mathbb{Z}$, so we add one:

  $\top$ ("top" = "don't know")

$\oplus$	-	0	+	$\top$
-	-	-	$\top$	$\top$
0	-	0	+	$\top$
+	$\top$	+	+	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$

Loss of Precision

- Abstract computation may lose information:

$$[\![(1 + 2) + -3]\!] = 0$$

but: $\sigma((1+2) + -3) =$

$$(\sigma(1) \oplus \sigma(2)) \oplus \sigma(-3) =$$

$$(+ \oplus +) \oplus - = \top$$

- We lost some precision
- But this will simplify the computation of the abstract answer in cases when the precise answer is not computable

Adding Division

- Straightforward except for division by 0
  - We say that there is no answer in that case
  - $\gamma(+ \oslash 0) = \{ n \mid n = n_1 / 0 , n_1 > 0 \} = \emptyset$
- Introduce $\perp$ to be the abstraction of the $\emptyset$
  - We also use the same abstraction for non-termination!

$\perp$ = “nothing”

$\top$ = “something unknown”

$\oslash$	-	0	+	$\top$	$\perp$
-	+	0	-	$\top$	$\perp$
0	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
+	-	0	+	$\top$	$\perp$
$\top$	$\top$	$\top$	$\top$	$\top$	$\perp$
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$

Game Criticism

- This term refers to a conflict between the mechanics or dynamics of a game and its story. For example, *Bioshock* was viewed as promoting selflessness through story but selfishness through gameplay, a disconnect that pulled some players out of the game. The term is often viewed as “highbrow” or “pretentious”.

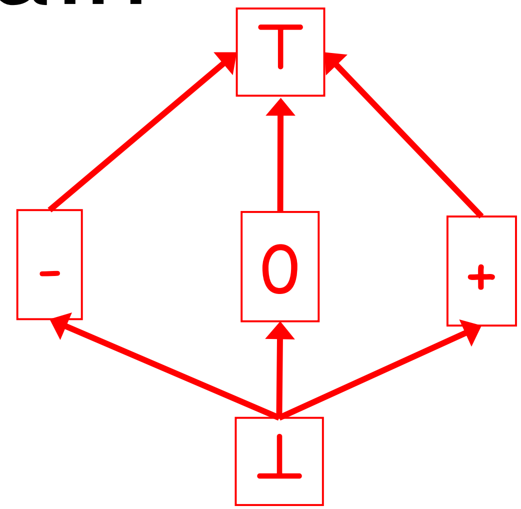
Q: Books (750 / 842)

- This 1962 Newbery Medal-winning novel by Madeleine L'Engle includes Charles Wallace, Mrs. Who, Mrs. Whatsit, Mrs. Which and the space-bending Tesseract.

Computer Science

- This American Turing-award winner is known for developing Speedcoding and FORTRAN (the first two high-level languages), as well creating a way to express the formal syntax of a language and using that approach to specify ALGOL. He later focused on function-level (as opposed to value-level) programming. His first major programming project calculated the positions of the Moon.

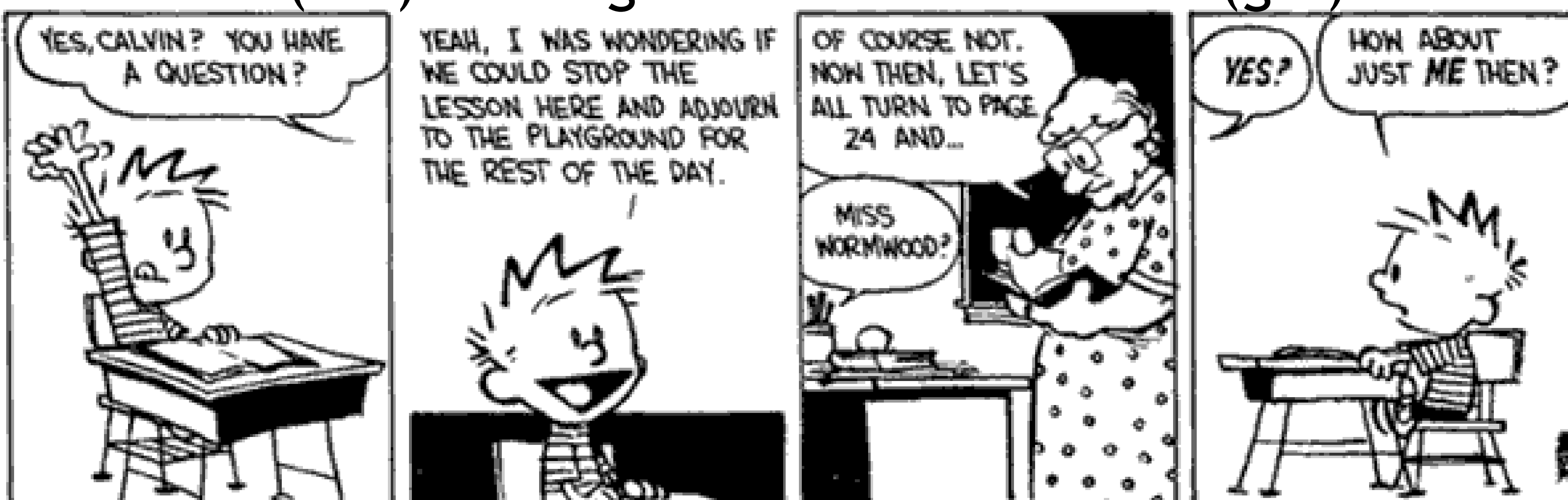
The Abstract Domain



- Our abstract domain forms a lattice
- A partial order is induced by γ

$$a_1 \le a_2 \quad \text{iff } \gamma(a_1) \subseteq \gamma(a_2)$$

  - We say that a_1 is more precise than a_2!
- Every finite subset has a least-upper bound (lub) and a greatest-lower bound (glb)

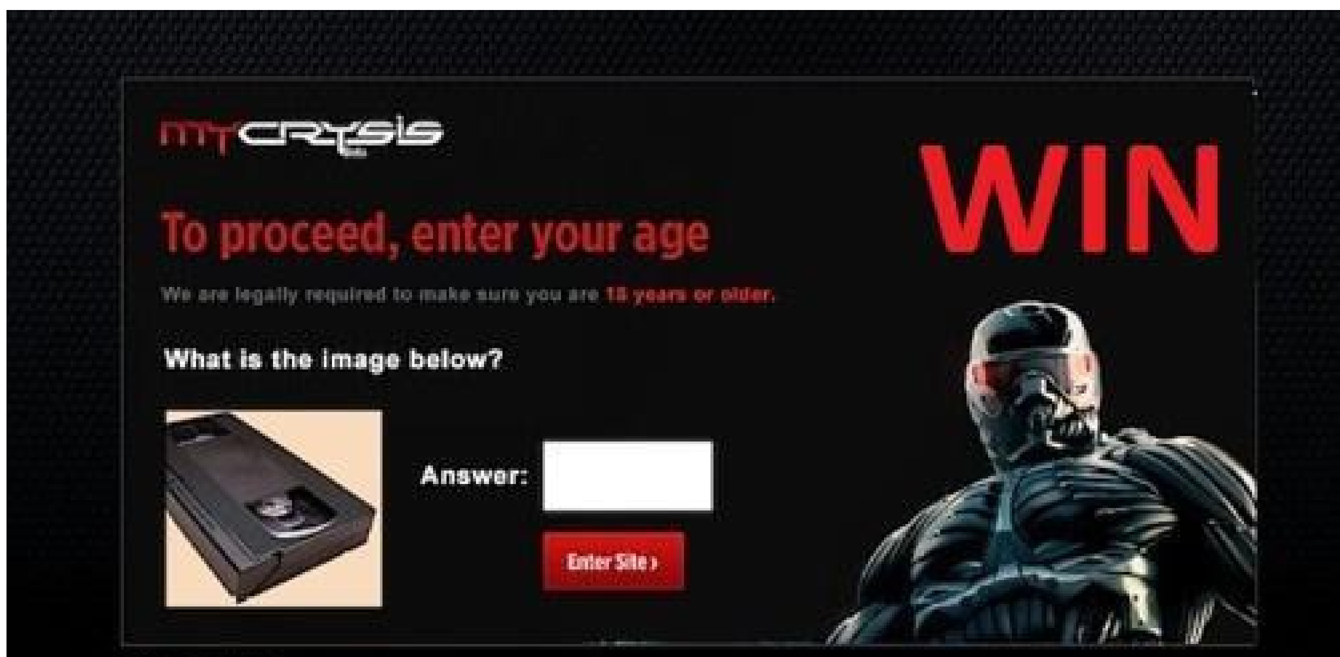


Lattice Facts

- A lattice is <u>complete</u> when every subset has a lub and a gub
  - Even infinite subsets!
- Every finite lattice is (trivially) complete
- Every complete lattice is a complete partial order (recall: proof techniques: induction!)
  - Since a chain is a subset
- Not every CPO is a complete lattice
  - Might not even be a lattice at all

Lattice History

- Early work in denotational semantics used lattices (instead of what?)
  - But only chains need to have lubs
  - And there was no need for ⊤ and glb



Lattice History

- Early work in denotational semantics used lattices (instead of what?)
  - But only chains need to have lubs
  - And there was no need for $\top$ and glb

- In abstract interpretation we'll use $\top$ to denote *"I don't know"*.
  - Corresponds to all values in the concrete domain

From One, Many

- We can start with the abstraction function β

$$\beta : C \to A$$

(maps a concrete value to the best abstract value)

  - A must be a lattice

- We can derive the concretization function γ

$$\gamma : A \to \mathcal{P}(C)$$

$$\gamma(a) = \{ x \in C \mid \beta(x) \leq a \}$$

- And the abstraction for sets α

$$\alpha : \mathcal{P}(C) \to A$$

$$\alpha(S) = \text{lub} \{ \beta(x) \mid x \in S \}$$

Example

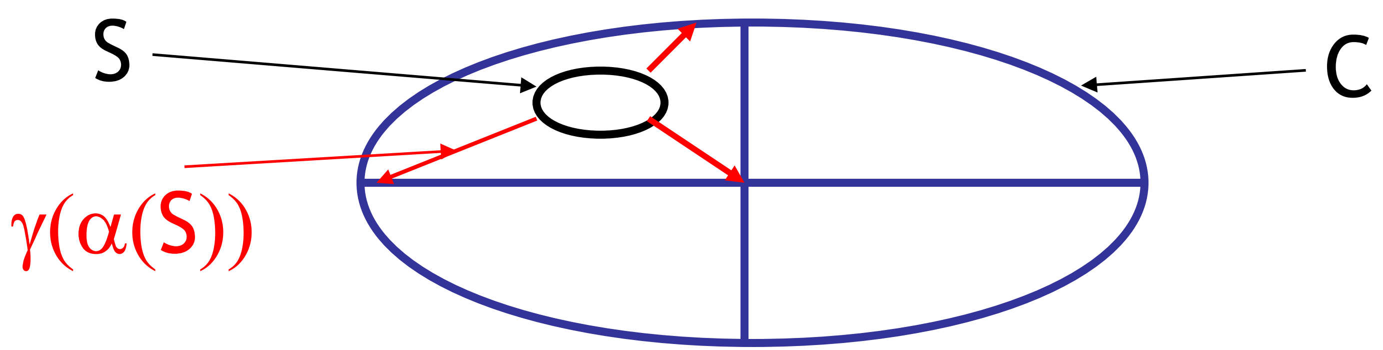
- Consider our sign lattice

$$\beta(n) = \begin{cases} + & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ - & \text{if } n < 0 \end{cases}$$

- $\alpha(S) = \text{lub } \{ \beta(x) \mid x \in S \}$
  - Example: $\alpha(\{1, 2\}) = \text{lub } \{ + \} = +$
    $\alpha(\{1, 0\}) = \text{lub } \{ +, 0 \} = \top$
    $\alpha(\{\}) = \text{lub } \emptyset = \bot$
- $\gamma(a) = \{ n \mid \beta(n) \leq a \}$
  - Example: $\gamma(+) = \{ n \mid \beta(n) \leq + \} =$
    $\{ n \mid \beta(n) = + \} = \{ n \mid n > 0 \}$
    $\gamma(\top) = \{ n \mid \beta(n) \leq \top \} = \mathbb{Z}$
    $\gamma(\bot) = \{ n \mid \beta(n) \leq \bot \} = \emptyset$

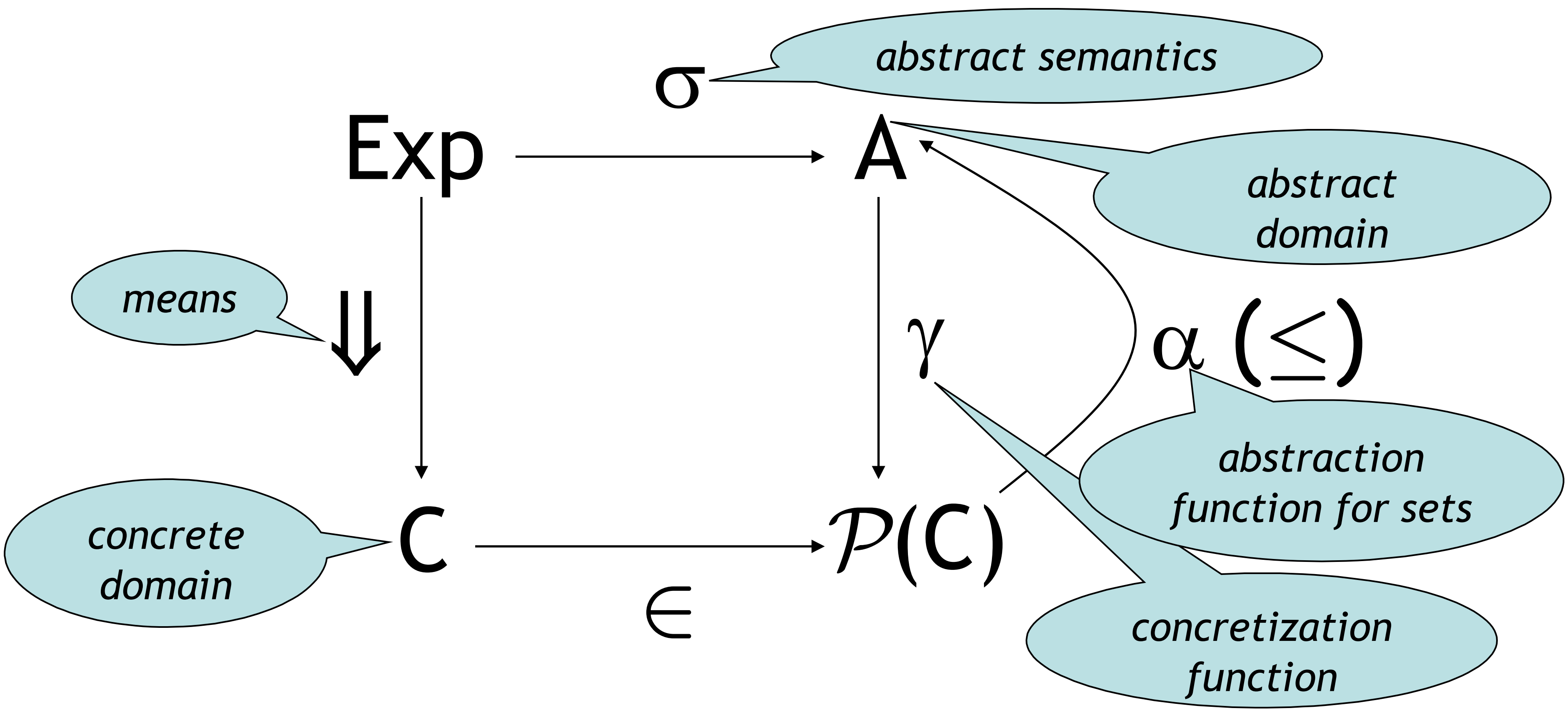
Galois Connections

- We can show that
  - γ and α are monotonic (with $\subseteq$ ordering on $\mathcal{P}(C)$)
  - $\alpha(\gamma(a)) = a$ for all $a \in A$
  - $\gamma(\alpha(S)) \supseteq S$ for all $S \in \mathcal{P}(C)$
- Such a pair of functions is called a Galois connection
  - Between the lattices A and $\mathcal{P}(C)$



Correctness Condition

- In general, abstract interpretation satisfies the following (amazingly common) diagram



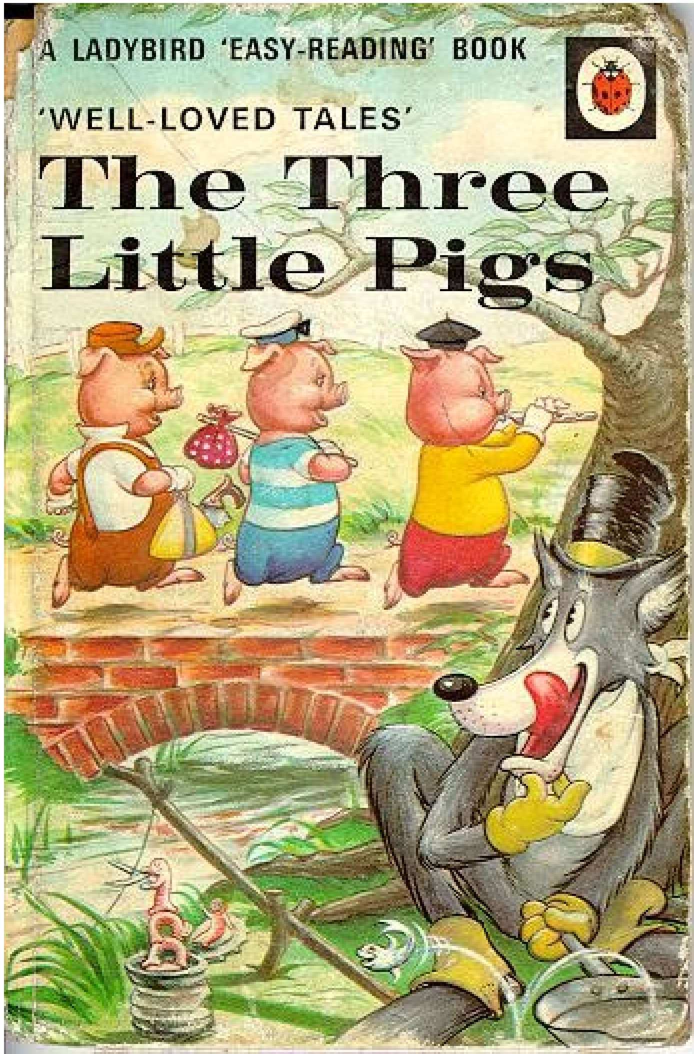
Three Little Correctness Conditions

- Three conditions define a correct abstract interpretation
- α and γ are monotonic
- α and γ form a Galois connection

  = "α and γ are almost inverses"

1. Abstraction of operations is correct

$$a_1 \underline{op} a_2 = \alpha(\gamma(a_1) \text{ op } \gamma(a_2))$$



"On The Board" Questions

- What is the VC for:

$$\text{for } i = e_{low} \text{ to } e_{high} \text{ do } c \text{ done}$$

- This axiomatic rule is unsound. Why?

$$\frac{\vdash \{A \wedge p\}\ c_{then}\ \{B_{then}\} \qquad \vdash \{A \wedge \neg p\}\ c_{else}\ \{B_{else}\}}{\vdash \{A\} \text{ if } p \text{ then } c_{then} \text{ else } c_{else}\ \{B_{then} \vee B_{else}\}}$$

Homework

- Read Cousot & Cousot Article